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Scattering process without conserving plasmon number in one-dimensional Wigner crystal

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In this paper we consider a quantum wire, the electrons in which form a one-dimensional Wigner crystal. One-dimensional electrons in a crystal are equivalent to plasmon modes describing the long-wavelength fluctuations of the charge density. We have studied the basic process of the plasmon scattering that is not conserved the total number of particles, – the scattering of two plasmons in three, and vice versa. For this process, the scattering rate is calculated and an equation for the relaxation of the artificial chemical potential are derived.

Keywords: one-dimensional Wigner crystal, plasma waves, collision integral, chemical potential.

В работе рассмотрен квантовый провод, электроны в котором формируют одномерный вигнеровский кристалл. Одномерные электроны в таком кристалле эквивалентны системе плазмонных мод, описывающих длинноволновые колебания зарядовой плотности. В работе исследован основной процесс рассеяния плазмонов, при котором не сохраняется полное число частиц, – процесс рассеяния двух плазмонов в три, и наоборот. Для этого процесса вычислена амплитуда рассеяния и получено уравнение для релаксации искусственно созданного химического потенциала.

Ключевые слова: одномерный вигнеровский кристалл, плазменные волны, интеграл столкновений, химический потенциал.

У роботі розглянуто квантовий дріт, електрони в якому формують одновимірний вігнерівський кристал. Одновимірні електрони у такому кристалі еквівалентні системі плазмонних мод, що описують довгохвильові коливання зарядової густини. У роботі досліджено основний процес розсіювання плазмонів, при якому не зберігається повне число частинок, – процес розсіювання двох плазмонів в три, і навпаки. Для цього процесу обчислена амплітуда розсіювання та отримано рівняння для релаксації штучно створеного хімічного потенціалу.

Ключові слова: одновимірний вігнерівський кристал, плазмові хвилі, інтеграл зіткнень, хімічний потенціал.

Introduction

One of the key areas of modern research is the kinetics of the nearly integrable quantum many-body systems. From this perspective the one-dimensional (1D) systems are particularly relevant as some exact solutions are known [1-2] that can be used to solve more general models in which integrability is broken weakly. Integrability ensures that the scattering of particles in 1D many-particle system is exactly equivalent to the sequence of the pair-particle collisions, and hence the set of initial momenta for each scattering event coincides with the set of finite momenta. Such scattering does not change the distribution function and are unable to lead the system to thermal equilibrium. A striking example of such long-lived non-equilibrium quantum states is a quantum Newton pendulum created using 1D-Bose gas in a trap [3].

To describe the 1D electron crystals (Wigner crystals) that are formed in the quantum wires, the nanotubes and the edge states, we can use exactly solvable Tomonaga-Luttinger model [4-6]. This model predicts the special

properties of 1D electron systems: the power-law anomalies in the tunneling density of states [7] and the effect of charge and spin separation [8]. However, this model has some serious deficiencies. Particularly in the framework of the model excitations have an infinite lifetime, which implies a lack of equilibration.

Renewed interest in 1D electron crystals stems from the new experimental results that do not fit into the paradigm of the Tomonaga-Luttinger model. Tunneling spectroscopy of quantum wires [9,10] and thermometry of quantum edge states [11,12] are a direct proof of thermal equilibrium in 1D electron systems. The deviations from perfect conductance quantization [13-17] and a violation of the Wiedemann-Franz law [18,19] in the wires with a low electron density are found. These observations have attracted considerable attention and require construction of a new theoretical model.

In the present paper we study the microscopic mechanisms of relaxation in the generalized Tomonaga-Luttinger description of one-dimensional electron liquids,

which amounts to keeping anharmonic interactions between plasmons. We follow and extend the way used in [20]. The 1D Wigner crystal [21-22] represents an extreme case of the Tomonaga-Luttinger liquid with small interaction parameter $\kappa = \pi \hbar v^2 / ms \ll 1$. Here v is particle density, m is electron mass, s is sound velocity of plasmons.

