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THE PARADIGM OF ÉVARISTE GALOIS

The article is dedicated to the scientific work of brilliant thinker, philosopher, mathematician Évariste Galois. The author discloses complicated contradictory way of formation a new paradigm in modern mathematics — the theory of groups. He reveals the role of the theory of groups in quantum mechanics, theory of relativity, crystallography, in solution the biggest mathematics riddle — Ferm's Great theorem.

Key words: *mathematics, science, the theory of groups, theoretical-group method, symmetry, Ferm's Great theorem.*

Пугач Б. Я. ПАРАДИГМА ЭВАРІСТА ГАЛУА *Стаття присвячена науковій творчості геніального мислителя, філософа, математика Евариста Галуа. Розкрито складний, суперечливий шлях формування нової парадигми сучасної математики — теорії груп. Виявлена роль теорії груп у квантовій механіці, теорії відносності, вирішенні найбільшої загадки математики — Великої теореми Ферма.*

Ключові слова: *математика, наука, філософія, теорія груп, теоретико-груповий метод, симетрія, Велика теорема Ферма.*

Пугач Б. Я. ПАРАДИГМА ЭВАРИСТА ГАЛУА *Статья посвящена научному творчеству гениального мыслителя, философа, математика Эвариста Галуа. Раскрыт сложный, противоречивый путь формирования новой парадигмы современной математики — теории групп. Выявлена роль теории групп в квантовой механике, теории относительности, в решении самой большой загадки математики — Великой теоремы Ферма.*

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The 26th of October is the momentous date in the history of science, the day when Évariste Galois, one of the thinkers and mathematicians of genius, was born (1811—1832). Like Nicolaus Copernicus who made a breakthrough in physics and astronomy and Immanuel Kant who made a revolution in philosophy, Galois made fundamental, revolutionary changes in mathematics.

Galois introduced a completely *new approach, a new point of view* into mathematics. He took a most important and necessary step into abstraction. In his works, "mathematics ceased to be the study of numbers and shapes-arithmetic, geometry, and ideas that developed out of them like algebra and trigonometry. It became the study of structure. What had been a study *of things* became a study *of processes*." Such a fundamental conclusion is made by a modern British mathematician Ian Steward. The importance of Galois's works lies in the fact that in his works he fully discloses entirely new and profound mathematical laws and that gave rise to *a new mathematical paradigm, i.e. theory of fields, concept of groups or Galois theory*. An entire branch of mathematics, a calculus of symmetry called group theory, came into being and has since invaded every corner of mathematics and many domains of scientific cognition. This is the *philosophical and mathematical role, and actuality of Galois's paradigm*. In his book *"The Ten Great*

Ideas of Science", Peter Atkins emphasizes that this programme "may have application of almost unlimited generality".
Rise of Galois' concept goes hand in hand with his daring reasoning on new methods and the path of development for mathematics. He calls attention to a tendency of some mathematicians to avoid any calculations at all. "Instead of algebraic formulae, they make use of lengthy arguments," writes Galois, "and, to lengthiness of mathematical transformations, they add lengthiness of wording of these transformations, using language not fit for such tasks. *These mathematicians are a hundred years behind the times*".
They were not the example Galois followed in his work. He has nothing like it. The thinker states, *"I speak of the analysis of analysis*. In this context *the highest calculations* that have been so far performed *(elliptical functions)* will be regarded simply as particular cases, the treatment of which was useful, even essential, but which it would be disastrous not to go beyond and embark on a broader search. There will be time to carry out *the detailed calculations* envisaged by *this higher analysis*, in which concepts are *classified according to their difficulties*, but not specified in their form, when special questions call for it". Furthermore, Galois gives a most important statement that *"the general thesis"* which may be understood as the general method, can be *"deciphered"* as a certain heuristic code only after a thorough consideration of all the work (pattern) where all the power of the study results reveals itself.
Galois profoundly ponders on *the philosophical foundations of the scientific quest, courses to obtain new scientific results*, as well as on problems of ethics, e.g. *honesty and fidelity* in science. During his incarceration in Sainte Pélagie prison he defines a goal to create his own philosophical programme. Galois writes: "We are going *to present* in several papers the most general and *most philosophical part* of our *investigations* which could not have been published earlier because of a thousand reasons. We will present only these provisions without obscuring them with examples and supplements among which analytics tend to lose their general ideas. They will be presented most diligently and we will honestly recount about the way that led us to these provisions and about the obstacles that we met. When this aim is achieved we can consider that we have shown an example of diligence of the degree never known before".
The strikingly lucid mind of the young mathematician takes interest not only in solution of most intricate and unsolvable problems of science but addresses the problem of solidarity of the scientist of the future: "*Scientists* are made for solitary existence no more so than all the other people. They also belong to their time and sooner or later they *will begin to act together*. How much extra time will be left for science!"
It is plausible to assume that no scientist in the history of science enjoyed such *a unity of scientific and social ideals as Évariste Galois*. Perhaps, never before this unity excited such an anger of men of scientific and state power.
The focus of Galois's attention was his main problem, i.e. the solvability of general algebraic equations by radicals. And not just the quintic equations that were dealt with by a wonderful Norwegian mathematician Niels Abel (1802—1829). Galois's aim was to find a criterion, method of solvability for all algebraic equations. Let us turn to some aspects of Galois theory.
First and foremost, Galois attempts to determine the concept of value rational in relation to other values. He defines it in *his "Memoir on the Conditions for Solvability of Equations by Radicals"* (1831): "One can agree to regard as rational all rational functions of a certain number of determined quantities, supposed to be known *a priori*. For example, one can choose a particular root of a whole number and regard as rational every rational function of this radical.
When we agree to regard certain quantities as known in this manner, we shall say that we *adjoin* them to the equation to be resolved. We shall say that these quantities are *adjoined* to the equation.

With these conventions, we shall call any rational any quantity which can be expressed as a rational function of the coefficients of the equation and of a certain number of *adjoined* quantities arbitrarily agreed upon". Here, these concepts of rational value are definitely expressed, and Galois closely approaches the concept of *field* generated by the set of algebraic numbers.

Then Galois introduces his key concept of *"equation group"*: "Let an equation be given whose *m* roots are *a, b, c, ...* There will always be a group of permutations of the letters *a, b, c, ...* which will have the following property:

1. that each function invariant under the substitution of this group will be known rationally;
2. conversely, that every function of the roots which can be determined rationally will be invariant under these substitutions".

