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Massive graviton in Minkowski and de Sitter space-time

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Пряме спостереження гравітаційних хвиль у 2015 році привернуло увагу до питання, чи є у гравітонів маса. Звичайно вважається, що у часопросторі Мінковського 2S + 1 лінійно незалежних станів поляризації релятивістської масивної частинки у безмасовій границі редукуються у два стани зі спіральностями ±S (тобто із проекціями спіну ±S на напрямок руху частинки). Хибність цього розповсюдженого уявлення продемонстрована у випадку масивного гравітону, представника частинок зі спіном 2. Також проаналізовані властивості гравітаційної хвилі у часопросторі де Сіттера, який є добрим наближенням до моделі сучасного стану нашого Всесвіту, що розширюється з прискоренням. З'ясовано, що у всесвіті де Сіттера (а це означає, що і у нашому реальному Всесвіті) гравітон не може бути безмасовим: квадрат його маси є від'ємним числом $m_g^2 = -(\hbar/c)^2 2\Lambda/3$, де Λ – айнштайнівська космологічна стала, тобто маса гравітона є суто уявним числом, яке по модулю дорівнює $|m_g| = 1,70\cdot10^{-33}$ eV/c^2.
Ключові слова: гравітаційні хвилі; гравітон; де Сіттер; спиральність; безмасова частинка.

Прямое наблюдение гравитационных волн в 2015 году привлекло внимание к вопросу о существовании массы гравитона. Принято считать, что в пространстве-времени Минковского 2S + 1 линейно независимых состояний поляризации релятивистской массивной частицы редуцируются в два состояния со спиральностями ±S (то есть с проекциями спина ±S на направление движения частицы). Ошибочность этого равпространенного представления продемонстрирована в случае массивного гравитона, представителя частиц со спином 2. Также проанализированы свойства гравитационной волны в пространстве-времени де Ситтера, которое представляет собой хорошее приближение к модели современного состояния нашей, расширяющейся с ускорением Вселенной. Показано, что во вселенной де Ситтера (а, следовательно, и в нашей реальной Вселенной) гравитон не является безмассовым: квадрат его массы отрицателен, $m_g^2 = -(\hbar/c)^2 2\Lambda/3$, где Λ – эйнштейновская космологическая постоянная, то есть масса гравитона представляет собой чисто мнимое число, модуль которого $|m_g| = 1,70\cdot10^{-33}$ eV/c^2.
Ключевые слова: гравитационные волны; гравитон; де Ситтер; спиральность; безмассовая частица.

The direct observation of gravitational waves in 2015 has drawn attention to a problem on existence of graviton mass. In the case of Minkowski space-time it is considered to be that the 2S + 1 linearly independent states of polarization of a relativistic massive particle are reduced to two states with helicities ±S (that is with projections of spin ±S on a direction of movement of a particle). The inaccuracy of this opinion is shown in the case of massive graviton, the special case of particles with spin 2. Also properties of a gravitational wave in the de Sitter space-time which represents good approach model of a modern state of our extending with accelera-tion Universe are analysed. It is shown, that in the de Sitter universe (and, hence, and in our real Universe) the graviton is not massless: the square of its mass is negative, $m_g^2 = -(\hbar/c)^2 2\Lambda/3$, where Λ is Einstein cosmo-logical constant, i.e. graviton mass is purely imaginary number modulo $|m_g| = 1,70\cdot10^{-33}$ eV/c^2.
Keywords: gravitational waves; graviton; de Sitter; helicity; massless particle.

Introduction

There were five direct detections of gravitational waves in the last three years. They gave a lot of information for modern cosmology. These events let one to ask an old question: do gravitons have mass? This was known as massive graviton problem. The particle with spin S in Minkowski space-time has 2S+1 linear independent states of polarization. It is usual to believe that in massless limit these states reduce to two helicity states . The other states "extinct". But for massive graviton in Minkowski space-time there is a contra-example, which saves all five helicity components.

