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The Dynamics of Processes of Resonant Scattering and Generation of Waves by a Three-Layer Dielectric with a Nonlinear Medium

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A mathematical model and results of numerical computations of the dynamics of processes of resonant scattering and generation of waves by a three-layer dielectric containing absolutely transparent decanalizing and canalizing nonlinear media are presented. The approach is based on a self-consistent solution of systems of nonlinear boundary value problems near the eigen-frequencies of the corresponding linearized spectral problems. The analysis of the relative Q-factors of the oscillations has shown the possibility of an indirect description of the processes of energy exchange.

Key words: *cubically polarizable medium, resonance scattering, wave generation, self-consistent analysis, spectral problems, relative Q-factor.*

Запропонована математична модель і наведені результати чисельного аналізу динаміки процесів резонансного розсіяння та генерації хвиль трьохшаровими діелектриками, що містять абсолютно прозору деканалізуючу та каналізуючу нелінійну середу. В основі розвиненого підходу лежить самоузгоджене рішення систем нелінійних граничних задач поблизу власних частот відповідних лінеаризованих спектральних задач. Аналіз величин відносних добротностей коливань показав можливість непрямого опису процесів енергетичного обміну.

Ключові слова: *кубічне поляризуєма середа, резонансне розсіяння, генерація хвиль, взаємоузгоджений аналіз, спектральні задачі, відносна добротність.*

Предложена математическая модель и приведены результаты численного анализа динамики процессов резонансного рассеяния и генерации волн трехслойными диэлектриками, содержащими абсолютно прозрачную деканализирующую и канализирующую нелинейную среду. В основе развитого подхода лежит самосогласованное решение систем нелинейных граничных задач вблизи собственных частот соответствующих линеаризованных спектральных задач. Анализ величин относительных добротностей колебаний показал возможность косвенного описания процессов энергетического обмена.

Ключевые слова: *кубически поляризуемая среда, резонансное рассеяние, генерация волн, самосогласованный анализ, спектральные задачи, относительная добротность.*

1. Introduction

Nonlinear dielectrics with controllable permittivity have been intensively studied over the recent decades and have now found wide applications in optics and electronics [1-6]. In most of the published papers, the scattering properties of nonlinear media and objects are analyzed. These include studies that investigate resonant scattering properties on nonlinear layered media with Kerr nonlinearity. In the resonance region, the analysis of such problems reduces to the solution of nonlinear boundary value problems. The next stage of research is connected with the investigation of the resonant properties of scattering and generation of waves by nonlinear layered structures [7, 8]. This causes the necessity to solve systems of nonlinear boundary value problems in a self-consistent way.

This paper presents a mathematical model as well as computational results of the dynamics of resonant scattering and wave generation for the wave packet excitation of a layered dielectric object which is formed from nonlinear cubically polarizable absolutely transparent (in the linear approximation) decanalizing and canalizing media. Calculations near the resonance frequencies of scattering and generation have shown that the minimum of the function, which characterizes the ratio of the quality factors of the eigen-oscillations of the induced nonlinear structure at the frequencies of excitation and generation, respectively, corresponds to the maximum value of the generated energy.

2. Statement of the problem

The problem of resonant scattering and generation of waves by a nonlinear, nonmagnetic, isotropic, E-polarized $\mathbf{E}=(E_x,0,0)^\mathsf{T}$, $\mathbf{H}=(0,H_y,H_z)^\mathsf{T}$, cubically polarizable $\mathbf{P}^{(\mathrm{NL})}=(P_x^{(\mathrm{NL})},0,0)^\mathsf{T}$, layered dielectric structure (see Fig. 1) is investigated in a self-consistent formulation [3]. The time dependency has the form $\exp(-in\omega t)$, $n\in N$. Here the variables x,y,z,t denote dimensionless spatial-temporal coordinates such that the thickness of the layer is equal to $4\pi\delta$, with $\delta>0$; $n\omega=n\kappa c$ are the circular frequencies, $n\kappa=2\pi/\lambda_{n\kappa}$ are frequency parameters; $\lambda_{n\kappa}$ are the lengths of the incident waves: $c=(\varepsilon_0\mu_0)^{-1/2}$, $\operatorname{Im}c=0$, ε_0 and μ_0 are the free space permittivity and permeability, respectively.