Here we should notice that *invariance* is an integral and required *property of any group*. It is this attribute that determines *the methodological value* and generality of a certain mathematical concept as well *as the possibility of its application in other fields of scientific cognition*. It is interesting to note that not just certain values are invariant but mathematical functions as well. "Here we call a function invariant not only if its form is unchanged by the substitutions of the roots, but also if its numerical value does not vary when these substitutes are applied", states Galois.

Évariste Galois revolutionized mathematics. He is the author, inventor of a language that made it possible to describe symmetries in mathematical structures and deduce their eff ects.

Presently, this language is called *"group theory"*. It is widely used in pure *and applied mathematics* as well as to express regularities of the physical world. Symmetry plays a central role in the modern physics, in the infinitesimal quantum world and in the infinite world of the Universe. There is a reason to believe that symmetry can pave the way to the future "Theory of Everything", i.e. a mathematical integration of the two main streams in the modern physics — quantum theory and relativity theory.

What is symmetry? *Symmetry* is not a number or a form; it is *a special type of transformation*, a certain unity of preservation and transformation. Thus, the laws of physics must be the same, invariant, in any place and at any time. We may say that the laws of nature must be symmetrical relative to movement through space and through time.

The Quantum physics states that everything in the Universe is composed of a set of elementary particles. Their behavior is described by mathematical equations expressing the laws of nature and those laws possess symmetries. As Ian Stewart notices: "Particles can be transformed mathematically into quite diff erent particles, and these transformations also leave the laws of physics unchanged".

Observe that these concepts — and more recent ones at the frontiers of today's physics as well — could not have been discovered without a deep mathematical understanding of symmetry. This understanding came from pure mathematics. Extraordinarily useful ideas can arise from purely abstract considerations —something that the American theoretical physicist and Nobel Prize winner (1963) Eugene Paul Wigner (1902—1995) referred to as *"the unreasonable effectiveness of mathematics in the natural sciences"*. Commenting on James Maxwell's (1831—1879) role in development and creation of the general theory of electromagnetic field — classic electrodynamics — the outstanding theorist and practitioner in physics Heinrich Hertz (1857—1894) made this cardinal conclusion: "It is impossible to study this wonderful theory without feeling as if *the mathematical equations had an independent life and an intelligence of their own*, as *if they were wiser than ourselves*, indeed wiser than their discoverer, as if *they gave forth more than he had put into them*".

Notice that as a dominant idea symmetry emerged in an unusual way that is not through geometry. The concept of symmetry manifested in mathematics and physics came from algebra.

According to Galois, solvability of a certain equation ceased to be an absolute problem demanding a ready definite answer. He considers it to be *a link between* a certain algebraic object — *equation* — and its "medium" — *a field* or a realm of rationality it belongs to. As soon as the equation's realm of rationality changes its Galois's group changes, too.

So the concept of group in the Galois theory becomes a powerful and versatile instrument. Thus, in particular, Augustin Cauchy never thought of ascribing such a role to the concept of group. "System of ajoint substitutions" was an indecomposable concept for him and he never singled out concepts of a subgroup and a normal group.

The idea of group's relativity is Galois' own invention. Later it penetrated all the mathematical and physical theories originated from the group theory. This idea in action can be seen in the Erlangen program of Felix Klein. As has been said by Galois's editors R. Bourgne and J. P. Azra: "Paradoxical in its concision, his thought was not made, in order that it be a starting place, but in order that it be a place of reuniting".

Ian Stewart, revealing an inconceivable efficiency of mathematics for comprehension of the nature's secrets, formulated a profound idea on a connection between philosophy and mathematics stating that "No one could have predicted that *questions about the solvability* of equations would lead to *one of the core concepts of mathematics*, that *of a group*, or that *groups* would prove to be the *language of symmetry*. Even less could anyone have known that *symmetry would unlock the secrets of the physical world*".

The complicated and contradictory process of unlocking entire realms of the physical world continues. However, the implications of symmetry for the whole of science are not explored and studied enough. There is much to understand for us. But we can state that *symmetry groups are our path through the unexplored wilderness and vast continents* until a still more powerful concept comes along. We are sure that the science of the 21st century will develop under the sign of the heuristic paradigm of the daring genius of mankind Évariste Galois.

Had not been Galois and his predecessors possessed with the *task of finding the conditions an equation can be solved by radicals*, the mankind's *discovery of the theory of group would have been blocked dramatically and possibly have never taken place*. No one could have predicted that an intricate problem of equation would help to clarify the deep structure of the physical world.

In modern mathematized theory, the concept of group elaborated on the base of Galois's ideas is a means of theorization of various branches of knowledge because the principle of invariance is expressed in it as a methodological imperative. The role of the principle manifests itself in the process of construction of mathematical and physical theories and is a crucial condition of efficiency and fruitfulness of mathematical ideas in the modern cognition of natural sciences. Thus, for example, in the most clear and vivid form, this is evident in non-Euclidean geometries.

Presently, Galois is one of the most popular names in mathematics. Such fundamental concepts as *Galois group, Galois field, Galois theory, Galois correspondence* and others bear his name. His ideas had a decisive impact on the development of algebra during the century and penetrated into the other branches of mathematics. The classic Galois theory has been generalized and elaborated in many domains.

The outstanding scientist Felix Klein (1857—1894) writes in his "Lectures on the Development of Mathematics in the 19th century": "Galois's great achievements are in two directions.

1. He created the first thorough classification of the irrationalities defined by algebraic equations, the subject known today as *Galois theory*.

2. He did extensive work on the integrals of arbitrary functions of one variable — *Abelian integrals* as we call them today — and left behind certain results that show him to be a forerunner of Riemann.

