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Determination of probabilistic type intervals for constructing antagonistic game kernel defined on a hyperparallelepiped enclosed within the unit hypercube

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A method of constructing a definitional domain for the kernel of an antagonistic game is suggested. The domain is a hyperparallelepiped enclosed within the unit hypercube. The game is intended for reducing interval uncertainties, where a pure strategy component is between 0 and 1, but the sum of the components is always equal to 1. Such normalization allows to distribute capacities among “rooms” optimally, whose needs are uncertain and enclosed within intervals. In a special case, when the available capacity is not sufficient, an additional capacity that a room may need is found.

Key words: *interval uncertainty, antagonistic game, construction of kernel.*

Пропонується метод побудови області визначення ядра антагоністичної гри. Цією областю є гіперпаралелепіпед усередині одиничного гіперкуба. Гра призначена для усунення інтервальних невизначеностей, де компонента чистої стратегії знаходиться між 0 та 1, але сума усіх компонент завжди дорівнює 1. Така нормалізація дозволяє оптимально розподіляти потужності між “відділеннями”, чиї потреби невизначені та вкладені в інтервали. У спеціальному випадку, коли наявної потужності недостатньо, визначається додаткова потужність, яку може потребувати відділення.

Ключові слова: *інтервальна невизначеність, антагоністична гра, побудова ядра.*

Предлагается метод построения области определения ядра антагонистической игры. Этой областью является гиперпараллелепипед внутри единичного гиперкуба. Игра предназначена для снижения интервальных неопределённостей, где компонента чистой стратегии находится между 0 и 1, а сумма всех компонент равна 1. Такая нормализация позволяет оптимально распределять мощности между “отделениями”, чьи потребности неопределенны и вложены в интервалы. В специальном случае, когда имеющейся мощности недостаточно, определяется дополнительная мощность, в которой может нуждаться отделение.

Ключевые слова: *интервальная неопределённость, антагонистическая игра, построение ядра.*

Construction of game kernels

The kernel of a game is a function, which is the core of modeling interaction among players. Construction of a game kernel is the crucial point in building the game model. Values of the kernel reflect consequences of applying the strategies chosen by players. Those values may be very susceptible to changes in strategies [1]. Therefore, the kernel must be constructed carefully. It is done quite well for finite games, where every player has a finite number of strategies [2]. Such games have a finite number of situations, which can be evaluated easily. Infinite games, more complicated ones, have infinite number of situations, so it is impossible to survey all the situations without an algorithmic approach. However, such an approach is not always available [3, 4].

Background of building game models

Sets of the players’ pure strategies are defined at the start of building the game model. Strategies that are equal/equivalent in the Helly metric are defined as a single strategy (see, e. g., [2]). The game kernel is defined on the Cartesian product of the strategy sets [2]. Then values of the kernel are obtained either analytically or empirically. In addition, a lot of games model distribution of resources/capacity [1, 3, 5, 6]. So it is convenient to normalize all data, having components of a pure strategy between 0 and 1, where the sum of the components is always equal to 1. In this way, each component belongs to an interval whose values resemble probabilities [2, 4, 7]. But when demands for resources are fluent, normalization of their grand total to 1 calls for a tricky approach.

The goal of the article and the tasks to be accomplished

Given the initial data, the goal is to determine probabilistic type intervals. They will constitute a hyperparallelepiped, on which the game kernel is defined for modeling distribution of a capacity. This hyperparallelepiped will be enclosed within the unit hypercube of the nonnegative orthant in Euclidean space of the corresponding dimensionality. For reaching the goal, the following tasks are to be accomplished:

1. To describe and set conventions of the initial data (related to the capacity units).
2. To state a method of mapping them into probabilistic type intervals.
3. Based on the game model, to give formulae for optimally distributing a capacity.

Eventually, a few examples are to be given. They shall illustrate how to use the mapping method in various cases for distributing a capacity.