We model the system of strongly interacting spinless electrons of mass m by the Hamiltonian (hereafter $\hbar = 1$)

$$H = \sum_l \frac{p_l^2}{2m} + \frac{1}{2} \sum_{l \neq l'} U(x_l - x_{l'}), \quad (1)$$

where p_l and x_l are the momentum and coordinate of the l th particle, and $V(x)$ is the interaction potential.

In order to involve the standard second-quantized representation we expand (1) with respect to small $u_l - u_{l'}$, which measures deviations of electrons from their respective equilibrium positions $x_l = l/v + u_l$, and introduce collective coordinates

$$\begin{aligned} u_l &= \sum_q \sqrt{\frac{1}{2mN\omega_q}} (b_q + b_{-q}^\dagger) e^{iq l}, \\ p_l &= -i \sum_q \sqrt{\frac{m\omega_q}{2N}} (b_q - b_{-q}^\dagger) e^{iq l}, \end{aligned} \quad (2)$$

where N is a number of electrons in a crystal, and plasmon creation and annihilation operators obey canonical commutation relations

$$[b_q, b_{q'}] = 0, \quad [b_q^\dagger, b_{q'}^\dagger] = 0, \quad [b_q, b_{q'}^\dagger] = \delta_{qq'},$$

The resulting Hamiltonian consists of the usual Wigner crystal part

$$H_0 = \sum_q \omega_q (b_q^\dagger b_q + 1/2), \quad (3)$$

and anharmonic terms discussed in the next section in details. Here the plasmon dispersion is given by

$$\omega_k^2 = \frac{2}{m} \sum_l V''(x = l/n) [1 - \cos(kl/n)]. \quad (4)$$

At the sufficiently small temperatures T the characteristic value of plasmon momentum q is small also, and we can simplify the dispersion relation to

$$\omega_q = s |q| (1 - \alpha q^2), \quad q \sim T/s \ll 1.$$

Here $s = \sqrt{V_{22}/m}$, $\alpha = V_{24}/24V_{22}$, and

$$V_{nm} = \sum_{l=1}^{\infty} V_l^{(m)} l^n, \quad V_l^{(m)} = V^{(m)}(x = l/n).$$

The nonlinearity of the dispersion strictly prohibits the decay of a single boson into two or more because of the momentum and energy conservation. The simplest scattering process involves two bosons both in the initial

and in the final states and the typical relaxation rate scales as the fifth power of temperature [20]. Essential feature of the two-into-two process is the conserving of the number of plasmons. So, this process leads the system to not-exact equilibrium state. Indeed, the resulting “equilibrium” distribution function is expected to have a chemical potential. Below we discuss the two-into-three process which changes the number of plasmons and thereby relaxes the chemical potential.

Two-into-three scattering process

As discussed in the introduction we expand the Hamiltonian H with respect to deviations $u_l - u_{l'}$ keeping terms up to the fifth order, $H \approx H_0 + H_3 + H_4 + H_5$,

$$\begin{aligned} H_{n=3,4,5} &= \frac{1}{2 \cdot n!} \sum_{l'} V_l^{(n)} (u_{l+l'} - u_{l'})^n \\ &= \frac{i^n N}{n(2mN)^{n/2}} \sum_{q_1, \dots, q_{n-1}} \frac{\phi_n(q_1, \dots, q_{n-1})}{\sqrt{\omega_{q_1} \dots \omega_{q_{n-1}} \omega_{\bar{q}_n}}} \\ &\quad \times (b_{q_1} + b_{-q_1}^\dagger) \dots (b_{q_{n-1}} + b_{-q_{n-1}}^\dagger) (b_{\bar{q}_n} + b_{-\bar{q}_n}^\dagger), \end{aligned} \quad (5)$$

where $\bar{q}_n = q_1 + \dots + q_{n-1}$ and

$$\begin{aligned} \phi_n(q_1, \dots, q_{n-1}) &= -2^{n-1} \sum_{l=1}^{\infty} V_l^{(n)} \sin(\bar{q}_n l / 2) \\ &\quad \times \sin(q_1 l / 2) \dots \sin(q_{n-1} l / 2). \end{aligned}$$