There is a hint of yet a third area, although one cannot determine its precise content because of the scanty references to it. In his farewell letter to his friend Chevalier, Galois spoke of investigations into the *"ambiguity of functions"*; it is possible that this referred to the idea of Riemann surfaces and multiple connectivity.
Galois's achievements cannot be appraised correctly without some knowledge of "Galois theory".
What are the other characteristics of the concept of group? Having appeared in algebra the concept acquired such a generality in the mathematical research that this allowed to use it in various branches of mathematics. Later, the notion of groups would go beyond the boundaries of mathematics and become a crucial concept in the theoretical constructs in physics and other natural sciences. Aleksandr Gennadievich Kurosh (1908—1971), an outstanding expert in many branches of algebra including group, ring and structure theories and the author of world-famous monographs "The Theory of Groups" and "A Course in Higher Algebra", states: "The concept of group is one of the most fundamental concepts of contemporary mathematics: it combines an affinity to familiar operations on numbers with an exceptionally wide domain of applicability".
The theory of groups allows defining symmetry of a geometrical figure in accurate terms. Thus, each geometrical figure can be correlated with a collection of all the space transformations. The latter defines the symmetry of a figure. Exactly based upon this position, Russian mineralogist and crystallographer Evgraf Stepanovich Fedorov (1853—1919), one of the founders of modern structural crystallography and mineralogy, solved a mathematical problem of classification of regular spatial point systems which is one of the fundamental problems of crystallography. There are 17 Fedorov wallpaper (or plane symmetry) groups and 230 space groups. These results are presented in Fedorov's classic work "The Symmetry of Regular Systems of Figures" (1890). Independently of Fedorov, in 1890— 1891, German mathematician Artur Moritz Schoenflies (1853—1928) classified all crystal spatial lattices by means of group theory.
It was the group theory that made it possible to propose a fairly complete classification. This is historically the first example of the application of group theory to natural sciences.
German theoretical physicist Max von Laue (1879—1960) developed the theory of X-ray interference in crystals: he proposed to use crystals in diff raction grating. The same year this theory was verified experimentally by German physicists Walter Friedrich (1883—1968) and Paul Knipping (1883—1935). As Max von Laue predicted, the scientists discovered the X-ray interference phenomenon caused by spatial crystal lattice. The discovery led to creation of a powerful instrument for the matter structure research, namely X-ray structure analysis. For his discovery of the diff raction of X-rays by crystals, Max von Laue was awarded the Nobel Prize in physics.
Subsequent development of structure analysis proved theoretical conclusions of E. S. Fedorov and A. Schoenflies. Elaborating on the ideas of the classic theory of crystal symmetry, the Soviet physicist-crystallographer Aleksey Vasilyevich Shubnikov (1887—1940) extended the symmetry class proposed by Fedorov and constructed 1651 symmetry types by including a complementary symmetry operation. These new predictions have been verified in varied forms of crystals. Classic examples of heuristic role of symmetry ideas belong to the domain of geometrical symmetry. It seems that, in the research of crystal forms, the Nature itself reveals this symmetry and the Man just needs to register it in the simplest geometric proportions and forms.
In the 20th century the role of symmetry in elaboration of physical theories increased. German mechanician and mathematician Georg Hamel (1877—1954) discovered the connection between conservation laws and basic symmetries of space and time. Later, German mathematician Amalie Emmy Noether (1882—1935) discovered the correspondence between invariance of a physical system relative to transformations of

symmetry described by continuous symmetry group and independent parameters, and a number of conserved quantities in a given system (Noether's theorem).
English physicist and astrophysicist James Jeans (1877—1946) discussing the reform of the mathematical curriculum at Princeton University said: *"We may as well cut out group theory. That is a subject which will never be of any use to physics."* But theory of group continued to be taught at Princeton and this fact was of no small importance to the history of science. By the irony of fate group theory grew later into one of the central themes of physics, and it still dominates the thinking of all of us who are struggling to understand the fundamental particles of nature. Curiously, but Princeton professors Eugene Paul Wigner and Hermann Weyl happened to be the pioneers in using group theory in physics. Group theory found its application in nuclear physics, quantum theory, elementary particle physics and relativity theory (Lorentz transformation group).
The modern physics uses so called inner or dynamical symmetries that— to use the expression of Eugene Wigner: *"are formulated in terms of the laws of nature"*. He was one of the first to demonstrate the eff ectiveness of applying the apparatus of group theory to quantum mechanics and he did much for the ideas of symmetry and group theory to be adopted by the modern theoretical science and also introduced ideas and methods applying them to fundamental problems. Applications of symmetry in quantum theory demonstrate that group theory makes it possible to look deeply into the nature of things, despite its creation as an answer to a certain pure mathematical problem.
The form of symmetry most widely used in the modern elementary particles physics is gauge symmetry. This term was introduced by German physicist and mathematician Hermann Weyl (1885— 1955). Using methods of group theory, he obtained certain results related to theory of atomic spectra. The scientist investigated theory of continuous groups and found their application to diff erential geometry, physics and relativity theory. Weyl's works played a great role in understanding the importance of symmetry ideas for both mathematics and physics. He did a lot for the concept of symmetry to become a physical one as well. Weyl found the correspondence between group theory and physics, stating that he "always tried to unite the true with the beautiful; but when I had to choose one or the other, I usually chose beautiful."
Electromagnetic fields were found to have gauge symmetry. This symmetry is characteristic of systems whose Lagrangian (Lagrange function) is invariant relative to the group of continuous transformations with parameters depending on the space-time coordinates.
In modern physics, the symmetry properties are used for classification problems, to identify new conservation laws, for construction of new generalized theories and to simplify actual calculations, for example, in spectroscopy, in order to obtain the selection rules.
The concept of symmetry expresses a fundamental law of nature and knowledge. The principle of symmetry is one of the most important regulators of methodological construction of a scientific theory. This imperative, according to the outstanding scientist Vladimir Ivanovich Vernadsky (1863—1945), is universal, it manifests itself in various subject areas of the physical world. Vernadsky states:
"The symmetry principle embraced and embraces yet newer fields in the twentieth century. From the field of matter it penetrated into the field of energy, from the field of crystallography, the physics of solid matter, it came into the field of chemistry, molecular processes and atomic physics. Undoubtedly, we shall discover its manifestations in the world of electrons, existing ever more deeply within the complex world surrounding us, and it will subordinate the properties of quanta. Undoubtfully and diversified, it embraces phenomena of life and of the world space". Thus, we have *a currently central problem of establishing the generality of symmetry principle based upon comprehensive study of the*

history of its emergence in science, its manifestation forms as well as *"the necessity of its philosophical investigation"* (V. I. Vernadsky).