The latest investigations show that de Sitter metric is one of the reasonably close ap-proximations to the metric of real Universe. So the investigation of the massive graviton prob-lem in de Sitter space-time is useful for understanding of the role of massive graviton in real expanding Universe. The graviton in de Sitter space-time is not massless, but its mass proper-ties are somewhat unusual.

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Gravitational quanta of Matvei Bronstein (1936)

Two important papers on gravitational quanta were published by prominent soviet phys-icist Matvei Bronstein in 1936. The first paper „Quantentheorie schwacher Gravitationsfelder" ("Quantum theory of weak gravitational field") appeared in Kharkov in German [1] (scientific monthly „Physikalische Zeitschrift der Sowjetunion" was published by "Ukrainian Institute of Physics and Technology", УФТІ). The expanded version of "Quantization of gravitational waves" was published in Russian [2]. Both papers are almost unknown abroad. Only in 2012 the first paper of Bronstein was republished in English [3].

Bronstein was considered the small deviations from Minkowski space-time with Rie-mann-Christoffel tensor

$$R_{\mu\rho\nu\sigma}=\frac{1}{2}(h_{\mu\nu,\rho\sigma}+h_{\rho\sigma,\mu\nu}-h_{\mu\sigma,\rho\nu}-h_{\rho\nu,\mu\sigma}),\quad (1)$$

where $h_{\mu\nu}$ are the small deviation of the fundamental metric tensor $g_{\mu\nu}(x)=g^0_{\mu\nu}+h_{\mu\nu}(x)$ from its Minkowskian value $g^0_{\mu\nu}$ ($g^0_{00}=1$, $g^0_{11}=g^0_{22}=g^0_{33}=-1$; $g^0_{\mu\nu}=0$, if $\mu\neq\nu$). Like Bronstein we admit the following summation convention for Greek indices in tensor values: ...μ...μ... ≡ ...0...0...-...1...1...-...2...2...-...3...3..., so we can define the Ricci tensor as $R_{\mu\nu}=R_{\mu\sigma\nu\sigma}=0$ and write down Einstein equations of gravitation (without cosmological constant Λ) in empty space,

$$R_{\mu\nu}=g^{0\rho\sigma}R_{\mu\rho\nu\sigma}\equiv R_{\mu\sigma\nu\sigma}=0. \quad (2)$$

In view of "gauge conditions" and tracelessness of $h_{\mu\nu}$-tensor

$$g^{0\rho\sigma}h_{\mu\rho,\sigma}\equiv h_{\mu\sigma,\sigma}=0,\quad g^{0\rho\sigma}h_{\rho\sigma}\equiv h_{\sigma\sigma}=0,\quad (3)$$

we have the wave equations

$$\Box h_{\mu\nu}=0. \quad (4)$$

The Riemann-Christoffel tensor satisfies two Bianchi identities,

$$R_{\mu\rho\nu\sigma}+R_{\mu\nu\sigma\rho}+R_{\mu\sigma\rho\nu}=0, \quad (5)$$

$$R_{\mu\rho\nu\sigma,\lambda}+R_{\mu\rho\sigma\lambda,\nu}+R_{\mu\rho\lambda\nu,\sigma}=0. \quad (6)$$

Let us introduce like Bronstein new notations

$$E_{ik}=R_{0i0k},\; B_{ik}=\frac{1}{2}\varepsilon_{imn}R_{0jmn}.$$

Ten quantities (E_{ik} and H_{ik} are symmetric traceless tensors) use up all ten components of the Riemann-Christoffel tensor $R_{\mu\rho\nu\sigma}$. In his paper Bronstein presented the linearized Einstein equations in the following form

$$\begin{aligned}\frac{\partial}{\partial t}E_{ij}&=\varepsilon_{ikl}\frac{\partial}{\partial x_k}B_{lj}, & \frac{\partial}{\partial x_i}B_{ij}&=0,\\ \frac{\partial}{\partial t}B_{ij}&=-\varepsilon_{ikl}\frac{\partial}{\partial x_k}E_{lj}, & \frac{\partial}{\partial x_i}E_{ij}&=0.\end{aligned}\quad (7)$$