The leading order inelastic scattering process that not conserving the plasmon numbers, involves two plasmons in the initial state and three plasmons in the final, or vice versa. The corresponding rate is generated to the first order in the interaction Hamiltonian H_5 , to the second order in the crossed terms between H_3 and H_4 , and finally from H_3 iterated to the third order. For the purpose of finding this rate we introduce \mathcal{T} -matrix

$$\mathcal{T} = H_5 + H_4 G_0 H_3 + H_3 G_0 H_4 + H_3 G_0 H_3 G_0 H_3, \quad (6)$$

where $G_0 = (\Omega_i - H_0)^{-1}$, with energy Ω_i of the initial plasmons. Then the transition rate for plasmon scattering is defined by \mathcal{T} -matrix and given by the Fermi golden rule expression

$$W_{Q_i}^{Q_f} = 2\pi |\langle Q_f | \mathcal{T} | Q_i \rangle|^2 \delta(\Omega_f - \Omega_i), \quad (7)$$

where $Q_{i/f}$ and $\Omega_{i/f}$ abbreviate total initial/final momenta and energies of the plasmons respectively.

Scattering rate

The detailed calculation of the transition rate is cumbersome and technically. Here we shortly describe the steps taken to calculate the rate. After applying Eq. (5) into Eq. (6) and then into Eq. (7) we express the matrix element

$\langle Q_f | \mathcal{T} | Q_i \rangle$ as a sum of the averaged products of the creation and annihilation operators. Each of these averaged products can be reduced by commutation relations to the composition of Kronecker's delta symbols. As a result we get the transition rate in the following form

$$W_{Q_i}^{Q_f} = 2\pi |A_{Q_i}^{Q_f}|^2 \delta_{Q_i, Q_f} \delta(\Omega_i - \Omega_f), \quad (8)$$

where amplitude of the scattering rate can be written in the following form,

$$A_{q_1, q_2, q_3}^{q_1', q_2', q_3'} = A_{q_1, q_2}^{q_1', q_2', q_3'} = \frac{\lambda_{q_1, q_2}^{q_1', q_2', q_3'} \nu^3}{(Nm)^{3/2} s^{1/2}} |q_1 q_2 q_1' q_2' q_3'|^{1/2}, \quad (9)$$

and

$$\begin{aligned} & \lambda_{q_1, q_2}^{q_1', q_2', q_3'} \\ &= \frac{(2s)^{1/2}}{4m^3 \nu^3} \prod_{\substack{q=q_1, q_2, \\ q_1', q_2', q_3'}} |q\omega_q|^{-1/2} [m^2 \phi_5(q_1, q_2, -q_1', -q_2') \\ & - \frac{m}{6} \sum_{\{p\}=\{q\}} \Phi^\pm(p_1, p_2) \phi_4(p_3, p_4, p_5) \\ & + \sum_{\{p\}=\{q\}} \Phi^\pm(p_1, p_2) \Phi^\pm(p_3, p_4) \phi_3(p_5, p_1 + p_2)]. \end{aligned} \quad (10)$$

Here the summation over $\{p\}=\{q\}$ means the summation over variables $p_{1,2,3,4,5}$ taking different value from set of initial momenta $q_{1,2}$ and final momenta with the opposite sign $-q'_{1,2,3}$. In other words the sets of $\{p\}$ and $\{q\}$ coincide,

$$\{p_1, p_2, p_3, p_4, p_5\} = \{q_1, q_2, -q_1', -q_2', -q_3'\}.$$

Then the sign “ \pm ” in superscript of $\Phi^\pm(p_1, p_2)$ should be chosen as “+” if p_1 and p_2 is the both initial or final momenta and “-” otherwise,

$$\Phi^\pm(p_1, p_2) = \frac{\phi_3(p_1, p_2)}{(\omega_{p_1} \pm \omega_{p_2})^2 - \omega_{p_1+p_2}^2}.$$

Equation (8), (9), and (10) describe the scattering of plasmons with arbitrary possible momenta. However expression for $\lambda_{q_1, q_2}^{q_1', q_2', q_3'}$ is crucially simplified in the low temperature limit. In this limit the plasmon momentum $q \sim T/s \ll 1$, and within the leading logarithmic accuracy

$\lambda_{q_1, q_2}^{q_1', q_2', q_3'}$ is independent of momenta.