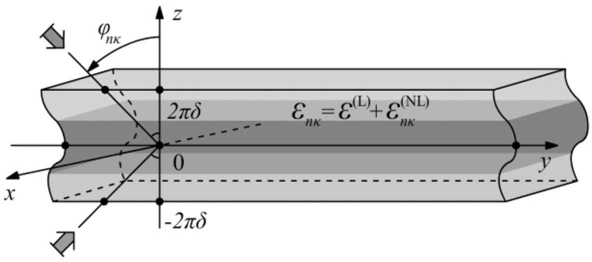

Fig. 1. The nonlinear layered dielectric structure.

The incidence of a packet of plane waves

$$\left\{\overline{E}_1^{\mathrm{inc}}(n\kappa;y,z)\right\}_{n=1}^{3}\cup\left\{\underline{E}_1^{\mathrm{inc}}(n\kappa;y,z)\right\}_{n=1}^{3},$$

with

$$\left\{\begin{Bmatrix}\overline{E}_1^{\mathrm{inc}}(n\kappa;y,z)\\ \underline{E}_1^{\mathrm{inc}}(n\kappa;y,z)\end{Bmatrix}=\begin{Bmatrix}a_{n\kappa}^{\mathrm{inc}}\\ b_{n\kappa}^{\mathrm{inc}}\end{Bmatrix}\exp\left[i\left(\Phi_{n\kappa}y\mp\Gamma_{n\kappa}(z\mp2\pi\delta)\right)\right],\quad z\,\substack{>\\<}\,\pm2\pi\delta\right\}_{n=1}^{3},$$

onto the layered structure at the angles $\{\varphi_{n\kappa},\pi-\varphi_{n\kappa}:|\varphi_{n\kappa}|<\pi/2\}_{n=1}^{3}$ and with respect to the amplitudes $\{a_{n\kappa}^{\mathrm{inc}},b_{n\kappa}^{\mathrm{inc}}\}_{n=1}^{3}$ at the frequencies $\{n\kappa\}_{n=1}^{3}$ is considered.

Here the excitation field consists of a strong field at the frequency κ (generating a field at the triple frequency) and of weak fields at the frequencies 2κ, 3κ (having an impact on the process of third harmonic generation due to the contribution of the weak electromagnetic fields).

In such a situation the problem under consideration can be described by a system of one-dimensional nonlinear integral equations w.r.t. the unknown functions $U(n\kappa;\cdot)\in L_2(-2\pi\delta, 2\pi\delta)$, see [8],

$$
\begin{aligned}
&U(n\kappa;z)+\frac{i(n\kappa)^2}{2\Gamma_{n\kappa}}\int_{-2\pi\delta}^{2\pi\delta}\exp(i\Gamma_{n\kappa}\,|z-\xi\,|)\\
&\times\left[1-\varepsilon_{n\kappa}\left(\xi,\alpha(\xi),\{U(m\kappa;\xi)\}_{m=1}^{3}\right)\right]U(n\kappa;\xi)\,d\xi\\
&=\frac{i(n\kappa)^2}{2\Gamma_{n\kappa}}\int_{-2\pi\delta}^{2\pi\delta}\exp(i\Gamma_{n\kappa}|z-\xi|)\,\alpha(\xi)\Big[\delta_n^1U^2(2\kappa;\xi)U^*(3\kappa;\xi)\\
&+\delta_n^3\left\{\frac{1}{3}U^3(\kappa;\xi)+U^2(2\kappa;\xi)U^*(\kappa;\xi)\right\}\Big]d\xi\\
&+\overline{U}^{\,\mathrm{inc}}(n\kappa;z)+\underline{U}^{\,\mathrm{inc}}(n\kappa;z),\quad n=1,2,3,
\end{aligned}
\tag{1}
$$

where

$$
\begin{aligned}
&\overline{U}^{\,\mathrm{inc}}(n\kappa;z)=a_{n\kappa}^{\mathrm{inc}}\exp\left[-i\Gamma_{n\kappa}(z-2\pi\delta)\right],\\
&\underline{U}^{\,\mathrm{inc}}(n\kappa;z)=b_{n\kappa}^{\mathrm{inc}}\exp\left[i\Gamma_{n\kappa}(z+2\pi\delta)\right],
\end{aligned}
$$

