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Quantum approach by the Lindblad master equation to the autonomous oscillator in hard excitation regime

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We propose the simple quantum model of nonlinear autonomous oscillator in hard excitation regime. We originate from classical equations of motion for similar oscillator and quantize them using the Lindblad master equation for the density matrix of this system. The solution for the populations of the stationary states of such oscillator may be explicitly found in the case when nonlinearity parameters of the problem are small. It was shown that in this situation there are three distinct regimes of behavior of the model. We compare properties of this model with corresponding ones of another closely connected open system, namely quantum oscillator in soft excitation regime. We discuss a possible applications of the results obtained.

Keywords: an auto-oscillator in hard excitation regime, the Lindblad master equation, density matrix, population inversion.

В роботі вперше запропонована квантова модель осцилятора з жорстким режимом збудження. Ми виходимо з класичних рівнянь руху для такого осцилятора і знаходимо їх квантовий аналог, використовуючи відповідне рівняння Ліндблада для еволюції матриці густини квантової системи. Отримано аналітичний розв'язок в стаціонарному випадку, коли нелінійні параметри задачі малі. Знайдено чотири різних режими поведінки даної системи. Порівнюються властивості розглянутої моделі з відповідними властивостями аналогічної системи – квантового осцилятора з м'яким режимом збудження. Обговорюється можливе використання отриманих результатів.

Ключові слова: жорстке збудження, рівняння Ліндблада, матриця густини, інверсія заселеності.

В работе впервые предложена квантовая версия модели для автономного осциллятора с жестким режимом возбуждения. Мы исходим из классических уравнений движения для такого осциллятора и находим их квантовый аналог, используя соответствующее уравнение Линдблада для эволюции матрицы плотности квантовой системы. Получено аналитическое решение в стационарном случае, когда нелинейные параметры задачи малы. Найдены четыре различных режима поведения данной системы. Сравниваются свойства рассмотренной модели с соответствующими свойствами аналогичной системы – квантового осциллятора с мягким возбуждением. Обсуждается возможное применение полученных результатов.

Ключевые слова: жесткое возбуждение, уравнение Линдблада, матрица плотности, инверсия населенностей.

The main goal of the paper is to introduce and consider the quantum model of nonlinear autonomous oscillator (AO) in hard excitation regime. Our basic tool for solving this problem is the Lindblad master equation (LME) which describes the evolution of any (closed or open) Markov quantum system. Clearly, the first aspiration that arises when one starts to study the behavior of certain complex quantum open system (OS) is the desire to reduce it to some more simple standard model that permits the rigorous mathematical analysis. In the theory of open systems there are at least two similar models namely 1) AO in soft excitation and 2) AO in hard excitation regimes. The first case has been studied in [1] where to this end the semi-classical method of quantization of classical non-Hamiltonian systems was proposed. Therefore in the present paper we will focus our attention on the case of AO in hard excitation regime. Note that AO both in soft and hard excitation regimes are widely used in physics, biology and other sciences. For example, in physics, an oscillator in soft excitation regime used as the standard model of a generator of electromagnetic oscillations. As regards to AO in hard excitation this system finds various applications aside from physics as well for example in biology where similar model can be applied for the description of activity of the giant axon of a squid in sea water [2]. Now let us describe briefly the method of transition from known classical equations of motion to quantum dynamics by means of the LME. The basic idea in this way is the correspondence principle in the form proposed by P. Dirac in his prominent book [3].

It turns out that the broad interpretation of correspondence principle allows one under certain conditions to quantize (at least in the semi-classical approximation) the equations of motion not only for closed but also for open systems using the LME which realizes the quantum description of the evolution of quantum OS in the Markov approximation. This equation for the evolution of the density matrix of quantum OS $\hat{\rho}$ has the following general form [4]:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] + \sum_{j=1}^{N}\{[\hat{R}_j\hat{\rho},\hat{R}_j^+] + \{[\hat{R}_j,\hat{\rho}\ \hat{R}_j^+]\}, \quad (1)$$

where $\hat{H}$ is - an hermitian operator (Hamiltonian), which describes the internal dynamics of quantum OS, and a set of non-hermitian operators $\{\hat{R}_j,\hat{R}_j^+\}$ - models its interaction with the environment.