Acknowledging the achievements and results of the titanic work of Felix Klein in various branches of the mathematical science, Hermann Weyl emphasized that: "Klein's concept of geometry is nothing but the theory of relativity in its universal, mathematically formalized sense. He understood and applied the group concept as a great organizing and simplifying principle in algebra, geometry and analysis. Klein, whose stamp of genius has marked the whole era, continues to have a powerful influence on modern mathematics that develops under the sign of group theory, topology and abstract algebra. *The flame that he lit* is not a greasy lamp of pedantic tradition, and *it warms the pots at all the mathematical kitchens and in the hearths of all the mathematical forges*, making a great work as well as a small daily one. His works continues to influence us, his name will not be forgotten".

The theory of continuous groups has a positive influence on the development of modern physics. This was expressively evidenced by discoveries in microcosm, made by Nobel Prize winners Louis de Broglie (1892—1987), Erwin Schroedinger (1887—1961), Paul Dirac (1902—1984). They made a considerable contribution into the creation of microparticle movement — wave quantum mechanics. In their works, the scientists widely applied the mathematical apparatus to explain regularities, properties and processes of a completely new subject area of the physical world. In their work the theory of group representation by linear operators proved its efficiency.

In particular, Luis de Broglie, in his doctor's thesis "Researches on the quantum theory" (1924) introduces a term "group velocity of the phase waves".

Erwin Schrodinger attempted (1926) to represent particle as a group of waves that occupy a certain point in space and move collectively (wave package).

Since the group theory is the basis of many studies in physics and mathematics, the problem of classification of groups is very relevant and important. To solve it, a voluntary international community of enthusiasts in the math was founded in 1970th. About 100 theorists identified specific questions and started to address this global scientific issue. Perhaps this is *the only example of a broad, massive and coordinated approach to solving a mathematical problem in the history of science*. Three infinite families of groups and 26 special cases of finite groups were gradually selected. Some of them have been possible to find only by computer-aided methods. As of 2004 only five of the twelve volumes of the complete proof have been published. The main reason for the slow solution of the problem is that the ranks of the participants have decreased significantly (because of age, death of many members of the association).

In the prison of St. Pelagie Évariste Galois edited his most important results. In the preface to one of his papers he claims: "Firstly, you will notice the second page of this work is not encumbered by surnames, Christian names or titles. Absent are eulogies to some prince whose purse would have opened at the smoke of incense, threatening to close once the incense holder was empty. Neither will you see, in characters three times as high as those in the text, homage respectfully paid to some high-ranking official in science, or to some savant-protector, a thing thought to be indispensable (I should say inevitable) for someone wishing to write at twenty.

I tell no one that I owe anything of value in my work to his advice or encouragement. I do not say so because it would be a lie". I would like to stress that *Galois obtained the latest results in mathematics independently*. Therefore, he had no reason to off er his gratitude to famous mathematicians. How did really *the great ones of the world and the great ones of science* (the terms of Galois) assist him in revealing a new knowledge in mathematics? On this issue there is Galois's totally accurate answer:

"I owe to *the great men of science* the fact that the first of these papers is appearing so late. I owe to *the great ones of the world* that the whole thing was written in prison, a place, you will agree, hardly suited for meditation, and where I have been dumbfounded at my own listlessness in keeping my mouth shut at my stupid, spiteful critics: and I think that I can say "spiteful critics" in all modesty because my adversaries are so low in my esteem. The whys and wherefores of my stay in prison have nothing to do with the subject at hand; but I must tell you how manuscripts go astray in the portfolios of the members of the Institute, although I cannot in truth conceive of such carelessness on the part of those who already have the death of Abel on their consciences. I do not want to compare myself with that illustrious mathematician but, suffice to say, I sent my memoir on the theory of equations to the Academy in February of 1830 (in a less complete form in 1829) and it has been impossible to find them or get them back. There are other anecdotes in this genre but I would be ungracious to recount them because, other than the loss of my manuscripts, those incidents do not concern me. Happy voyager, only my poor countenance saved me from the jaws of wolves".

In prison he also received an envelope from the Academy of Sciences. Inside, he found his manuscript and a note signed by the Secretary of the Academy Francois Arago:

"Dear m. Galois, Your paper on solvability of equations by radicals was sent to m. Poisson to referee. He has returned it with his report, which we quote: "We have made every eff ort to understand M. Galois's proofs. His *argument is neither* sufficiently *clear nor* sufficiently *developed to allow* us *to judge its rigor*; *it is not even possible for us to give an idea of this paper*.

The author claims that the propositions contained in his *manuscript* are *a part of a general theory which has rich application*.

For this reason, one should rather wait to form a more definite opinion, therefore, until the author publishes a more complete account of his work".

And here, for the *n*th time, the Academy rejected Galois's work. Is there any fault of his own in this rejection? Perhaps Galois did not always present his thoughts clear enough and formulated some of his theorems as if they were proved without really having proved them. Furthermore, his manner of presenting his work was rather unusual for the mathematicians of the first half of the 19th century. *The new style of mathematical thinking* formed by Galois has a high level of *novelty* and becomes *dominant* only *in the 20th century*.

Instead of lengthy calculations, he used absolutely unexpected ideas in his problem solving. Besides, his works contained too many new concepts. It is no wonder that Poisson and other mathematicians misinterpreted his works as not clear and accessible enough or even "incomprehensible".

Évariste Galois obtained many results of his theory when he was just 16—18 years old and submitted them to the Paris Academy of Science twice. However, even most prominent French mathematicians of the time A. Cauchy, J. Fourier and S. Poisson failed to understand Galois's works and recognize their importance.

Galois addresses most intricate mathematical problems and their history and emphasizes that, beginning from Leonhard Euler, calculations have become more and more necessary and more and more difficult at that. Modern mathematicians have to present the results of their researches so orderly that one could grasp a considerable number of operations at first glance. How can mathematicians fullfil such a difficult task? Galois has an unorthodox answer to this question: "To take a bold leap at these operations, to group them according to their difficulties and not according to their form; that is, according to my view, the mission of future mathematicians, that is the path I have taken in this work". Galois disliked lengthy calculations that obscured the main point. His desire for grouping the problems together according to their profound structural analogies and by their appearance — is not it *the programme of modern mathematics*, as it has turned out, formulated a hundred years ahead of the time?!

