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Topological methods in measurement and research of nonlinear dynamical systems

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The authors substantiate the need to create a special theory of measurement and analysis of measurement results for nonlinear dynamical systems. The theory should be based on the principles of the open systems theory, dynamic chaos theory and synergetics theory. The authors analyzed the main topological methods and tools for studying of nonlinear dynamic systems. The main characteristics of nonlinear dynamical systems (interval values of dynamic variables, strong dependence on initial conditions and noise, complex, often chaotic dynamics, evolution) were systematized. It was proposed the next topological tools for analysis of measurement results in nonlinear dynamical systems: measurement portrait, Shennon entropy, fractal dimension, forecasting time.
Keywords: nonlinear dynamical system; topology; chaos; Shannon entropy; fractal dimension.

Ю.С. Курський, Ю.П. Мачехін, О.С. Гнатенко

Авторами обґрунтовано необхідність створення спеціальної теорії вимірювання та аналізу результатів вимірювання в нелінійних динамічних системах. Теорія повинна ґрунтуватися на принципах теорії відкритих систем, динамічного хаосу, синергетики. Авторами виконано аналіз топологічних методів та інструментів дослідження нелінійних динамічних систем. Систематизовані основні характеристики нелінійних динамічних систем, серед яких: інтервальність значень динамічних змінних, сильна залежність від початкових умов і шумів, складна, часто хаотична динаміка, еволюція, Запропоновано інструменти аналізу результатів вимірювання: портрет вимірювання, ентропія Шеннона, фрактальна розмірність, час передбачуваності.
Ключові слова: нелінійна динамічна система; топологія; хаос; ентропія Шеннона; фрактальна розмірність.

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Авторами обоснована необходимость создания специальной теории измерения и анализа результатов измерения в нелинейных динамических системах. Теория должна основываться на принципах теории открытых систем, динамического хаоса, синергетики. Авторами выполнен анализ топологических методов и инструментов исследования нелинейных динамических систем. Систематизированы основные характеристики нелинейных динамических систем, среди которых: интервальность значений динамических переменных, сильная зависимость от начальных условий и шумов, сложная, часто хаотическая динамика, эволюция, Предложены топологические инструменты анализа результатов измерения динамических переменных нелинейных динамических систем: портрет измерения, энтропия Шеннона, фрактальная размерность, время предсказуемости.
Ключевые слова: нелинейная динамическая система; топология; хаос; энтропия Шеннона; фрактальная размерность.

Introduction.

One of the most important scientific tasks today is a study of the self-organization processes and complex hierarchical systems. They talk about research, forecast and management of characteristics: climate, ecosystems, biopopulations, physical devices (such as a laser) and living organisms (such as a human). All of these objects are classified as open nonlinear dynamic systems (NDS). Their general characteristics include: the nonlinearity of dynamics, strong dependence on the initial conditions and external influences, possibility of chaotic behavior and self-organization. Studies of nonlinear processes and systems are devoted to the works of A.N. Kolmogorov, E. Lorentz, S. Smale, I. Prigogine, H. Haken, V.L. Ginzburg et al. [1-6].

For research, forecast and management of NDS we must create the new methods for experimental research and measurement. Despite the urgency of the issue, the problem of measuring the NDS characteristics until recently was not considered. The authors pointed out the discrepancy between the physical and mathematical foundations of the deterministic classical measurement theory and the processes in NDS [10-12]. For research and measurement in NDS we develop the special measurement theory (Nonlinear Metrology) [10]. It is based on the principles of the next interdisciplinary theories: the information theory, open systems theory, dynamic chaos theory, synergetic theory, and a number of others.

The task of this paper is to make a classification of the dynamical systems, to study their common characteristics, that are important for measurement, and to choose the mathematical methods and tools for analysis of measurement results and forecast the dynamics of complex systems.