The equations (7) are very similar to the Maxwell ones,

$$\begin{aligned}\frac{\partial}{\partial t}E_{i}&=\varepsilon_{ikl}\frac{\partial}{\partial x_k}B_{l}, & \frac{\partial}{\partial x_i}B_{i}&=0,\\ \frac{\partial}{\partial t}B_{i}&=-\varepsilon_{ikl}\frac{\partial}{\partial x_k}E_{l}, & \frac{\partial}{\partial x_i}E_{i}&=0.\end{aligned}\quad (8)$$

It isn't accidental because in the both cases the equations with time derivatives in (7) and (8) can be written in the form satisfying for arbitrary spin S

$$\left[\frac{\partial}{\partial t}+\frac{1}{\lambda}\left(\frac{\partial}{\partial\vec{x}}\vec{S}\right)\right]\psi=0, \quad (9)$$

where $\vec{S}$ is the spin operator of particle with spin S and λ is the helicity of the particle under consideration. The famous Italian physicist Ettore Majorana was the first who presented Maxwell equations (8) in the form (9) (1928-1932, unpublished, [4])

$$i\frac{\partial}{\partial t}\vec{\psi}=\pm\mathrm{rot}\vec{\psi},\quad \mathrm{div}\vec{\psi}=0, \quad (10)$$

where $\vec{\psi}=\vec{E}\pm i\vec{B}$, and understood that the equations (10) are the wave equations for photons. The wave equations (10) for right and left photons have very simple physical meaning: pho-ton's Hamiltonian $H=\pm\vec{S}\hat{\vec{p}}.\equiv\pm\mathrm{rot}$ The arbitrary spin wave equations for the massless fields can be written in the following relativistically covariant form [5,6]

$$(iS_{\mu\nu}+S\delta_{\mu\nu})\frac{\partial}{\partial x_\nu}\psi(x)=0, \quad (11)$$

where $S_{\mu\nu}$ are infinitesimal operators of the (s_1,s_2)-representation of the proper Lorentz group, $S=s_1+s_2$. The well known Weinberg theorem [7] states that the helicity of the massless field under consideration $\lambda=s_1-s_2$.

Massive graviton in Minkowski space-time

The massless graviton is described with complex analogue Riemann-Christoffel symbols $R_{\mu\nu\rho\sigma}$ and Einstein equations (2) in the flat limit. If we want to describe massive graviton we must also introduce two tensors $H_{\mu\nu\rho}$ and $H_{\mu\nu}$[1] with some symmetry properties of them.

[1] The massive graviton is usually defined with Pauli-Fierz formalism [8]. Our definition based on the Bargman-Wigner ones [9].

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho}, \qquad (12)$$

$$R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0, \qquad (13)$$

$$H_{\mu\nu\rho} = -H_{\nu\mu\rho}, \qquad (14)$$

$$H_{\mu\nu\rho} + H_{\nu\rho\mu} + H_{\rho\mu\nu} + 0, \qquad (15)$$

$$H_{\mu\nu} = H_{\nu\mu},\ H_{\mu\mu} = 0, \qquad (16)$$

$$R_{\mu\sigma\rho\sigma} = R_{\mu\nu} = H_{\mu\nu.} \qquad (17)$$

Next we write down the series of equation with mass of the graviton.

$$R_{\mu\nu\rho\sigma,\sigma} = mH_{\mu\nu\rho}, \qquad (18)$$

$$H_{\mu\nu\sigma,\rho} - H_{\mu\nu\rho,\sigma} = mR_{\mu\nu\rho\sigma}, \qquad (19)$$

$$H_{\mu\sigma\nu,\sigma} = mH_{\mu\nu}, \qquad (20)$$

$$H_{\mu\rho,\sigma} - H_{\mu\sigma,\rho} = mH_{\sigma\rho\mu}. \qquad (21)$$

Then we give the examples without mass.

$$R_{\mu\nu\rho\sigma,\lambda} + R_{\mu\nu\sigma\lambda,\rho} + R_{\mu\nu\lambda\rho,\sigma} = 0, \qquad (22)$$

$$H_{\rho\sigma\nu,\lambda} + H_{\sigma\lambda\nu,\rho} + H_{\lambda\rho\nu,\sigma} = 0, \qquad (23)$$

$$H_{\mu\nu,\nu} = 0. \qquad (24)$$

In massless limit we obtain the independent equations for the particles with helicities ±2, ±1, 0 which led to according to Eddington [10] TT-waves, TL-waves and LL-waves.