Galois states that scientific truth should not be considered as something finished and unchangeable. It reveals itself in the eternally uncompleted motion of discoveries and constant process of acquirement of a more profound and exact knowledge. Galois's works let us make a unique contact with the live creation of the young mathematician in the form it has appeared after his manuscripts were deciphered. They let us feel his groping his way for the so much desired truth, experience the birth of something completely new. To embrace the creation that bears the imprint of ruthless circumstances of his life.

This is a unique example of a mathematical work that is closely interwoven with the personality of the author who did not want and could not separate himself from his works.

But this time the ideas he had presented to the eminent mathematicians A. Cauchy (1789) and J. Fourier (1768—1839) were hidden behind algebraic calculations. The night was far spent when he finished his calculations and wrote his famous letter — his scientific last will and testament — to his only friend August Chevalier and asked him, in case of his death in a duel, to pass his works on to the greatest mathematicians of Europe:

"My Dear Friend,

I have made some new discoveries in analysis. The first concerns the theory of equations, the others integral functions.

In the theory of equations I have researched the conditions for the solvability of equations by radicals; this has given me the occasion to deepen this theory and describe all the transformations possible on an equation even though it is not solvable by radicals. All of this will be found in three memoirs.

In my life I have often dared to advance propositions about which I was not sure. But all I have written down has been clear in my head for over a year, and it would not be in my interest to leave myself open to the suspicion that I announce theorems of which I do not have complete proof. Make a public request of Jacobi or Gauss [German mathematicians Carl Gustav Jacob Jacobi (1894—1851); Johann Carl Friedrich Gauss (1777—1855). — *B. P.*] to give their opinions not as to *the truth* but as *to the importance of these theorems*. After that, I hope some men will find it profitable to sort out this mess".

Galois puts the cardinal question in his scientific testament: if the future *mathematicians* discover what Galois found they *would have to know the name of the pioneer, the true creator of the group theory Évariste Galois*. He was also the first to see the importance of his revolutionary results for the future development of science. *Well, immortal fame of one's own name costs too much. "The last battle is the battle for recognition and establishment in science."* — Thus probably Galois thought during his final night before the fatal duel. — "*The last battle is the battle for immortality.* Perhaps *the only one I am destined to win. I shall win but I am never to enjoy the sweet fruits of victory."* What lucid, striking, and prophetic words! *Galois's discoveries belong not only to algebra and even mathematics, philosophy, and science but to the whole universal world of culture.*

Carrying out Galois's last will, August Chevalier and Alfred Galois (Évariste's younger brother) sent copies of his manuscript to Carl Gauss, Carl Jacobi and other prominent mathematicians. But it was not until almost ten years later that his work was given its due. It happened in 1846 when one of the copies was given to an outstanding French mathematician Joseph Liouville (1809—1882). The scientist sensed a spark of genius contained in the work and gave much of his time to thoroughly look into Galois's notes. Liouville edited Galois's memories and published them in his influential *"Journal de Mathématiques pures et appliquées"*. In the preface to this publication Joseph Liouville wrote: "The main object of Évariste Galois's investigations is the conditions of solvability of equations by radicals. The author constructs the fundamentals of a general theory which he applies in detail to any equation whose power is a prime number". Many mathematicians appreciated the publication where Galois revealed the mechanism for the solution of quintal equations positively. Firstly, Galois divided all quintal equations into two types — solvable and unsolvable equations — and then proposed a method for finding solutions for such equations. Moreover, he turned to the equations of higher power

containing x^6, x^7, etc. and succeeded to prove which of them are solvable. *The fundamental work of Évariste Galois is a mathematical masterpiece of the 19th century.*
In his preface to Galois works Joseph Liouville called attention not only to the flaws of Galois's texts but to their merits as well. In particular, he emphasized that "my diligence was rewarded and I felt extraordinary content at the moment when, having made some minor corrections, *was convinced in the validity of the method Galois* used to prove this beautiful theorem".
Galois's works are highly sophisticated. That is why prominent French mathematicians had studied his scientific oeuvres for 25 years and then admitted that they did not understand them at all. The first one to succeed was a well-known French mathematician Camille Jordan (1838—1922) who devoted many years to this cause. In 1870 he published the first systematic course on *group theory and Galois theory*. In this book of 667 pages titled "The Treatise on Substitutions and Algebraic Equations" he clarified and comple mented Galois's short and concise researches, revealed the true sen se of his theory as a whole, and made it available to the wide mathe matical circles. Jordan's treatise contains Galois theories concerned with study of substitution groups proper and supplements to Galois theory to equations in various branches of mathematics.
In his preface to the book Jordan writes: "It was reserved for Galois to give a coherent proof of the theory of solvability of equations [by radicals. — *B. P.*]. The problem of solvability that seemed to form the only object of the theory of equations now appears as the first link in the long chain of questions related to transformations of irrationals and their classification. Galois applied his general methods to this particular problem to find the property characteristic of groups of equations solvable by radicals without any difficulty".
In his works Jordan singles out normal subgroups and the concept of the simple group; he is the first to investigate mathematical groups that would become the object of study in the 20th century. When presenting Galois's theory the author turns to the modern method of associating an equation not with a certain number of root transmutations but with a group of substitutions, and the criterion of the solvability of an equation by radicals is expressed by the solvability of its Galois group. Jordan's treatise becomes a text-book on both the theory of group and Galois theory.
Thus, C. Jordan's monograph "The Treatise on Substitutions and Algebraic Equations" presented Galois theory in the systematic form and became an important element of mathematical education and the foundation for further mathematical investigations. After reading this book of Jordan two talented young mathematicians — the Norwegian Sophus Lie and German Felix Klein — took a great interest in the ideas of group theory. The former applied Galois's ideas to the theory of diff erential equations (Lie groups), the latter used them in geometry (Erlangen program).
The theory of groups being the central in modern mathematics has *developed by* way of successive generalizations. This theory originates from a specific problem that has attracted mathematical minds since as early as the Middle Ages, that is the problem of finding solutions for an algebraic equation of degree higher than the second by algebraic methods, i.e. by addition, substraction, multiplication, division, and extraction of roots. The theory of quadric equations was known as far back as the ancient Babylon. Italian mathematicians of the Renaissance Girolamo Cardano (1501—1576) and Niccoló Tartaglia (1500—1557) found the solution for equations of the third and the forth degree in general. However, in its quest for solution for equations of degree five, or higher the science met with insurmountable obstacles.
History of the group theory begins in the middle of 19th century after publication of Galois's works. The works of French mathematicians Joseph-Louis Lagrange (1736—1813) and Alexandre Vandermonde (1755—1796) on the theory of algebraic equations introduced the first group object, namely substitutions.