The recipe of quantization proposed in [1] consists of three successive steps (its justification and all details see in this paper). Firstly, the input dynamical equations should be presented in the special form allowing the quantization (FAQ). In the simplest case of a system with one degree of freedom with dynamical variables x and p or equivalently with complex coordinate $z = \frac{x+\dot{p}}{\sqrt{2}}$ the desired equation in FAQ looks as follows:

$$\frac{dz}{dt} = -\frac{i}{\hbar}\frac{dH}{dz^*} + \sum_{j=1}^{N}\{\bar{R}_j\frac{dR_j}{dz^*} - R_j\frac{d\bar{R}_j}{dz^*}\}, \quad (2)$$

This representation, in the case where it is found determines automatically the classical functions $H(z,z^*)$, $R(z,z^*)$ and $\bar{R}(z,z^*)$ entered in Eq. (2).

The second step is to find the quantum analogs of classical functions $\hat{H}$, $\hat{R}$. $\hat{R}^+$. To this end the simple rule can be proposed: one should replace in all classical variables the coordinates z and z^* by the Bose operators $\hat{a}$ and $\hat{a}^+$. After this procedure the operators $\hat{H}$, $\hat{R}$ and $\hat{R}^+$ thus obtained should be substituted into the LME. Now let us demonstrate in detail how the method of quantization operates in the case of AO in hard excitation regime. We will consider the simplest model of such oscillator that can be described by the following equation of motion for the complex coordinate z [5]:

$$\dot{z} = -i\omega z - \varepsilon_1 z + \varepsilon_2 z\,|\,z\,|^2 - cz\,|\,z\,|^4, \quad (3)$$

where ε_1, ε_2 and c - are the constants, describing the behavior of the oscillator. We are interested mainly in possible stationary regimes of the behavior of the oscillator as functions of these constants. One can easily verify that Eq. (3) can be represented in the FAQ. Indeed let us introduce the functions $H = \omega z^* z$, $R_1 = \sqrt{\varepsilon_1}z$, $R_2 = \sqrt{\frac{\varepsilon_2}{2}}z^{*2}$, $R_3 = \sqrt{\frac{c}{3}}z^3$. After that r.h.s. of Eq. (3) may be written down as:

$$-i\omega z - \varepsilon_1 z + \varepsilon_2 z\,|\,z\,|^2 - cz\,|\,z\,|^4 = -\frac{i}{\hbar}\frac{dH}{dz^*} + \sum_{j=1}^{N}\{\bar{R}_j\frac{dR_j}{dz^*} - R_j\frac{d\bar{R}_j}{dz^*}\}. \quad (4)$$

In what follows we will assume that $c = 1$ since this case is always may be achieved by choosing of appropriate time scale. According to the above mentioned recipe of quantization the LME for the AO in hard excitation regime takes the following form:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] + \sum_{j=1}^{N}\{[\hat{R}_j\hat{\rho},\hat{R}_j^+] + \{[\hat{R}_j,\hat{\rho}\ \hat{R}_j^+]\},$$

where $\hat{R}_1 = \sqrt{\varepsilon_1}\hat{a}$, $\hat{R}_2 = \sqrt{\frac{\varepsilon_2}{2}}\hat{a}^{+2}$, $\hat{R}_3 = \sqrt{\frac{1}{3}}\hat{a}^3$. (5)

From physical reasons we expect that steady regimes of classical system (3) in quantum case correspond to stationary states of its quantum analogue described by the LME (5). We'll seek the stationary solutions of Eq. (5) in the form $\hat{\rho}_s = \sum_{n=0}^{\infty}|n\rangle\rho_n\langle n|$, where $|n\rangle$ - are eigenvectors of the operator $\hat{n}$ or in other words we assume that $\hat{\rho}_s$ is a certain function of operator $\hat{n}$. Using the standard rule of commutation: $[\hat{a},\hat{a}^+] = 1$ after the simple algebra we obtain the following difference equation for the unknown coefficients ρ_n:

$$\varepsilon_1((n+1)\rho_{n+1} - n\rho_n) + \varepsilon_2((n-1)n\rho_{n-2} - (n+2)(n-1)\rho_n) + ((n+3)(n+2)(n+1)\rho_{n+3} - (n-2)(n-1)n\rho_n) = 0 \quad (6)$$

Let us introduce the generating function for these coefficients according the definition: $G(u) = \sum_{n=0}^{\infty}\rho_n u^n$.