1. Classification of the dynamical systems.

A dynamical system is any object (a set of objects) or process (a set of processes). For them the concept of a state is unambiguously defined as a set of the quantities values $[X_1(t),...,X_n(t)]$ at any time t and the law of evolution $F(X_i, t)$ of the initial state $[X_1(t_0),...,X_n(t_0)]$ is given:

$$F[X_1(t_0),...,X_n(t_0)] \to [X_1(t),...,X_n(t)]. \quad (1)$$

A dynamical system can be described by a differential equation of the next form:

$$\frac{dX_i(t)}{dt} = F[X_1(t),...,X_n(t)]. \quad (2)$$

The space of all possible states of the system described by expression (1) forms a phase space. The dimension of phase space, as well as of the system dimension, is determined by number of the dynamic variables $X_i(t)$ (DV).

The dynamic systems include the systems of any nature: physical, chemical and biological objects, societies and populations, ecosystems and financial markets, computing processes and information transformation processes [13]. Classification of dynamic systems can be made based on the nature of origin and the basic properties of the systems.

By the nature of origin, the dynamical systems can be classified as: physical, chemical, biological, information and other systems. By the basic properties, their classification can be performed on the following grounds:

- by nature of dynamics - deterministic, stochastic and chaotic, linear and nonlinear systems;
- by interaction with an external environment - open and closed systems:
- by possibility to converse an energy into a heat - dissipative and conservative systems;
- by the nature of a state change - continuous and discrete systems;
- by possibility of self-organization - evolving and not-evolving systems;
- by structure - single-level and multi-level, complex hierarchical systems.

Let's consider the main features of these systems. Deterministic systems are the systems whose DPs change over time according to a strictly defined law $F(X_i, t)$. Stochastic systems are characterized by the random DVs behavior, the values of which can be described by the mathematical apparatus of probability theory. Chaotic systems are the systems with a chaotic dynamics. Linear systems are the systems with a linear or linearized law of evolution $F(X_i, t)$. Nonlinear dynamical systems are the systems whose evolutionary law can't be described by a linear or linearized equation. The values of NDS DVs change in a nonlinear way. Moreover, the evolution law $F(X_i, t)$ of real NDS can be described analytically extremely rarely. Therefore, as a rule, we can't to make a long-term predication of the NDS state. We should to note that a linear system is also deterministic system, but a nonlinear system, because of complex dynamic, can't be referred to either deterministic or stochastic systems. It can be classified as a partially deterministic system.

Open systems, according to I. Prigogine, are the systems through which the flows of energy and entropy can flow [4]. In case of the large flows, the nonlinear self-organization (evolution) processes can take place in such systems. They are characterized by the spontaneous appearance of a complex, often chaotic, structure. Closed systems, respectively, have properties that are opposite to open ones.

Dissipative systems are the open systems that operate far from the thermodynamic equilibrium and are characterized by the possibility of dissipation (dissipation) of energy coming from outside. Conservative systems are the systems with conservation of energy.

Continuous and discrete systems are characterized by a continuous or discrete, respectively, character of the DVs values change. But in the case of discrete measurement even the continuous systems are considered as the discrete ones.

Evolutionary systems are the systems with the evolution and self-organization functions, which are expressed in decreasing of entropy and increasing of order. A distinctive feature of a hierarchical, complex system is a multilevel structure, each level of which includes the interconnected subsystems.

This classification is incomplete. In a number of publications we can find such types of systems as concentrated and distributed, autonomous and non-autonomous; self-oscillatory and other systems.

If the object of research can be classified as a linear, closed, conservative and deterministic system that for measurement and evaluate their results we can use the methods and tools of the classical measurement theory. The cornerstones of it are: the principle of the existence of the single value of the measured quantity, the satisfaction of the measurement results with the central limit theorem and correctness of the ergodic hypothesis [14]. The nonlinear, dissipative, chaotic, evolutionary systems require a fundamentally different approach to the measurement [15].