Among the followers of Lagrange and Vandermonde there must be mentioned the name of Italian mathematician Paolo Ruffini (1795—1822). In his studies on the theory of equations, he examines not only a group of substitutions but its subgroups as well. The famous German mathematician and physicist Carl Friedrich Gauss (1777—1855) in his 1801 book “Disquisitiones Arithmeticae” (Latin, “Arithmetical Investigations”) makes an important move towards the establishment of the theory of groups. He defines the first example of construction of the factor-ring in history. His proof is very general and applies to any case of finite field and this fact was readily noticed by Galois when he began to build the theory of finite field.

English mathematician Arthur Cayley (1821—1895) published the definition and the first studies of abstract groups. Cayley’s most important results belong to the domains of algebraic geometry, linear algebra and group theory. Felix Klein argues that Cayley is “the founder of modern algebraic geometry in both invariant theory and its geometric part”.

Cayley points out that substitutions can be elements of a group. He used the term “group’ in homage to Galois. His works were not well-known immediately after their publication, but later became a model of group definition and were included into all text-books. The work on group theory was continued by French mathematician Joseph Alfred Serret (1819—1885). In particular, his two-volume “Course on Higher Algebra” deals with elements of Galois theory.

Felix Klein is an outstanding German mathematician of the last third of the 19th century. His main works encompass non-Euclidean geometry, theory of continuous group, theory of algebraic equations, theory of elliptic functions, and theory of automorphic functions. Having found that group of motions of Lobachevsky space as well as group of motions of Euclidean space and other projective metrics are subgroups of space projective transformations, Klein arrived at the general idea of the role of groups of transformations in geometry which he presented in the lecture on his appointment as professor at the University of Erlangen (1872). The lecture is titled “A Comparative Review of Recent Researches in Geometry”, known as “The Erlangen Program”.

So, what is the idea of the Erlangen Program? It is a new paradigm of the geometric world, the uniform view on various (e. g. Euclidean, affine and projective) geometries. Euclidean geometry considers the properties of figures that do not change under motion; equal figures are defined as those that can be transformed into one another by a motion. But instead of motions one may choose any other collection of geometric transformations. When choosing groups of certain transformations one may obtain certain geometries. Affine and projective transformations result in affine and projective geometries. Felix Klein proved that if one starts from projective transformations that carry a certain circle (or any other conic), one comes to the non- Euclidean Lobachevsky geometry. *The Erlangen Program has given impetus to the further development of geometry.*

Thus, every geometry may be regarded as the theory of invariants of a particular group of transformations. By broadening or narrowing the group one can convert one geometry to another. This approach made it possible to combine many diff erent geometries: Euclidean, affine, projective and all non-Euclidean ones.

Felix Klein becomes a follower of Galois group theory. Hence, the Erlangen Program opens with the definition of *group of transformations*:

“The most essential idea required in the following discussion is that of a group of space transformations.

The combination of any number of transformations of space is always equivalent to a single transformation. If now a given system of transformations has a property that any transformation obtained by combining any transformations of the system belongs to that system, it shall be called *a group of transformations”*. As an example of a group of transformations, Klein gives “the totality of motions”. He also argues that the rotations about one point form a subgroup of the group of motions.

Stating that geometric properties of configurations are independent of the position they occupy in space, of its absolute magnitude and of orientation and arrangement of its parts, i.e. remain unchanged by any motions of space and by all its configurations, Klein concludes that *"geometric properties are not changed by the transformations of the principal group. And, conversely, geometric properties are characterized by their remaining invariant under the transformations of the principal group"*.

From "space", i.e. three-dimensional Euclidean space, the scientist proceeds to an arbitrary "manifoldness": "By analogy with the transformations of space we speak of transformations of the manifoldness; they also form groups. But there is no longer, as there is in space, one group distinguished above the rest by its signification; each group is of equal importance with every other. As a generation of geometry arises then the following comprehensive problem:

Given manifoldness and a group of transformations of the same; to investigate the configurations belonging to the manifoldness with regard to such properties as are not altered by the transformations of the group.

To make use of a modern form of expression, which to be sure is ordinarly used only with reference to a particular group, the group of all the linear transformation, the problem might be stated as follows:

Given manifoldness and a group of transformations of the same; to develop the theory of invariants relating to the group". This is the general, central problem of Erlangen Program.

The concept of group *has united* analytical and projective geometry. Aims and results of the programme *have united* algebra and geometry. Hence, "The theory of binary forms and projective geometry of the plane with reference to a conic are identical", or "The theory of binary forms and general projective metrical geometry in the plane are one and the same".

Geometry and algebra were mutually enriched by identification of conics — lines of second order generated by intersection of two space figures — with the binary (from Latin *binarius* meaning "double, consisting of two parts". — *B. P.*) quadric forms. Owing to geometry, the invariant theory acquired a convenient and orderly representation, and algebra gave generality to geometry methods. *The boundaries between both branches of mathematics have been smoothed out* and this can be considered as *one of the first trends of modern mathematics*.

Consequently, the Erlangen Program gave an impetus to the further development of geometry. According to French mathematician F. Russo, the novelty of the program is that it has brought together the concept of transformations and that one of groups, and it is "a most important moment in the history of geometry and, even more globally, in the history of mathematics".

Owing to Klein's works, theory of group became one of the most important divisions of mathematics. Its development is closely associated with the synthesis of geometric and algebraic concepts treated in the works of C. F. Gauss, G. Monge, B. Riemann and H. Grassmann.

The Erlangen Program made it possible to understand all the diversity of geometric systems from the uniform invariant theory position. This program, along with theory of group, had a considerable influence on the other branches of mathematics as well (e. g. theory of functions, theory of diff erential equations, etc.). Introduction of the Erlangen program into physics, that is the establishment of invariant theory approach, occurred with the emergence of the special relativity theory. First, H. Minkowski formulated it as an invariant theory and then F. Klein directly linked the special relativity theory and classic mechanics to his paradigm. The book by V. G. Vizgin "The Erlangen Program and Physics" deals with the analysis of two problems: formation of invariant theory approach to construction of physical problems and finding the physical foundations of the Erlangen Program.