δ_n^k – Kronecker's symbol; * – complex conjugation; $\Gamma_{n\kappa}=\sqrt{(n\kappa)^2-\Phi_{n\kappa}{}^2}$ and $\Phi_{n\kappa}=n\kappa\sin(\varphi_{n\kappa})$ – the transverse and longitudinal propagation constants of the nonlinear structure,

$$
\begin{aligned}
&\varepsilon_{n\kappa}=\left\{1,|z|>2\pi\delta;\ \text{and}\ \varepsilon^{(\mathrm{L})}+\varepsilon_{n\kappa}^{(\mathrm{NL})},\ |z|\le 2\pi\delta\right\},\\
&\varepsilon^{(\mathrm{L})}=1+4\pi\chi_{11}^{(1)}(z),\\
&\varepsilon_{n\kappa}^{(\mathrm{NL})}=\alpha(z)\left[\sum_{m=1}^{3}\left|E_1(m\kappa;y,z)\right|^2\right.\\
&\left.+\left\{\delta_n^1\frac{\left[E_1^*(\kappa;y,z)\right]^2}{E_1(\kappa;y,z)}+\delta_n^2\frac{E_1^*(2\kappa;y,z)}{E_1(2\kappa;y,z)}E_1(\kappa;y,z)\right\}E_1(3\kappa;y,z)\right],
\end{aligned}
\tag{2}
$$

$\alpha(z)=3\pi\chi_{xxxx}^{(3)}(z)$ – the function of cubic susceptibility of the nonlinear medium, $\chi_{xx}^{(1)}$ and $\chi_{xxxx}^{(3)}$ – components of the susceptibility tensors of the nonlinear medium.

The solution of the problem is represented as

$$E_x(n\kappa;y,z)=U(n\kappa;z)\exp(i\Phi_{n\kappa}y)$$
$$=\begin{cases} a_{n\kappa}^{\text{inc}}\exp\left\{i\left[\Phi_{n\kappa}y-\Gamma_{n\kappa}(z-2\pi\delta)\right]\right\} \\ \qquad +a_{n\kappa}^{\text{scat/gen}}\exp\left\{i\left[\Phi_{n\kappa}y+\Gamma_{n\kappa}(z-2\pi\delta)\right]\right\}, & z>2\pi\delta, \\ U(n\kappa;z)\exp(i\Phi_{n\kappa}y), & |z|\le 2\pi\delta, \\ b_{n\kappa}^{\text{inc}}\exp\left\{i\left[\Phi_{n\kappa}y+\Gamma_{n\kappa}(z+2\pi\delta)\right]\right\} \\ \qquad +b_{n\kappa}^{\text{scat/gen}}\exp\left\{i\left[\Phi_{n\kappa}y-\Gamma_{n\kappa}(z+2\pi\delta)\right]\right\}, & z<-2\pi\delta, \end{cases} \tag{3}$$

and can be obtained from (1) using the formulas

$$U(n\kappa;2\pi\delta)=a_{n\kappa}^{\text{inc}}+a_{n\kappa}^{\text{scat/gen}},\ U(n\kappa;-2\pi\delta)=b_{n\kappa}^{\text{inc}}+b_{n\kappa}^{\text{scat/gen}},\ n=1,2,3.$$

3. Self-consistent analysis and spectral problems

The application of suitable quadrature rules to the system of nonlinear integral equations (1) leads to a system of complex-valued nonlinear algebraic equations of the second kind [8]

$$\begin{aligned}&\left[\mathbf{I}-\mathbf{B}_{n\kappa}(\mathbf{U}_{\kappa},\mathbf{U}_{2\kappa},\mathbf{U}_{3\kappa})\right]\mathbf{U}_{n\kappa}\\ &=\delta_n^1\mathbf{C}_{\kappa}(\mathbf{U}_{2\kappa},\mathbf{U}_{3\kappa})+\delta_n^3\mathbf{C}_{3\kappa}(\mathbf{U}_{\kappa},\mathbf{U}_{2\kappa})+\overline{\mathbf{U}}_{n\kappa}^{\text{inc}}+\underline{\mathbf{U}}_{n\kappa}^{\text{inc}},\quad n=1,2,3,\end{aligned} \tag{4}$$