The most difficult objects for the research, correct measurement and mathematical description are the open, dissipative, hierarchical NDS with the chaotic dynamic and self-organization possess. Such systems include the laser, human, ocean and other complex systems. At the same time, the study of chaotic processes in dissipative NDS is one of the fundamental tasks of modern natural science. DVs of such systems are characterized by interval values, the central limit theorem is not satisfied, the ergodic hypothesis is not always confirmed. Dynamic variables must be correctly measured using the measurement models and approaches that are maximum appropriate to the properties and processes in NDS. A correct measurement of the NDS DVs is an obligatory condition for the estimation of current status but allow us to forecast and manage the real systems.

2. Methods and tools for NDS research.

For NDS research it was created a number of interdisciplinary theories. The brightest of they are: the theory of dynamic chaos [13], synergetics [5], theory of dynamical systems [12]. They solve problems of research, modeling and forecasting of the NDS dynamics. Their methods are widely used in applied problems of the broadest direction - from laser engineering to arrhythmology and neurodynamics [16, 17]. The analysis of the main provisions and tools of these theories will allow us to construct a new theory for measurement of the NDS DVs.

The researchers apply two methods for NDS study, that are differed in the type of mathematical model [12]. The first method is based on the mathematical modeling of a system and searching of the evolution function $F(X_i,\, t)$ (2). The state of the system at the time is a point in the phase space, given by the DVs values $[X_1(t),...,X_n(t)]$ and evolution function $F(X_i,\, t)$. The system state change corresponds with the movement of the "depicting" point, which describes the phase trajectory. A set of phase trajectories forms a phase portrait of a system. The phase portrait and evolution function make up the mathematical model of a system. The phase portrait serves as an object for studying the dynamics. The evolution function allows us to predict the DVs values. The problem of the described method is a complex mathematical problem of searching of the evolution function $F(X_i,\, t)$.

The second method focuses on the functional side of a system. It does not allow us to study all features of the internal structure of a system. The system is interpreted as a "black box" with input $[X_1(t_0),...,X_n(t_0)]$ and output $[X_1(t),...,X_n(t)]$ the DVs values. In this case, the "black box" plays the role of the evolution function, transforming the inputs into the outputs, and the mathematical model is determined by the spaces of the inputs and outputs.

The first method has comprehensive information about a system, but in a practice it can be implemented only in rare cases. The second method does not allow us to investigate all the features of a system, but it allow us to determine the DVs values at the time intervals and to construct an incomplete, discrete phase portrait. We think that the second method is most suitable for constructing the models for measurement in the real NDSs.

2.1. Phase portrait.

A phase portrait is the most popular tool of the qualitative theory of dynamical systems [18]. The researching of it allows us to know: the type of system dynamics (deterministic, stochastic or chaotic), Lyapunov exponents, forecast time et al [13]. The values of the system states can be represented by a matrix of dimension $n \times m$ (here n is the number of DVs and m is the number of DVs measurements) in the next form:

$$\begin{vmatrix} X_1(t_0) \ldots X_n(t_0) \\ \ldots \\ X_1(t_m) \ldots X_n(t_m) \end{vmatrix}. \qquad (3)$$

A phase portrait can be limited and unlimited in space, can increase or decrease. The phase volume of the conservative systems is conserved but the phase volume of the dissipative systems is not.

A special kind of a phase portrait is an attractor. It is the state of the dynamical system to which a system aspires in the time during their development. The presence of an attractor indicates a "special" dynamics of a system. There is a strange attractor witch often is a testament of the chaotic dynamics of NDS. Its distinguishing feature is the exponential instability, which is expressed in the exponential discrepancy of the phase portrait trajectories and the fractal dimension [13].

The analysis of a phase portrait is often used in the applied researches of NDS [17]. In the framework of the nonlinear metrology the authors propose to use the "measurement portrait" instead of the classical measurement equation (model equation). It is a graphical and numerical display of the DVs measurement results DV. The measurement portrait is a phase portrait of the NDS trajectory, constructed with the uncertainties or measurement errors [20]. This approach allows us don't find the evolution function.