Introduction of Erlangen Program into physics not as a particular geometric and *group theory method*, but *a fundamental principle of construction of physical theories* became possible only after the emergence of formulations of physical theories specifically *as invariants* of certain *groups of transformations* these theories are based upon.
Norwegian mathematician Sophus Lie (1842—1899) is the author of many outstanding discoveries in mathematical analysis and geometry. However, primarily, he is renown as the *founder of continuous groups of transformations*. The peculiarity of the theory built by S. Lie, besides its importance and beauty, lies in the fact that it had been developed in a most meticulous and fundamental way. Its statement took three voluminous books with two volumes of supplements. It is hardly possible to find another example of creation of such a developed discipline by one researcher in the history of science. That is why no one ever objected the name — "Lie group" — that became associated with it in the beginning of the 20th century.
It is interesting that Lie's work is associated with two leading theories of our time — relativity theory and particle theory. This is evident, in particular, in the fact that Lorentz group which lies in the foundation of relativity theory represents a special case of Lie group. Let us turn to relativity theory. The principle of covariance A. Einstein attached special significance to (mathematical expression of invariance of physical laws relative to the choice of a coordinate system), refers to application to the physical reality of F. Klein postulate stating that every geometry is an invariant theory of one or another group. Klein's scheme was further elaborated by Lie in his theory of diff erential invariants. Moreover, in his work, Lie touched upon geometry of Riemannian spaces which was the foundation of Einstein's relativity theory. In the 1950th the problem of finding specific solutions for Einstein field equations and classification of these equations demanded methods that were directly associated with Lie studies.
As early as on the first stage of quantum mechanics development, theory of continuous groups proved its importance for the new branch. It derives from invariance of its fundamental equation — Schroedinger equation — relative to a group of space motions. On its next stage, an important role is played by a Lorentz group associated with the name of Dirac. Here, everywhere, as in the would-be particle theory, the instrumental role belongs to theory of various Lie group representations, based upon the powerful infinitesimal method created by Lie (Lie algebra is a term introduced by H. Weyl in 1934, 35 years after Lie's demise).
David Hilbert, in his famous report to the Second International Congress of Mathematics (Paris, 1900), formulated so called "Hilbert Problems" (one of them, the Fifth Problem, concerned Lie groups and was partially proved by J. von Neumann (1933), L.S. Pontryagin (1934) and finally proved by A. Gleason, D. Montgomery and L. Zippin (1952)), argued that Lie groups would be of great importance to the theoretical physics of the 20th century. His prediction came true. S. Lie always relied on his analytical gift but enjoyed a striking geometrical insight.
The studies on theory of continuous groups brought Lie the world-wide fame. His three-volume treatise on transformation groups is still Lie's most popular work. It is a classical one, the "Bible" of theory of continuous groups and their invariants. The new domain of mathematics he discovered was based on geometry, algebra, topology, and analysis, diff erential equations, i.e. was a cross-disciplinary one. It is due to this fact that Lie's problematic leaped over the century and firmly took its well deserved place in the 20th century. Hundreds of modern works on diff erential geometry, group theory, topology, and diff erential equations have used concepts of *Lie group* and *Lie algebra*.
His amazing and unorthodox works have called forth admiration and won a high appraisal from both geometricians and analysts. These discoveries would have a beneficial eff ect on the development of science. The great successor of Sophus Lie, Élie Cartan argued: "He will pass into history as the great creator of continuous groups." Friedrich Engel, Klein's

follower, stated: "In Lie we have lost not only one of the most outstanding mathematicians of our time but of all times."
To take one striking example of Galois group theory's role in solving the most important problems of modern mathematical cognition: The famous French mathematician Pierre de Fermat (1601—1665) is known as the man who laid the foundation of analytical geometry and number theory. His name is associated with one of the most well-known and phenomenal mathematical problems, *namely the Fermat's Last Theorem* (1637). It links the fundamentals of mathematics going back to Pythagoras, with the most cardinal ideas of modern mathematics. Fermat's theorem has a special place among other unsolved problems due to its deceptive simplicity.
Many a mathematician had tried to solve this riddle for three and a half centuries. As known, Fermat's Last Theorem states that the equation
$x^n + y^n = z^n$
has no integer solutions for n greater than 2 (i. e. $n > 2$). In the margins of "Arithmetic" (probably 3 AD.) by Diophantus Fermat scribbled a note which reads: "I have discovered a truly marvelous demonstration of this proposition that this margin is too narrow to contain" (Latin, *"Cuis rei demonstrationem mirabilem sane aetex hanc marginis exiguitas non caparet."*) Fermat's words gives the grounds for conclusion that he was very much content with his "truly marvelous" proof that he never told. However Fermat's Last Theorem gained an extraordinary fame in the centuries to come.
Neither physics, nor chemistry or mathematics have a problem that is stated so simple and so definitely and could have been unsolved for such a long time. It has been called the most difficult "riddle" in mathematics. All attempts to solve it successively failed. Thus, for example, L. Euler, A. Cauchy and other mathematicians of the 17—19th centuries proved many of the theorems formulated by Fermat but his principal theorem was still unsolved. In the beginning of the 20th century one of the great mathematicians David Hilbert was asked why he had never tried to solve Fermat's Last Theorem. Hilbert said: "Before beginning I should put in three years of intensive study, and I haven't that much time to squander on a probable failure." In 1920 during a public lecture where he reviewed the theorem's proofs, D. Hilbert expressed his hope that his young listeners would possibly witness its solution. And probably his predictions were well grounded.
Let us get back to *Galois group theory* which its author turned into *a powerful method* capable to *solve problems* that seemed unsolvable before. It has taken a deserved place in the solution of Fermat's Last Theorem. Rational numbers contains infinite number of elements, and one could suppose that the greater this group, the greater interest it attracts in mathematics. But Galois followed the principle "the less, the better". As the English physicist and journalist Simon Singh writes in his wonderful book "Fermat's Last Theorem", "instead of using infinite groups, Galois began with a particular equation and constructed his group from a handful of solutions to the equation. It was groups from the solutions to quintic equations which allowed Galois to derive his results about these equations. A century and a half later, Wiles would use Galois's work as the foundation for his proof of the Taniyama—Shimura conjecture".
In the second half of the 20th century a powerful arsenal of mathematical instruments made it really possible to prove Fermat's Last Theorem. In their quest for the solution of the greatest problem in the history of mankind Japanese mathematicians Yutaka Taniyama and Goro Shimura formulated a hypothesis on a link between modular forms and elliptic lines (Taniyama-Shimura conjecture). This needs some clarification. Elliptic curves (lines) received their name because some functions closely associated with these lines were needed for measuring length of ellipses (and, therefore, lengths of planetary orbits). Such equations require *E-series*. The latter contains a high percentage of information on the equation it describes. Just as biological DNA carries in itself all the information needed for building a living organism, *E-series* carries the most relevant information on an elliptic

curve. Andrew Wiles was a brilliant expert on arithmetic of elliptic curves. Each new result brought him experience that later would lead him to the opportunity to prove Fermat's Last Theorem.