Mapping the initial data into probabilistic type intervals

Let V be a capacity of some object (energy, water, gas, funds, heat, oil, etc.). This capacity is to be distributed among N “rooms” or to be divided into N parts, where $N \in \mathbb{N}$. The i-th room needs its part of the capacity within a segment

$$\left[V_i^{<\min>};\, V_i^{<\max>}\right]$$

by

$$V_i^{<\max>} > V_i^{<\min>} > 0\,.$$

So its capacity to be delivered is

$$V_i \in \left[V_i^{<\min>};\, V_i^{<\max>}\right],\ \ i = \overline{1,\, N}\,.$$

Also the i-th room has its nominally required capacity r_i (see, e. g., [8]).

As the capacities $\{V_i\}_{i=1}^{N}$ are fluent, there are three cases of treating them to V. If

$$V \leqslant \sum_{i=1}^{N} V_i^{<\min>} \tag{1}$$

then it is divided among N rooms proportionally to their nominal demands $\{r_i\}_{i=1}^{N}$:

$$V_i^{*} = r_i V_i^{<\min>} \Big/ \sum_{k=1}^{N} V_k^{<\min>} \quad \forall\ i = \overline{1,\, N}\,. \tag{2}$$

Otherwise, there are another two cases, which are not as naive as (1) and solution (2):

$$\sum_{i=1}^{N} V_i^{<\min>} < V < \sum_{i=1}^{N} V_i^{<\max>} , \tag{3}$$

$$\sum_{i=1}^{N} V_i^{<\max>} \leqslant V . \tag{4}$$

The case of inequality (4) is easier. Instead of the initial segments with endpoints $V_i^{<\min>}$ and $V_i^{<\max>}$, we get them as

$$\left[V_i^{<\min>} \big/ V ;\, V_i^{<\max>} \big/ V \right] = [a_i ;\, b_i] \subset (0;\, 1) \;\text{ by }\; i = \overline{1,\, N} . \tag{5}$$

Denote a demand of the i-th room by x_i and its supply by y_i. Here $\mathbf{X} = (x_i)_{1\times N}$, $\mathbf{Y} = (y_i)_{1\times N}$ by $x_i \in [a_i ;\, b_i]$ and $y_i \in [a_i ;\, b_i]$. Let $\eta(z)$ be a function that maps a positive z into a positive. Then the kernel of a game model for the distribution is

$$K(\mathbf{X},\, \mathbf{Y}) = \max \left\{ \left\{ \eta(x_k) / \eta(y_k) \right\}_{k=1}^{N} ,\, \eta\left(1 - \sum_{n=1}^{N} x_n \right) \Bigg/ \eta\left(1 - \sum_{n=1}^{N} y_n \right) \right\} . \tag{6}$$

This kernel (6) is defined on a hyperparallelepiped

$$\left(\bigtimes_{k=1}^{N} [a_k ;\, b_k] \right) \times \left(\bigtimes_{n=1}^{N} [a_n ;\, b_n] \right) \subset \bigtimes_{j=1}^{2N} (0;\, 1) \subset \bigtimes_{j=1}^{2N} [0;\, 1] \subset \mathbb{R}^{2N} . \tag{7}$$

It is solved easily for the function $\eta(z) = z$, where the game becomes strictly convex [2, 7]. The solution $\mathbf{Y}^* = \left(y_i^* \right)_{1\times N}$ is a pure strategy [2] of the second player (distributor). Then [7]

$$y_i^* = b_i \Bigg/ \left(1 + \sum_{m=1}^{N} b_m - \sum_{m=1}^{N} a_m \right) \tag{8}$$

by

$$b_i \Bigg/ \left(1 + \sum_{m=1}^{N} b_m - \sum_{m=1}^{N} a_m \right) \geqslant a_i \quad \forall\, i = \overline{1,\, N} . \tag{9}$$