Substituting this expression into the Eq. (6) we obtain the following third order differential equation for the $G(u)$:

$$(1-u^3)\frac{d^3G}{du^3} + \varepsilon_2(u^2-1)\frac{d^2(u^2G)}{du^2} + \varepsilon_1(1-u)\frac{dG}{du} = 0 \quad (7)$$

It is impossible to find out analytical solution of Eq. (7) in analytical form therefore we restrict ourselves to the case when coefficients ε_1 and ε_2 are small but their ratio can be of arbitrary value namely $\frac{\varepsilon_2}{\varepsilon_1} = \gamma$. In the lowest approximation (when both ε_1 and ε_2 tend to zero), $G(u)$ is a certain polynomial of the second order: $G(u) = \rho_0 + \rho_1 u + \rho_2 u^2$, where populations ρ_n should be found as follows. Substituting the expression for $G(u)$ in Eq. (7) and taking into account that all $\rho_i = 0$ when $i > 2$, and by virtue of normalization condition $\rho_0 + \rho_1 + \rho_2 = 1$ we obtain the closed system of equations for the nonzero coefficients ρ_n that takes the form:

$$\begin{cases} \rho_1 = 2\gamma\rho_0 \\ \rho_2 = (6\gamma + 1)\rho_1 / 2 \\ \rho_0 + \rho_1 + \rho_2 = 1 \end{cases} \tag{8}$$

The solution of Eq. (8) looks as follows:

$$\begin{cases} \rho_0 = \dfrac{1}{6\gamma^2 + 3\gamma + 1} \\ \rho_1 = \dfrac{2\gamma}{6\gamma^2 + 3\gamma + 1} \\ \rho_2 = \dfrac{6\gamma^2 + \gamma}{6\gamma^2 + 3\gamma + 1} \end{cases} \tag{9}$$

Having in hands this solution we can analyze possible regimes of behavior for AO in hard excitation regime as the function of the parameter γ. First of all let us clarify two limiting cases a) $\gamma \to 0$ and b) $\gamma \to \infty$.