where $\mathbf{U}_{n\kappa}=\{U(n\kappa;z_l)\}_{l=1}^{N}$ – the vectors of the unknown approximate values of the solution, $\{\{z_l\}_{l=1}^{N}: z_1=-2\pi\delta<...<z_l<...<z_N=2\pi\delta\}$ – a discrete set of interpolation nodes, $\mathbf{I}=\{\delta_l^m\}_{l,m=1}^{N}$ – the identity matrix, $\mathbf{B}_{n\kappa}(\mathbf{U}_{\kappa},\mathbf{U}_{2\kappa},\mathbf{U}_{3\kappa})$ – nonlinear matrices, $\mathbf{C}_{\kappa}(\mathbf{U}_{2\kappa},\mathbf{U}_{3\kappa})$, $\mathbf{C}_{3\kappa}(\mathbf{U}_{\kappa},\mathbf{U}_{2\kappa})$ – the vectors of the right-hand sides determined by the choice of the quadrature rule, $\overline{\mathbf{U}}_{n\kappa}^{\text{inc}}=\left\{a_{n\kappa}^{\text{inc}}\exp\left[-i\Gamma_{n\kappa}(z_l-2\pi\delta)\right]\right\}_{l=1}^{N}$ and $\underline{\mathbf{U}}_{n\kappa}^{\text{inc}}=\left\{b_{n\kappa}^{\text{inc}}\exp\left[i\Gamma_{n\kappa}(z_l+2\pi\delta)\right]\right\}_{l=1}^{N}$ – the vectors induced by the incident wave packets at the multiple frequencies $n\kappa$, $n=1,2,3$.

A self-consistent solution of (4) can be found numerically by the help of a block-iterative method, where at each iteration step in each block of the system (4) a system of linearized algebraic equations is solved [8].

The analytic continuation of the linearized problems into the region of complex values of the frequency parameter allows us to switch to the analysis of spectral problems [8, 9]. The determination of the eigen-frequencies κ_n and eigen-fields $\mathbf{U}_{\kappa_n}$ reduces to the solution of the following equations:

$$\begin{cases} f_{n\kappa}(\kappa_n)=\det\left[\mathbf{I}-\mathbf{B}_{n\kappa}(\kappa_n)\right]=0, \\ \left[\mathbf{I}-\mathbf{B}_{n\kappa}(\kappa_n)\right]\mathbf{U}_{\kappa_n}=\mathbf{0}, \end{cases} \tag{5}$$

where $\kappa_n \in \Omega_{n\kappa} \subset H_{n\kappa}$, $\kappa = \text{const} \in [0, +\infty)$, $n = 1, 2, 3$, $\Omega_{n\kappa}$ are the sets of eigen-frequencies and $H_{n\kappa}$ denote two-sheeted Riemann surfaces [8], $\mathbf{U}_{\kappa_n} = \{U(\kappa_n; z_l)\}_{l=1}^{N}$ - the vector of unknown values of the nontrivial solution at the nodes in the layer corresponding to the eigen-frequency κ_n, $\mathbf{B}_{n\kappa}(\kappa_n) = \mathbf{B}_{n\kappa}(\kappa_n; \mathbf{U}_\kappa, \mathbf{U}_{2\kappa}, \mathbf{U}_{3\kappa})$ - the matrix with the given vectors $\mathbf{U}_{n\kappa}$ (cf. (4)).