If a condition in (9) is violated, the solution has special forms [7]. Anyway, here

$$V_i^* = y_i^* V \;\text{ and }\; V_{\text{off}} = \left(1 - \sum_{i=1}^{N} y_i^* \right) V \tag{10}$$

are the optimal capacity delivered to the i-th room and capacity that is drawn off (or ignored/rejected), respectively. Particularly, when $N = 1$ goes into (6), we simply have the kernel [2]

$$K(x, y) = \max\{x/y, (1-x)/(1-y)\} \text{ by } x = x_1, \quad y = y_1, \tag{11}$$

which is an example of the simplest model for reducing interval uncertainty [2, 6, 7]. The optimal strategy of the distributor in the game with kernel (11) is

$$y^* = b/(1+b-a).$$

For solving the case of inequality (3), we take a fictional capacity V_+ and intervals:

$$V_+ = \sum_{i=1}^{N} V_i^{<\max>}, \quad \left[V_i^{<\min>}/V_+ ; V_i^{<\max>}/V_+\right] = [a_i; b_i] \subset (0; 1) \text{ by } i = \overline{1, N}. \tag{12}$$

When y_i^* is found in the game with kernel (6) on (7), then we check whether

$$y_i^* \leqslant \left(V_i^{<\max>} V\right)/V_+^2 . \tag{13}$$

If (13) is true, then the optimal capacity delivered to the i-th room is $V_i^* = y_i^* V_+$. If (13) is false, then

$$y_i^{**} = \left(V_i^{<\max>} V\right)/V_+^2, \quad V_i^* = y_i^{**} V_+ = \left(V_i^{<\max>} V\right)/V_+, \quad V_i^{<\mathrm{add}>} = \left(y_i^* - y_i^{**}\right)V_+, \tag{14}$$

where $V_i^{<\mathrm{add}>}$ is an additional capacity that this room needs.

Examples of using the mapping method in various cases

Considering a few examples, we should not forget that the solution depends on kernel (6). Firstly, let $V_1 \in [20; 30]$ for a single room. Then, using normalization (5) and solution (8) by (9),

$$[a_1; b_1] = [20/V; 30/V], \quad y_1^* = 30/(V+10),$$
$$V_{\mathrm{off}} = (V-20)V/(V+10) \text{ for any } V \geqslant 30 .$$

If $20 < V < 30$, then we use (12), (13), (14):

$$y_1^* = 30/(V+10) \leqslant (30V)/30^2 = V/30$$

by

$$V^2 + 10V - 900 \geqslant 0,$$

whence these inequalities are true by $V \geqslant 5\sqrt{37} - 5$. This implies that

$$y_1^* = 30/(V+10) \text{ by } V \in \left[5\sqrt{37} - 5; 30\right]$$

and

$$y_1^{**} = V/30 \text{ by } V \in \left(20; 5\sqrt{37} - 5\right].$$

Suppose that $V_1 \in [20; 30]$ and $V_2 \in [25; 40]$ for $N = 2$, $V \geqslant 70$. We have hyperparallelepiped (7) as

$$[a_1;\, b_1]\times[a_2;\, b_2]=[20/V;\, 30/V]\times[25/V;\, 40/V].$$

Here we get just strategies

$$y_1^*=30/(V+25)$$

and

$$y_2^*=40/(V+25),$$

inasmuch as

$$y_1^*=30/(V+25)\geqslant 20/V$$

is followed with $V\geqslant 50$, and

$$y_2^*=40/(V+25)\geqslant 25/V$$

is followed with $V\geqslant 125/3$. If $45<V<70$, then the fictional capacity $V_+=70$, hyperparallelepiped (7) is

$$[a_1;\, b_1]\times[a_2;\, b_2]=[2/7;\, 3/7]\times[5/14;\, 4/7],$$

whereupon points $y_1^*=6/19$ and $y_2^*=8/19$ are checked whether (13) is true:

$$6/19\leqslant(30V)/70^2=3V/490$$

is followed with $V\geqslant 980/19$, and $8/19\leqslant 4V/490$ is followed with $V\geqslant 980/19$. Hence,

$$y_1^*=6/19,\quad y_2^*=8/19\ \text{ by }\ V\in[980/19;\, 70)$$

and

$$y_1^{**}=3V/490,\quad y_2^{**}=4V/490\ \text{ by }\ V\in(45;\, 980/19],$$

where

$$V_1^{<\mathrm{add}>}\approx 22.11-0.43V,\quad V_2^{<\mathrm{add}>}\approx 29.47-0.57V$$

by (14).