2.2. Topology and other characteristics of NDS.

S. Smale linked the topology of a phase space and the dynamics of a system [3]. He abandoned the idea of observation an individual trajectory that requires the solution of the equation (2), and proposed to investigate the integral phase space and its geometric structure. Studies have shown that topological transformations of a phase space are a reflection of the physical processes. Thus, the scattering and loss of energy by a system are expressed in the compression of the phase portrait. Approximately the same phase portraits indicate a similar dynamics of the systems. If the shape of the phase portrait is accessible to the visual representation, the system can be solved.

The geometric study of the phase portraits allows obtaining such data about NDS as: the nature of the dynamics, time horizon for the DVs values prediction, intervals of the DVs values. We can determine: the attractor volume, Lyapunov exponents, Shannon entropy and Kolmogorov entropy, attractor dimension, and other quantities. We suggest use some of these characteristics for analysis of the measurement results in the case of NDS.

2.2.1. Lyapunov exponents are used for study the dynamics of a system in the vicinity of an arbitrary trajectory. They characterize the degree of stretching and contraction of the phase portrait along the selected phase trajectories. If the two close trajectories $x_i(t)$ and $x_{i+1}(t)$ are chosen so that $x_{i+1}(t)=x_i(t)+\xi(t)$, $\xi(0)=\varepsilon, \varepsilon\to 0$ that the next function:

$$\Xi\left[\xi(0)\right]=\lim_{t\to\infty}\frac{1}{t}\ln\left[\frac{\xi(t)}{\xi(0)}\right] \quad (4)$$

takes a finite series of the Lyapunov exponents $\{\lambda_i\}, i=1,2,...,n$, the totality of which forms the Lyapunov spectrum [13]. The number of Lyapunov exponents corresponds with the attractor dimension D_A, which can be fractional:

$$D_A = j+\sum_{i=1}^{j}\frac{\lambda_i}{\left|\lambda_{i+1}\right|}, \quad (5)$$

here j is the Lyapunov dimension, it is determined from the expressions:

$$\lambda_1+\lambda_2+...+\lambda_j>0,\ \lambda_1+\lambda_2+...+\lambda_{j+1}<0.$$

The total Lyapunov exponent Λ can be considered as an indicator of a stability of a system dynamics. When $\Lambda=0$ it is Hamiltonian system. It has a stable dynamics, the processes occurring in it can be regarded as deterministic processes, the volume of the phase portrait is unchanged $\Delta V_A = const$. When $\Lambda>0$ the phase portrait volume is growing $\Delta V_A\uparrow$, the NDS dynamics is chaotic. If $\Lambda<0$ the phase portrait volume decreases , that typical for the dissipative systems.

2.2.2. Entropy. For topological analysis of the NDS phase portrait the Shannon (*H*-entropy(and the Kolmogorov-Sinay (*K*-entropy) are used. *H*-entropy (or information entropy) is one of the key concepts of the information theory [21]. For a system that can be in the states X_i with probability distribution density $p(X_i)$, Shannon entropy is given by the next formula:

$$H=-\sum_{i=1}^{N}p(X_i)\ln p(X_i)\,. \quad (6)$$

Entropy is a measure of the order or disorder of the system. According to (6), The Shannon entropy assumes large values when the distribution density $p=p(X_i)$ has the values small. If a number of values N is bounded, then the entropy is maximal for the uniform distribution law $H\to\ln N$ for $p(X_i)\to 1/N$. The entropy is minimal $H\to 0$ for the normal distribution law when $p(X_i)\to 1$. The entropy of a strange attractor is higher than the entropy of a regular attractor. The entropy of chaotic and random dynamics is higher than the entropy of an ordered motion. The change of the *H*-entropy values indicates a change in the NDS dynamics.

The using the Kolmogorov entropy allowed us to introduce a rigorous criterion of chaotic, as an unstable by Lyapunov motion with positive metric entropy $K>0$ [13]. Analyzing the phase portrait of a system, the *K*-entropy is defined as:

$$K=\lim_{\substack{d(0)\to 0\\ t\to\infty}}\frac{\ln[d(t)/d(0)]}{t}, \quad (7)$$

here *d(0)*, *d(t)* are the distances between two nearby trajectories at the initial and current time, respectively: $d(t)=\left|x_2(t)-x_1(t)\right|$.