For example,

$$\lambda_{q_1, q_2}^{q_1', q_2', q_3'} \equiv \lambda = 55\sqrt{2}/48 \approx 1.62,$$

in the case of the screened Coulomb interaction potential,

$$V(x) = \frac{e^2}{|x|} - \frac{e^2}{\sqrt{x^2 + 4d^2}}.$$

where $d \gg \nu^{-1}$ is a distance to screen and e is an elementary charge.

Relaxation of the artificial chemical potential

Now we assume that the crystal is brought out from equilibrium. The main scattering process that relaxes the crystal is the two-in-two process studied in [20]. This process conserves the number of plasmons and therefore drives the distribution function n_q of the plasmons to Fermi-like distribution

$$n_q = [e^{(\omega_q - \mu)/T} - 1]^{-1}, \quad (11)$$

where μ is artificial chemical potential that should relax to zero due to the two-to-three scattering process.

The evolution of the distribution function can be described by Boltzmann equation

$$\frac{\partial n_q}{\partial t} = \mathcal{I}[n_q], \quad (12)$$

where $\mathcal{I}[n_q]$ is the collision integral that corresponds to the two-to-three scattering process,

$$\begin{aligned} \mathcal{I}[n_q] &= \sum_{q_1 > q_2} \sum_{q_2' > q_3'} W_{q_1, q_2}^{q_1', q_2', q_3'} n_{q_1, q_2}^{q_1', q_2', q_3'} \\ & - \sum_{q_2} \sum_{q_1' > q_2' > q_3'} W_{q_1, q_2}^{q_1', q_2', q_3'} n_{q_1, q_2}^{q_1', q_2', q_3'}, \\ n_{q_1, q_2}^{q_1', q_2', q_3'} &= n_{q_1} n_{q_2} \bar{n}_{q_1'} \bar{n}_{q_2'} \bar{n}_{q_3'} - \bar{n}_{q_1} \bar{n}_{q_2} n_{q_1'} n_{q_2'} n_{q_3'}, \end{aligned}$$

with $\bar{n}_q = 1 + n_q$.

In order to simplify Eq. (12) we summarize it over q ,

$$\frac{\partial}{\partial t} \sum_q n_q = \frac{1}{12} \sum_{\substack{q_1, q_2 \\ q_1', q_2', q_3'}} W_{q_1, q_2}^{q_1', q_2', q_3'} n_{q_1, q_2}^{q_1', q_2', q_3'},$$

and substitute Eq. (11) in it. Finally we get

$$\frac{\partial T}{\partial t} \frac{1}{|\mu|} = \tau_0^{-1} \equiv \frac{\pi}{386} \frac{s}{T} \frac{1}{N} \sum_{\substack{q_1, q_2 \\ q_1', q_2', q_3'}} \frac{W_{q_1, q_2}^{q_1', q_2', q_3'}}{\prod_{\substack{q=q_1, q_2 \\ q_1', q_2', q_3'}} \sinh \frac{\omega_q}{2T}}. \quad (13)$$

Here we assume that the crystal is near equilibrium, so chemical potential is small enough, $|\mu| \ll T$, and it is negative to prevent the distribution function from singularity.

In order to calculate the characteristic time τ_0 that determine the relaxation of the artificial chemical potential, we should examine the kinematics of the two-in-three scattering process.