Surely, it is much easier when interval data are identical. Let $V_i\in[30;\, 40]$ for $i=\overline{1,\, 3}$. The case $V\geqslant 120$ is trivial:

$$y_i^*=40/(V+30)\quad \forall\ i=\overline{1,\, N}.$$

If $90<V<120$, then $y_i^*=4/15$ and $4/15\leqslant V/360$ is followed with $V\geqslant 96$, whence

$$y_i^*=4/15\ \text{ by }\ V\in[96;\, 120)$$

and

$$y_i^{**}=V/360\ \text{ by }\ V\in(90;\, 96].$$

An additional capacity that these rooms need is

$$V_i^{<\text{add}>} = 32 - V/3, \quad i = \overline{1, 3}.$$

The considered examples do not cover all the peculiarities in using the method of probabilistic type intervals, but they give a simple illustration of how to solve cases of inequalities (3) and (4). Cases with a larger amount of intervals are solved similarly.

Conclusion

The suggested approach allows determining probabilistic type intervals that constitute hyperparallelepiped (7). This is done either with (5) by (4) or with (12) by (3). Once the game with kernel (6) on hyperparallelepiped (7) is given, the second player's optimal strategy in this game's solution can be used for distributing a capacity among "rooms". This is executed either with (10) or with (14) by when inequality (13) fails. Moreover, in the case of inequality (4), superfluous capacity is ignored. If (3) is true and (13) is false, we know what capacity is needed for the i-th room. The easiest case (1), standing apart from the game model solutions, is nonetheless very important giving us capacities (2). However, if nominal demands are unknown, the capacity should be divided into equal parts. Thus, further study can be focused on violations of inequalities in (9). Such violations generate "cyclic" solutions given in a few steps [7].

REFERENCES

1. Ning Z. A message distribution mechanism based on distributed game theory in LTE-V network / Z. Ning, P. Wang, N. N. Van // Procedia Computer Science. – 2017. – Vol. 107. – P. 590 – 598.
2. Воробьёв Н. Н. Основы теории игр. Бескоалиционные игры / Воробьёв Н. Н. – М. : Наука, Главная редакция физико-математической литературы, 1984. – 496 с. – Title in English : Vorob'yov, N. N. (1984) Game theory fundamentals. Noncooperative games. Moscow: Nauka, 496.
3. Lozano S. Information sharing in DEA: A cooperative game theory approach / S. Lozano // European Journal of Operational Research. – 2012. – Vol. 222, Iss. 3. – P. 558 – 565.
4. Information, Inference and Decision / Ed. by G. Menges. – Dordrecht : D. Reidel Publishing Company, 1974. – 201 p.
5. Madani K. Game theory and water resources / K. Madani // Journal of Hydrology. – 2010. – Vol. 381, Iss. 3 – 4. – P. 225 – 238.
6. Azzedin F. Modeling BitTorrent choking algorithm using game theory / F. Azzedin, M. Yahaya // Future Generation Computer Systems. – 2016. – Vol. 55. – P. 255 – 265.
7. Romanuke V. V. Model of removing partial indeterminancies of the probabilistic type as maximal disbalance minimization / V. V. Romanuke // Scientific bulletin of CNU. Computer systems and components. – 2011. – Vol. 2, Iss. 1. – P. 32 – 40.
8. Feng G. Characterization of pore volume of cumulative water injection distribution / G. Feng, M. Yu // Petroleum. – 2015. – Vol. 1, Iss. 2. – P. 158 – 163.

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