TT-waves or Bronstein waves are:

$$R_{\mu\nu\rho\sigma,\sigma} = 0, \qquad (25)$$

$$R_{\mu\nu\rho\sigma,\lambda} + R_{\mu\nu\sigma\lambda,\rho} + R_{\mu\nu\lambda\rho,\sigma} = 0, \qquad (26)$$

$$R_{\mu\nu} = 0. \qquad (27)$$

TL-waves are:

$$H_{\rho\sigma\nu,\sigma} = 0\,, \qquad (28)$$

$$H_{\mu\nu\sigma,\rho} - H_{\mu\nu\rho,\sigma} = 0, \qquad (29)$$

$$H_{\rho\sigma\nu,\lambda} + H_{\sigma\lambda\nu,\rho} + H_{\lambda\rho\nu,\sigma} = 0, \qquad (30)$$

LL-waves:

$$H_{\mu\sigma,\sigma} = 0, \qquad (31)$$

$$H_{\nu\sigma,\lambda} - H_{\nu\lambda,\sigma} = 0 \qquad (32)$$

TT-gravitational wave in de Sitter space-time

De Sitter space-time is special solution of Einstein equations with cosmological con-stant for empty space and is the special case of Riemannian manifolds called by mathemati-cians as Einstein manifolds [11],

$$\boldsymbol{R}_{\mu\nu} = \Lambda g_{\mu\nu} \qquad (33)$$

$$\boldsymbol{R}_{\mu\nu} = g^{\rho\sigma}\boldsymbol{R}_{\mu\rho\nu\sigma} \qquad (34)$$

De Sitter space-time is described by the metric tensors

$$g_{\mu\nu}\ (g_{00} = 1,\ g_{ik} = -a(t)^2\delta_{ik}) \qquad (35)$$

and

$$g^{\mu\nu}\ (g^{00} = 1,\ g^{ik} = -(1/a(t)^2)\delta_{ik}) \qquad (36)$$

and the metric

$$ds^2 = dt^2 - a(t)^2 d\vec{x}^2, \qquad (37)$$

where

$$a(t) = \exp\sqrt{\frac{\Lambda}{3}}(t - t_0), \qquad (38)$$

Λ – is cosmological constant and t_0 – is the age of Universe. We see that $g^{\mu\nu}(t_0) = g^{0\mu\nu}$ is Minkowskian metric tensor.

Let us consider the small deviations from de Sitter space-time by the use Riemann-Christoffel tensor $\Re_{\mu\nu\rho\sigma} = \boldsymbol{R}_{\mu\nu\rho\sigma} + S_{\mu\nu\rho\sigma}$ and metric tensor $g_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$, where is small deviations of Riemann-Christoffel tensor $S_{\mu\nu\rho\sigma}$ from its de Sitter value $\boldsymbol{R}_{\mu\nu\rho\sigma}$. For small $h_{\mu\nu}$ tensor $\Re_{\mu\nu\rho\sigma}$ can be express by a formula

$$\begin{aligned}\Re_{\mu\nu\rho\sigma} &= \frac{\Lambda}{3}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) + R_{\mu\nu\rho\sigma} = \\ &= \frac{\Lambda}{3}[(g+h)_{\mu\rho}(g+h)_{\nu\sigma} - \\ &-(g+h)_{\mu\sigma}(g+h)_{\nu\rho})] + R_{\mu\nu\rho\sigma} = \\ &= \frac{\Lambda}{3}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) + \\ &+\frac{\Lambda}{3}(h_{\mu\rho}g_{\nu\sigma} + g_{\mu\rho}h_{\nu\sigma} - \\ &-h_{\mu\sigma}g_{\nu\rho} - g_{\mu\sigma}h_{\nu\rho}) + R_{\mu\nu\rho\sigma}.\end{aligned} \qquad (39)$$