Kinematic constraints

Here we study the kinematics of the two-to-three scattering process and determine the combination of momenta that contributes substantially into the collision

integral and τ_0 . Now we assume that $q_1 > 0$ (case of $q_1 < 0$ can be studied analogously). The momenta and energy conservation laws give

$$\begin{aligned} q'_1 + q'_2 + q'_3 &= q_1 + q_2, \\ |q'_1|(1 - \alpha q_1'^2) + |q'_2|(1 - \alpha q_2'^2) + |q'_3|(1 - \alpha q_3'^2) &= |q_1|(1 - \alpha q_1^2) + |q_2|(1 - \alpha q_2^2). \end{aligned} \quad (14)$$

The first equation sets $\delta_{Q_i, Q_f} = \delta_{q'_3, q_1 + q_2 - q'_1 - q'_2}$. In the case when all momenta is positive $q_{1,2}, q'_{1,2,3} > 0$ the nonlinearity plays the key role because the linear terms in Eq. (14) cancel each other and, excluding q'_3 , we get the quadratic equation for q'_2 ,

$$\begin{aligned} (q_1 + q_2 - q'_1)q_2'^2 - (q_1 + q_2 - q'_1)^2 q'_2 &+ (q_1 - q'_1)(q_2 - q'_1)(q_1 + q_2) = 0. \end{aligned} \quad (15)$$

Then the energy δ -function transforms into

$$\delta(\Omega_i - \Omega_f) = \frac{\delta(q'_2 - q_+) + \delta(q'_2 - q_-)}{3s\alpha(q_+^2 - q_-^2)}, \quad (16)$$

where q_{\pm} are the roots of the quadratic equation (15).

In all other cases when one or several momenta are negative linear terms in Eq. (14) do not cancel each other and do not produce such denominator as in Eq. (16). For example, assume that $q'_2 < 0$ and $q_{1,2}, q'_{1,3} > 0$. Then the energy conservation gives

$$\begin{aligned} \delta(\Omega_i - \Omega_f) &= \frac{1}{s} \delta[q'_2 + 3\alpha(q_1 + q_2)(q'_1 - q_1)(q'_1 - q_2) / 2]. \end{aligned}$$

Comparing the last equation with Eq. (16) one can conclude that the contribution in τ_0 , Eq. (13), is much smaller to factor $\alpha(q_+^2 - q_-^2) \sim (T/s)^2 \ll 1$ in the denominator in Eq. (16) when momenta are of different signs. Then in the further calculation we assume all momenta to be of the same sign.

Calculation of characteristic time

Now we can calculate the characteristic time τ_0 . For that purpose we apply the scattering rate in the form of Eqs. (8) and (9) into Eq. (13) and integrate only over momenta of the same sign,

$$\tau_0^{-1} = \lambda \mathcal{I} \kappa^3 \alpha^{-1} s (T / 2\pi s)^5.$$

where \mathcal{I} is a dimensionless integral originated from the sum over momenta in Eq. (13),

$$\mathcal{I} = \frac{8}{9} \iiint_{\substack{x_{1,2,3} > 0 \\ x_{\pm} > 0}} \frac{dx_1 dx_2 dx_3 (x_+^2 - x_-^2)^{-1}}{\zeta(x_1)\zeta(x_2)\zeta(x_3)\zeta(x_+)\zeta(x_-)} \approx 1.814, \quad (17)$$

with $\zeta(x) = x^{-1} \sinh x$, and x_{\pm} are roots of Eq. (15) with q changed to $-q$,

$$x_{\pm} = \frac{x_1 + x_2 - x_3}{2} \left[1 \pm \sqrt{\frac{4(x_1^3 + x_2^3 - x_3^3)}{3(x_1 + x_2 - x_3)^3} - \frac{1}{3}} \right].$$

It should be noted that integration in Eq. (17) performed over domain where $x_{1,2,3} > 0, x_{\pm} > 0$.

Conclusions

In the present paper we study the scattering of the plasmons in the one-dimensional Wigner crystal. The leading process involves two plasmons in the both initial and final states and, therefore, conserves the number of the plasmons. This process drives the crystal to the non-exact equilibrium that can possess the artificial chemical potential. This chemical potential should be relaxed by the process that not conserving the plasmon number. The main process of such a kind involves two plasmons in the initial state and three in the final, or vice versa. We determine the scattering rate of this process, derive equation of relaxation of the artificial chemical potential, and calculate the characteristic time of the relaxation.

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