The Japanese theorists considered investigation of modular forms that are some of the most peculiar and miraculous objects in mathematics an especially attractive topic. A distinguishing characteristic of modular forms is their high level of symmetry.

As concerns modular forms (or *M-series*), just as "the *E-series* is the DNA for elliptic equations, the *M-series* is the DNA for modular forms. The modular form is an enormously complicated beast, studied largely because of its symmetry and discovered only in the 19th century. Modular forms and elliptic equations live in completely diff erent regions of the mathematical cosmos, and nobody would ever have believed that there was the link between the two subjects. However, *Taniyama* and *Shimura* were to shock the mathematical community *by suggesting* that *elliptical equations and modular forms* were one and the same thing". But the authors did not off er any logical proof.

To prove theorem of Taniyama—Shimura, mathematicians have to demonstrate that each of infinitely many elliptic equations might be paired with a modular form. As concerns Wiles's method, it is interesting that elements of the E-series have a natural order. That is why when the first elements have been paired ($E_1=M_1$), the next step is to pair the second elements ($E_2=M_2$), etc. So, Wiles has to show that the first element of the *E-series* might be paired to the first element of the *M-series*. If first elements of the series *matched* then so must the second, third, etc. elements *match*.

Now we can see the outlines of Wiles's complex programme. So how to make the first step in practice? S. Singh makes an important conclusion that the mathematician recognized *the heuristic potential of Galois's results*, i.e. *Galois group*.

According to this point of view, "the first step towards the implementation of such programme Wiles made when he *realized all the power of Galois groups*. To create such a group, it was possible to use several solutions of an equation associated with an elliptic curve. After months of analysis Wiles could prove that Galois groups allow making one indubitable conclusion: the first element of each *E-series* did indeed match the first element of the associated *M-series*. The next step demanded of Wiles to find a method to show that if one element of the *E-series* matched the corresponding element in the *M-series* then so must the next elements match each other". The solution of this problem had taken two years and Wiles did not know how much time was needed to continue the process.

Next, a mathematician from Saarbrьcken (Germany) Gerhard Frey suggested (1984) that if someone could prove the Taniyama— Shimura conjecture, Fermat's Last Theorem would be proved as well. In other words, there was only one obstacle that blocked the way to proving Fermat's theorem, and that was the lack of the proof for the conjecture of the Japanese mathematicians. The next step is associated with the name of UC Berkeley mathematics professor Kenneth Ribet who found the connection between the Fermat's Last Theorem and the Taniyama—Shimura conjecture. At that, he considered this supposition to be absolutely unprovable and suggested that Andrew Wiles was one of a few men in the world who dared to prove this hypothesis as it remained in the mainstream of mathematics.

Having earned his Ph.D. in Cambridge, Andrew Wiles moves to Princeton University. He takes an important personal decision to set on systematic quest for the proof of the Taniyama-Shimura conjecture and work in complete isolation and secrecy.

After seven years of solitary work Wiles finished his work and was ready to declare the results to the world. In the summer of 1993, Wiles presented a series of lectures titled "Modular forms, elliptic curved and Galois representations" during an international conference at the Isaac Newton Institute in Wiles's home-town of Cambridge. After minor corrections there were no doubts about the proof. Two articles of 130 pages in total were thoroughly scrutinized and in May of 1996 published in "Annals of Mathematics".

John Coates who guided Andrew Wiles's gradual research announced: "In mathematical terms *the final proof* is an *equivalent* of splitting of the atom or *finding the structure of DNA*. A proof of Fermat is a great intellectual triumph and one shouldn't lose sight of the fact that it has revolutionized number theory in one fell swoop. For me the charm and beauty of Andrew's work has been that it has been a tremendous step for number theory."

Andrew Wiles has enriched mathematics with a whole number of new methods and strategies that can be used to prove other theorems. According to Simon Singh, "during Wiles's eight-year ordeal he had brought together virtually all the breakthroughs in twentieth-century number theory and incorporated them in one almighty one.

Via the Taniyama—Shimura conjecture Wiles *had unified the elliptic and modular worlds*, and in so doing provided mathematics with a shortcut to many other proofs — problems in one domain could be solved by analogy with problems in the parallel domain." — And then Singh continues: "After Wiles's success there is a renewed eff ort to prove other unifying conjectures between other areas of mathematics. Here was a breakthrough that could lead *mathematics into* the next *golden age*".

But how does one of the most prominent mathematicians of the 20th century feel about Fermat's Last Theorem? Wiles recalls:

"Fermat's Last Theorem … was my childhood passion. I have this rare privilege of being able to pursue in my adult life what had been my childhood dream. I know it's a rare privilege. Having solved Fermat's Last Theorem there's certainly a sense of loss but at the same time there is this tremendous sense of freedom. I was so obsessed by this problem that for eight years I was thinking about it all the time — when I woke up in the morning to when I went to sleep at night. That particular odyssey is now over. My mind is at rest."

Wiles's proof is a triumph of mathematics in solving the hardest problem in the history of science. This proof is based on the fundamental results obtained by L. Euler, C. Gauss (mathematical kings of their time), É. Galois (creator of group and field theory), H. Poincaré, D. Hilbert, Y. Taniyama, and G. Shimura, R. Taylor and many other authors directly or indirectly associated with the most complicated problem that remained unsolvable for 358 years. The proof of Fermat's Last Theorem ranges with such major breakthroughs of the 20th century as invention of computer, space flight, etc. It was a supernova burst that would always shine brightly for mankind.

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