According (7) the *K*-entropy characterizes the degree of the trajectories divergence, and the degree of randomness of the system dynamics. It is related to the Lyapunov exponents (4) by the expression:

$$K=\int\sum_{\lambda_i\ge 0}\lambda_i(x)\mathrm{d}\mu\,. \quad (8)$$

So when the system has chaotic dinamycs its entropy $K>0$.

The Shannon entropy, *S*-theorem by Yu. Klimontovich [21], entropy scales, we consider as a tool for estimating of the deviation of a system from an equilibrium state. The entropy analysis was used before in the human health measurement model [23] and for estimating the temperature during the laser cooling of particles [24].

2.2.3. Fractal dimension. Many of the NDS processes have the property of self-similarity or scaling - invariance under multiplicative scale changes. Self-similarity can be strict or approximate. A self-similar object or process looks unchanged when you zoom in or out the scale. Such objects and processes include the Brownian particle motion, turbulent flows, strange attractors, time series of the measurement results [25].

The most striking feature of the objects self-similarity is their unusually fine structure. Such objects B. Mandelbrot called the fractals [26]. The importance of fractals lies in the fact that they are able to model a huge number of objects, phenomena and real-world processes, real NDSs.

The fractals are characterized by Hausdorff (or fractal) dimension D_H. It takes fractional values in the interval $0 < D_H \leq 3$. For a fractal curve 1 < D_H < 2, for a surface 2 < D_H < 3, a point has dimension D_H = 0, for a continuous line D_H = 1.

Fractal dimension is used in various practical applications to identify the objects and processes. The special interest is its use for analysis of NDS phase portraits and the measurement results time series $x(t_1),...,x(t_n)$ [27]. For determination of the time series fractal dimension D_H we use the statistical method of the normalized range (*R/σ* - analysis), derived empirically by P. Hurst [25]. The indicator H_R is associated with D_H by next expression:

$$D_H = 2 - H_R \tag{9}$$

The Hurst index H_R is determined using the value *R/σ*, here *R* is the range between the maximum and minimum values of the increment function $x(i,n)$, the value *σ* is the standard deviation:

$$R(t) = \max_{1 \leq i \leq m} x(i,n) - \min_{1 \leq i \leq m} x(i,n),$$
$$x(i,n) = \sum_{i=1}^{n} (x_i - \overline{x}) \quad , \tag{10}$$

here $\overline{x}_i$ is the arithmetic mean of the values $x(t_1),...,x(t_n)$.

The correlation *R/σ* is related with parameter H_R by formula:

$$R/\sigma = (n/2)^{H_R} . \tag{11}$$

In [27] the fractal analysis (9)-(11) was used for analyze the dynamics of the laser radiation frequency. The author proposed a fractal scale for evaluating the results of measurements with reference points D_H = 1, D_H = 1,5 , D_H = 2, separating different dynamics characters. If D_H = 1 it means that the dynamics of the system is strictly deterministic. If D_H = 2 the system behaves in a regular way, but the spread of the measured results is very large, that doesn't allow us to use the methods for processing of the measurement results. If D_H = 1,5 the process is random. The dynamics corresponds with Brownian motion with independent (Markov) increments. For analyze of such systems characteristics we can use the statistical methods. In the case when 1 < D_H < 1,5 or 1,5 < D_H < 2 the process is non-Markov, chaotic, persistent and antipersistent, respectively.

The fractal dimension allows us to estimate the trend of the DV dynamics of NDS and can be used to predict its values.

2.2.4. Forecasting time. One of the main and oldest tasks of analyzing systems and time series of DV measurement results has been the task of forecasting their dynamics. In some cases, the purpose of the forecasting is not the value of an individual DV, but forecasting of dynamics and its trend. For this, the fractal analysis (9)-(11) and the fractal scales [27] are applied.

The time interval when we can do the correct forecasting of the system dynamics is called the forecasting time or the forecasting horizon. The forecasting time depends on the degree of determinism of the NDS dynamics (the maximal for a deterministic system and minimal for random and chaotic systems) and metrological possibilities [29].