We mention that the radiation condition to the eigen-fields

$$E_1(\kappa_n; y, z) = \begin{Bmatrix} a_{\kappa_n} \\ b_{\kappa_n} \end{Bmatrix} \exp\left[i\left(\Phi_{n\kappa} y \pm \Gamma_{\kappa_n}(\kappa_n, \Phi_{n\kappa})(z \mp 2\pi\delta)\right)\right], \quad z \begin{matrix} > \\ < \end{matrix} \pm 2\pi\delta, \quad n = 1, 2, 3,$$

for real values of the parameters κ_n and $\Phi_{n\kappa}$ is consistent with the physically justified requirement of the absence of waves coming from infinity $z = \pm\infty$ in the radiation fields:

$$\operatorname{Im}\Gamma_{\kappa_n}(\kappa_n, \Phi_{n\kappa}) \geq 0, \ \operatorname{Re}\Gamma_{\kappa_n}(\kappa_n, \Phi_{n\kappa}) \operatorname{Re}\kappa_n \geq 0,$$

for $\operatorname{Im}\Phi_{n\kappa} = 0$, $\operatorname{Im}\kappa_n = 0$, $n = 1, 2, 3$.

The nontrivial solutions of the spectral problem (5) allow us to write the electric components of the eigen-fields as follows:

$$E_1(\kappa_n; y, z) = U(\kappa_n; z) \exp(i\Phi_{n\kappa} y) =$$

$$= \begin{cases} a_{\kappa_n} \exp\left[i\left(\Phi_{n\kappa} y + \Gamma_{\kappa_n}(\kappa_n, \Phi_{n\kappa})(z - 2\pi\delta)\right)\right], & z > 2\pi\delta, \\ U(\kappa_n; z) \exp(i\Phi_{n\kappa} y), & |z| \leq 2\pi\delta, \\ b_{\kappa_n} \exp\left[i\left(\Phi_{n\kappa} y - \Gamma_{\kappa_n}(\kappa_n, \Phi_{n\kappa})(z + 2\pi\delta)\right)\right], & z < -2\pi\delta. \end{cases} \tag{6}$$

Here: $\kappa_n \in \Omega_{n\kappa} \subset H_{n\kappa}$, $n = 1, 2, 3$, $\kappa \equiv \kappa^{\text{inc}}$ - a given constant value equal to the excitation frequency of the nonlinear structure, $a_{\kappa_n} = U(\kappa_n; 2\pi\delta)$ и $b_{\kappa_n} = U(\kappa_n; -2\pi\delta)$ - the radiation coefficients of the eigen-field, $\Gamma_{\kappa_n}(\kappa_n, \Phi_{n\kappa}) = \sqrt{\kappa_n^2 - \Phi_{n\kappa}^2}$ - the functions of the transverse propagation (depending on the complex spectral frequency parameters κ_n), $\Phi_{n\kappa} = n\kappa \sin(\varphi_{n\kappa})$ - the given real values of the longitudinal propagation constants.

The eigen-frequencies $\kappa_n \in \Omega_{n\kappa} \subset H_{n\kappa}$, $n = 1, 2, 3$, i.e. the characteristic numbers of the dispersion equations of problem (5), are obtained by solving the corresponding dispersion equations $f_{n\kappa}(\kappa_n) = \det\left[\mathbf{I} - \mathbf{B}_{n\kappa}(\kappa_n)\right] = 0$, $n = 1, 2, 3$, using Newton's iterative method or a modification of it. The nontrivial solutions $\mathbf{U}_{\kappa_n}$ of the homogeneous systems $\left[\mathbf{I} - \mathbf{B}_{n\kappa}(\kappa_n)\right]\mathbf{U}_{\kappa_n} = \mathbf{0}$, $n = 1, 2, 3$, of linear algebraic equations (5) corresponding to these characteristic numbers are the eigen-fields (6) of the linearized nonlinear layered structures with an induced dielectric permittivity (2). The sought solutions $\mathbf{U}_{\kappa_n}$ are unique except for an arbitrary multiplicative constant.

Since the irregular dynamics of the resonance processes of scattering and generation of waves is observed near the eigen-frequencies of the object under study, such excitation frequencies $\kappa \equiv \kappa^{\text{inc}} = \text{const} \in [0, +\infty)$ are of interest, which are determined, for example, by one of the following quantities:

$$\begin{aligned} &\kappa = \operatorname{Re}\kappa_n / n \quad \text{for} \quad \forall n \in \{1, 2, 3\}; \\ &\kappa = \operatorname{Re}(\kappa_1 + \kappa_2 + \kappa_3)/(1+2+3); \quad \kappa = \operatorname{Re}(\kappa_1 + \kappa_3)/(1+3). \end{aligned} \tag{7}$$

One of the requirements (7) can be satisfied by an iterative approach such that, at each step, iterative processes of the successive solution of problems (4) и (5) are carried out. As an initial approximation to the solution of the noted nonlinear problem (4), (5), (7), we can take the solution of the corresponding linear problem with the coefficient of nonlinear susceptibility $\alpha(z) \equiv 0$.