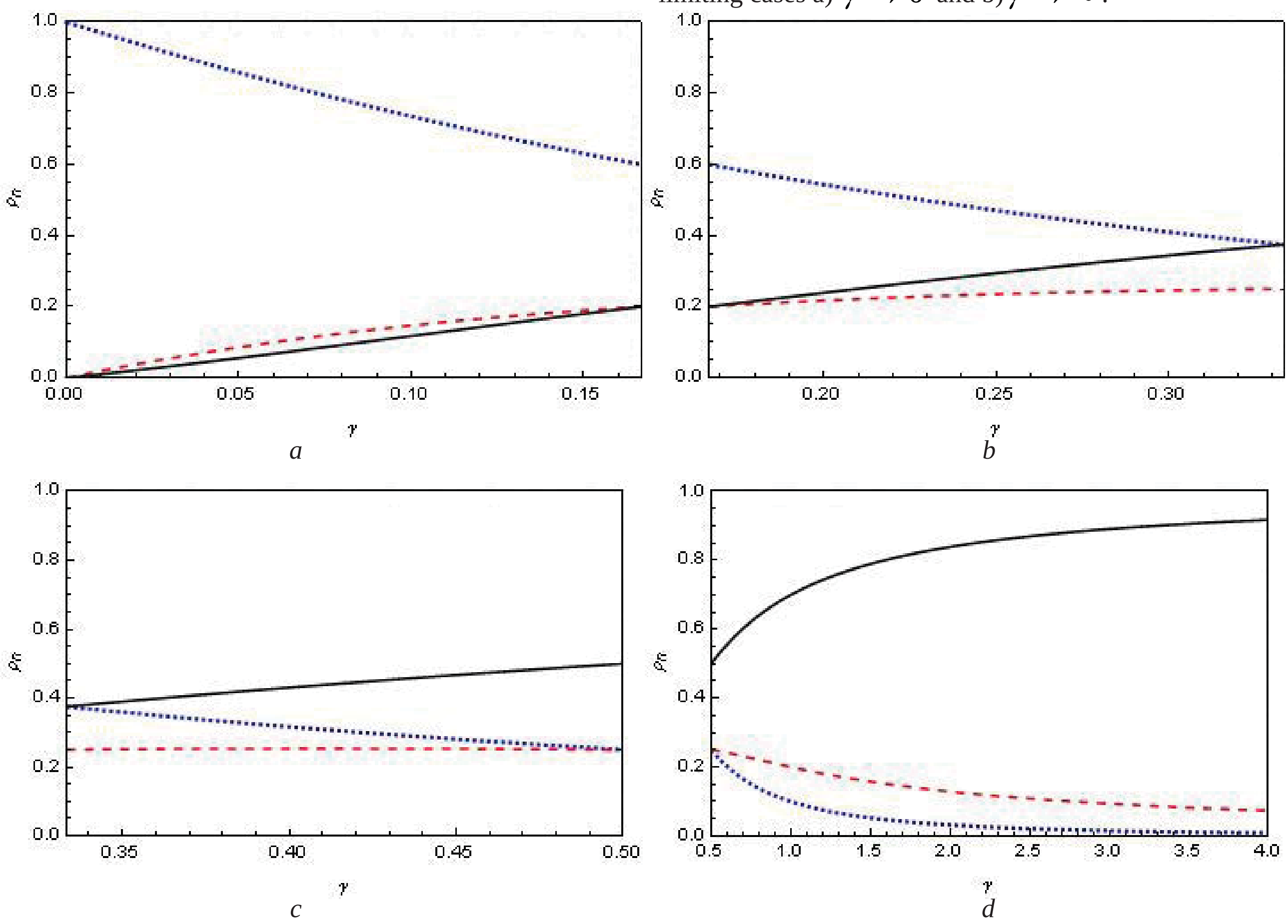

Fig. 1. Population levels ρ_0 (spotted curves), ρ_1 (dashed curves) and ρ_2 (solid curves) as the function of single physical constant γ. These four regimes correspond to the four ranges of γ: a) $\gamma \le \frac{1}{6}$, then $\rho_0 > \rho_1 > \rho_2$; b) $\frac{1}{6} < \gamma < \frac{1}{3}$ then $\rho_0 > \rho_2 > \rho_1$; c) $\frac{1}{3} < \gamma < \frac{1}{2}$ then $\rho_2 > \rho_0 > \rho_1$; d) $\frac{1}{2} < \gamma$ then $\rho_2 > \rho_1 > \rho_0$.

In the case a) $\rho_0 \to 1, \rho_1, \rho_2$ tend to zero. This case corresponds to the vacuum state of AO in hard excitation regime (or the state of rest in the classical case).

In the case b) $\rho_0 = \rho_1 = 0$, and $\rho_2 = 1$. It is the case of maximum possible excitation of the system in our approximation. It corresponds to the state above threshold in classical case.

Now one can specify the four distinct regimes of the AO under study depended on the parameter $\gamma = \frac{\varepsilon_2}{\varepsilon_1}$. These regimes are represented in Fig. 1.

It is interesting to compare the results obtained in the present paper with similar ones relating to AO in soft excitation regime. Remind that generation function $G(u)$ for stationary states of AO in soft excitation regime satisfies to the following second order differential equation (see Eq. (26) in Ref. [1]):

$$(1+u)\frac{d^2G}{du^2} - vu\frac{dG}{du} - vG = 0 \qquad (10)$$

where v is the only nonlinear parameter of this oscillator. Its solution that satisfies all physical conditions can be expressed as

$$G(u) = \frac{F(1, v, v(1+u))}{F(1, v, 2v)}, \qquad (11)$$

where $F(a,b,x)$ is the standard confluent hypergeometric function. Using the expansion of this function namely: $F(a,b,x) = 1 + \frac{a}{b}x + \frac{a(a+1)}{2!b(b+1)}x + \ldots$ one can easily see that if parameter of nonlinearity v tends to zero, corresponding generation function tends to:

$$G_0(u) = \frac{2+u}{3}. \qquad (12)$$

Thus the AO in soft excitation regime with small nonlinearity reduced to the two level system with population $\frac{2}{3}$ in the lower and $\frac{1}{3}$ in the upper level respectively. We see that compared with such primitive regime the case AO in hard excitation regime reveals considerably much more rich behavior.

Conclusions:

1. The quantum model of an AO in hard excitation regime is firstly proposed in this paper.
2. Using the methods of the quantum theory of the OQS, the Lindblad equation for the density matrix of the oscillator was obtained, and it was found the four distinct regimes of the oscillator in the case when the physical parameters of the model are small.
3. It was shown that the quantum model proposed here has much more rich behavior then AO in soft excitation regime.
4. It is worth to note that the model AO in hard excitation regime considered in present paper, if it should be implemented as physical device, naturally realizes the curious case of three level quantum system in which one can achieves (by varying only single parameter) population inversion on any desired pair of levels.

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