In the case of chaotic NDS, a weak impact of the initial conditions or a small change in the system parameters cause to unpredictability of the resulting motion in finite time, which J. Lighthill [30] called the "forecasting horizon" (or forecasting time). The forecasting time T_{for} is related to the Lyapunov exponent λ (7) as:

$$T_{for}(\lambda) = \frac{1}{\lambda_{\max}} \log \frac{1}{\varepsilon}, \tag{12}$$

here λ_{max} is the maximum Lyapunov exponent.

In practice, the forecasting time (12) is often calculated using the next simplified formulas:

$$T_{for}(K) = \frac{1}{K}, T_{for}(\lambda) = \frac{1}{\lambda_{\max}}. \tag{13}$$

The term "forecasting time" is important for the formulation of the measurement equation (model equation) of DVs. We suggest use this value as the correctness time of the measurement equation for NDS case.

3. Measurement principles for NDS case.

The measurement of the NDS DVs is a multi-factor experiment. The processing of the measurement results in a multifactorial experiment is aimed at obtaining the basic scientific data in the new form of mathematical models and their interpretation. We shouldn't only to calculate the average value of the measured quantity or its dispersion [14]. The theory of nonlinear measurements, measurement and analysis models should correspond with the properties of such systems. Let's consider the important for measurement procedures real NDS properties.

The dynamics of NDS has a complex, non-linear, including chaotic, character. NDS exchanges energy and information with the environment and other systems, it is influenced by external factors. The influence of some factors (and noises) is critical for the system, it is can changes the dynamics from random to regular, chaotic, and vice versa.

The state of the NDS at a time moment t is characterized by the n-dimensional state vector $X\left[X_1(t),...,X_n(t)\right]$. The DV $X_i(t)$ value changes in time, but stays in the interval $X_i^{\min} \le X_i \le X_i^{\max}$. This interval is due to the functionality of the system. If the DV value outputs of the interval it means that the system destroys.

The phase portrait of NDS in a chaos state is a strange attractor. The exponential dispersal of the phase trajectories leads to the fact that the measured quantities can take any values in the frame of the attractor. If at the moment of measurement t_0 the DV value X_i is in the interval $[y_i(t_0)-u_i(t_0), y_i(t_0)+u_i(t_0)]$ (here $y_i(t_0),u_i(t_0)$ are the estimation and uncertainty of the X_i measurement result at the time t_0) that in tine the DV value will located in attractor frame $[y_{\min}-u_{\min}, y_{\max}+u_{\max}]$ (here $y_{\min}, y_{\max}, u_{\min}, u_{\max}$ are the estimates and uncertainties of the measurement of the X_i minimum and maximum values):

$$[y_i(t_0)-u_i(t_0), y_i(t_0)+u_i(t_0)] \in \\ \in [y_{\min}-u_{\min}, y_{\max}+u_{\max}]. \quad (14)$$

The next, after t_0 time DV values become predictable within the attractor frame (14).

Systems can evolve, some of them have the self-organization function.

Based on the described properties, the authors offer the next topological tools for analyzing the measurement results for NDS case:

1. the time series of the DVs measurement results (3);
2. a measurement portrait (a phase portrait with the measurement uncertainties), constructed on the measurement results (3);
3. the Lyapunov exponents (4);
4. the Shannon (6) and Kolmogorov entropies (8);
5. a fractal dimension (9)-(11) of the measurement time series (3);
6. a forecasting time (13).

In this case, the all quantities values must contain an error (or uncertainty of the measurement results).

The application the physical approaches, topological mathematical methods and tools of nonlinear metrology makes it possible to provide studies of systems with complex, nonlinear dynamics. The topological methods and tools for measurement result analysis help to evaluate the reliability of the measurement data and give a possibility to predict the NDS dynamics.

Conclusions

1. The classification of dynamic systems by origin and properties is performed. It is shown that the dissipative, nonlinear dynamical systems are the most difficult for research, measurement and forecast.