It follows from (39) that

$$\begin{aligned}S_{\mu\nu\rho\sigma} &= \frac{\Lambda}{3}(h_{\mu\rho}g_{\nu\sigma} + g_{\mu\rho}h_{\nu\sigma} - \\ &-h_{\mu\sigma}g_{\nu\rho} - g_{\mu\sigma}h_{\nu\rho}) + R_{\mu\nu\rho\sigma}.\end{aligned} \qquad (41)$$

Calculating

$$\begin{aligned}\Re_{\mu\nu} &= g^{\rho\sigma}\Re_{\mu\rho\nu\sigma} = (g^{\rho\sigma} - h^{\rho\sigma})\times \\ &\times(\boldsymbol{R}_{\mu\rho\nu\sigma} + S_{\mu\rho\nu\sigma}) = \Lambda g_{\mu\nu} + \frac{\Lambda}{3}h_{\mu\nu} + \\ &+2\frac{\Lambda}{3}h_{\mu\nu} + g^{\rho\sigma}R_{\mu\rho\nu\sigma} = \Lambda(g_{\mu\nu} + h_{\mu\nu}),\end{aligned} \qquad (42)$$

we obtain

$$R_{\mu\nu} = g^{\rho\sigma} R_{\mu\rho\nu\sigma} = 0. \quad (43)$$

For the curvature tensor $R_{\mu\rho\nu\sigma}$, responsible to gravitational wave, in the case of the de Sitter background space-time we can obtain now, instead of (1) in the case of the Minkowski-an background space-time, the following relation,

$$\begin{aligned} R_{\mu\rho\nu\sigma} = {} & \frac{1}{2}(h_{\mu\nu,\rho\sigma} + h_{\rho\sigma,\mu\nu} - h_{\mu\sigma,\rho\nu} - h_{\rho\nu,\mu\sigma}) - \\ & - h_{\lambda\eta}\left(\left\{{\lambda \atop \mu\nu}\right\}\left\{{\eta \atop \rho\sigma}\right\} - \left\{{\lambda \atop \mu\sigma}\right\}\left\{{\eta \atop \rho\nu}\right\}\right) + \\ & + \frac{1}{2}\left\{{\lambda \atop \mu\nu}\right\}\left[h_{\lambda\rho,\sigma} + h_{\lambda\sigma,\rho} - h_{\rho\sigma,\lambda}\right] + \\ & + \frac{1}{2}\left\{{\lambda \atop \rho\sigma}\right\}\left[h_{\lambda\mu,\nu} + h_{\lambda\nu,\mu} - h_{\mu\nu,\lambda}\right] - \\ & - \frac{1}{2}\left\{{\lambda \atop \mu\sigma}\right\}\left[h_{\lambda\rho,\nu} + h_{\lambda\nu,\rho} - h_{\rho\nu,\lambda}\right] - \\ & - \frac{1}{2}\left\{{\lambda \atop \rho\nu}\right\}\left[h_{\lambda\mu,\sigma} + h_{\lambda\sigma,\mu} - h_{\mu\sigma,\lambda}\right] - \\ & - \frac{\Lambda}{3}(h_{\mu\nu} g_{\rho\sigma} + g_{\mu\nu} h_{\rho\sigma} - h_{\mu\sigma} g_{\rho\nu} - g_{\mu\sigma} h_{\rho\nu}), \end{aligned} \quad (44)$$

where Christoffel symbol

$$\left\{{\eta \atop \mu\nu}\right\} = \frac{1}{2} g^{\lambda\eta}(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda}), \quad (45)$$