2. The necessity of creating a special theory of measurement and measurement results analysis for nonlinear dynamical systems is substantiated.

3. The analysis of the main topological methods and tools (including topological methods and tools) for the study of nonlinear dynamical systems is performed:

4. The main characteristics of non-linear dynamical systems are systematized, among them: interval values of the dynamical variables, strong dependence on initial conditions and noises, complex, often chaotic dynamics, evolution,

5. In accordance with the main characteristics of nonlinear dynamical systems, the next topological tools for analyzing the measurement results are proposed: measurement portrait, entropy, fractal dimension, forecasting time.

References

1. A.N. Kolmogorov. DAN USSR, 98, 527 (1954).
2. E. Lorenz. Bulletin of the American Meteorological Society. 50, 345, (1969).
3. S.Smale, Bulletim of the Amer. Math. Soc. 66, 1 (1960).
4. I. Prigogine, Time, Structure and Fluctuations, Nobel Lecture. (1977).
5. H. Haken. The science of structure :synergetics:New York: Van Nostrand Reinhold, (1984), 255 p.
6. V.L. Ginzburg, UFN, 169, 4. 419 (1999).
7. Yu.P. Machekhin, V.A. Single, UMJ, 2, 49 (1996).
8. A.S. Gnatenko, Yu.P. Machekhin, Yu.S. Kurskoy, V.P. Obozna, J. Nano- Electron. Phys. 10 (2), 02033 (2018).
9. Yu.P. Machekhin, Yu.S. Kurskoy, A.S. Gnatenko, Radiotekhnica, 192, 2, 102 (2018).
10. Yu.P. Machekhin, Yu.S. Kurskoy, Fundamentals of Nonlinear Metrology: LAP (2014), 240 c.
11. Yu. Machekhin, Yu. Kurskoy, M&T. 60, 3, 49 (2017).
12. Yu. Neimark. Dynamic Systems and Controlled Processes, Editorial URSS, M. (2010). 336 c.
13. V.Yu. Loskutov. UFN, 177, 989 (2007).
14. P.V. Novitsky, I.A. Zograf, Error estimation of measurement results, Energoatomizdat, Leningrad, (1991), 304 p.
15. Yu.P. Machekhin, Yu.S. Kurskoy, Instruments and Methods of Measurement. 6, 2. 257 (2015).
16. G.G. Akchurin, A.G. Akchurin, Letters in JTP, 31, 10, 76 (2005).
17. A.A. Khadartsev, V.M. Eskov, Therapist, 1. 35 (2015).
18. A.A. Andronov, E.A. Leontovich, I.I. Gordon, A.G. Meier, Qualitative theory of second-order dynamical systems, Nauka, M. (1966), 568 c.

19. N.G. Van Campen. Stochastic processes in physics and chemistry. Vischaya shkola, M. (1990), 376 p.
20. Yu. Machekhin, Yu. Kurskoy, E. Prisich, M&T, 61, 5, 48 (2016).
21. Yu.L. Klimontovich, Letters in JTP, 8, 1412 (1983).
22. Yu. Machekhin, Yu. Kurskoy, Measuring technique, 57. 6. 609 (2014).
23. Yu. Machekhin, Yu. Kurskoy, M&T, 50, 6, 56 (2014).
24. Yu. Machekhin Yu. Kurskoy, M&T, 56, 06, 50 (2015).
25. M. Schroeder, Fractals, chaos, power laws, Regular and chaotic dynamics, Izhevsk, (2001), 316 p.
26. B. Mandelbrot, The Fractal Geometry of Nature, Freeman, San Francisco, (1982), 491 p.
27. Yu. Machekhin, Measurement Techniques. 52. 8. 835 (2009).
28. M. Born, Continuity, determinism, reality, Nauka, M. (1971), 415 p.
29. Yu.A. Kravtsov, UFN, 158, 1, 93 (1989).
30. J. Lighthill, Proc. Roy. Soc. Ser. A. 407. 35 (1986).