$\left\{{0 \atop ik}\right\} = \sqrt{\dfrac{\Lambda}{3}} a^2 \delta_{ik}$ and $\left\{{k \atop i0}\right\} = \left\{{k \atop 0i}\right\} = \sqrt{\dfrac{\Lambda}{3}}\, \delta_{ik}$, $i,k = 1,2,3$ are only nonzero values of $\left\{{\lambda \atop \mu\nu}\right\}$ for de Sitter space-time. Using the conditions (3) in de Sitter space-time

$$g^{\rho\sigma} h_{\mu\rho,\sigma} \equiv h_{\mu\sigma,\sigma} = 0, \quad g^{\rho\sigma} h_{\rho\sigma} \equiv h_{\sigma\sigma} = 0, \quad (46)$$

and taking into account that for TT-wave

$$h_{00} = h_{0i} = h_{i0} = 0, \quad (47)$$

finally we obtain from (43) and (44) the wave equation for gravitational TT-wave,

$$\left(\frac{\partial^2}{\partial t^2} - \sqrt{\frac{\Lambda}{3}}\frac{\partial}{\partial t} - \frac{1}{a(t)^2}\Delta - \frac{2}{3}\Lambda\right) h_{ik} = 0. \quad (48)$$

We can rewrite the equation (48) in the form

$$\left(\tilde{\Box} + \sqrt{\frac{\Lambda}{3}}\frac{\partial}{\partial t} - m_g^2\right) h_{ik} = 0, \quad (49)$$

where

$$\tilde{\Box} = \frac{1}{a(t)^2}\Delta - \frac{\partial^2}{\partial t^2} \quad (50)$$

is generalized d'Alembertian, and

$$m_g^2 = -\frac{2}{3}\Lambda \equiv -(\hbar / c)^2 \frac{2}{3}\Lambda. \quad (51)$$

We see, that "graviton mass" is purely imaginary number module $|m_g| = 1,70 \cdot 10^{-33}$ eV/c^2. This result was obtained in [12] but in [12] was lost important second term in the equation (49).

It is worthy of being noted that for Maxwell wave in the de Sitter space-time we can obtain the wave equation very similar to (48). The modified equation (10) in de Sitter space-time looks as follow

$$ia(t)\frac{\partial}{\partial t}\vec{\psi} = \pm \operatorname{rot}\vec{\psi}, \quad \operatorname{div}\vec{\psi} = 0. \quad (52)$$

Calculating

$$ia(t)\frac{\partial}{\partial t}\left(ia(t)\frac{\partial}{\partial t}\vec{\psi}\right) = \operatorname{rot}\operatorname{rot}\vec{\psi}, \quad (53)$$

we obtain

$$\left(\frac{\partial^2}{\partial t^2} + \sqrt{\frac{\Lambda}{3}}\frac{\partial}{\partial t} - \frac{1}{a(t)^2}\Delta\right)\vec{\psi} = 0. \quad (54)$$

The equation (54) is lacking in "mass" term and differs from (48) by the sign of second term, but we will not here go into problems on his physical meaning.

Summary

We are exploring a limiting Bargman-Wigner equations in opposite Pauli-Fierz ones usually used by investigation of massive graviton. To sum up, the massless limit of wave equa-tions saves all possible (five) polarization states. The TT graviton particle corresponds to Bron-stein ones.

It is obtained the generalization (48) of Bronstein equation in Minkowski space-time for gravitational TT-waves in de Sitter space-time. In the case of Λ=0 the equation (48) trans-forms to Bronstein equation (4). Let us compare the second term in the equation (48), that we have calculated, with the "mass" term $2\Lambda/3$. The frequency of last detected gravitational wave GW170817 was located between the frequency 24 Hz and a frequency of about few hundred Hz. Let us take for definiteness the frequency ν_g of gravitational wave as 100 Hz. The energy corresponds to this frequency is $h\nu_g = 2,07 \cdot 10^{-13}$ eV.

$$\frac{h\nu_g}{|m_g| c^2} = \frac{2,07 \cdot 10^{-13}}{1,70 \cdot 10^{-33}} \approx 10^{20}. \quad (55)$$

We see that the second term in the equations (48) by twenty orders of magnitude greater then the "mass" term. So this term is of significant importance in the case of real gravitational waves.

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