4. Three-layer dielectric objects with nonlinear media

In what follows we present results of numerical studies that demonstrate the resonance effect of the transfer of the oscillation energy at the scattering frequency to the oscillation energy at the generation frequency for three-layer objects containing nonlinear, *absolutely transparent* (in the linear approximation $\varepsilon^{(\text{L})} = 1$) layers, which decanalize ($\alpha < 0$) and canalize ($\alpha < 0$) the excitation field. The dynamics of the scattered/generated fields and of the nonlinear dielectric permittivities at the resonance frequencies of scattering/generation are given, the resonance frequencies being close to the eigen-frequencies of the considered nonlinear structures induced by the incident field.

We consider nonlinear layered objects, see (2) and Fig. 1, with the parameters:

$$\left\{\varepsilon^{(\text{L})}(z), \alpha(z)\right\} = \begin{cases} \{\varepsilon^{(\text{L})} = 1.5, \alpha = 0\}, & [z \in -2\pi\delta, -2\pi\delta/3) \cup (2\pi\delta/3, 2\pi\delta]; \\ \{\varepsilon^{(\text{L})} = 1, \alpha = \mp 0.01\}, & z \in [-2\pi\delta/3, 2\pi\delta/3]; \end{cases} \tag{8}$$

at $\delta = 0.5$ and $\varphi_{n\kappa} = 0^0$, $n = 1, 2, 3$. The excitation takes place from above by an electromagnetic field with $a_\kappa^{\text{inc}} = \text{const}$, $\{a_{n\kappa}^{\text{inc}} = 0\}_{n=2}^3$, $\{b_{n\kappa}^{\text{inc}} = 0\}_{n=1}^3$ at the basic frequency $\kappa \equiv \kappa^{\text{inc}} = \operatorname{Re}\kappa_1$.

In order to describe the scattering and generation properties of the nonlinear structure, we introduce the following notation:

$$R_{n\kappa}^{+} = \left|a_{n\kappa}^{\text{scat/gen}}\right|^2 \Big/ \sum_{s=1}^{3}\left(\left|a_{s\kappa}^{\text{inc}}\right|^2 + \left|b_{s\kappa}^{\text{inc}}\right|^2\right), \quad z > 2\pi\delta,$$

$$R_{n\kappa}^{-} = \left|b_{n\kappa}^{\text{scat/gen}}\right|^2 \Big/ \sum_{s=1}^{3}\left(\left|a_{s\kappa}^{\text{inc}}\right|^2 + \left|b_{s\kappa}^{\text{inc}}\right|^2\right), \quad z < -2\pi\delta, \qquad n = 1, 2, 3.$$

The quantities $R_{n\kappa}^{+}$, $R_{n\kappa}^{-}$ are called scattering/generation (or radiation) coefficients of the waves w.r.t. the total intensity of the incident packet.

Denote by

$$U_{n\kappa} = U(n\kappa; z)$$

the complex Fourier amplitudes of the total scattered/generated fields at the frequencies $n\kappa$, cf. (3).

We also define the total energy of the scattered and generated fields at the frequencies $n\kappa$ by

$$W_{n\kappa} = \left|a_{n\kappa}^{\text{scat/gen}}\right|^2 + \left|b_{n\kappa}^{\text{scat/gen}}\right|^2, \quad n = 1, 2, 3,$$

and the energetic quantities

$$W_{\{3:1\}} = W_{3\kappa}/W_{\kappa}, \qquad W_{\{3:123\}} = W_{3\kappa}\Big/\sum_{n=1}^{3} W_{n\kappa}.$$

In the numerical experiments, the function of the relative Q-factor of the eigen-oscillations is of particular interest:

$$Q_{\{1:3\}} = Q_{\kappa_1}/Q_{\kappa_3}.$$

Here

$$Q_{\kappa_n} = -\operatorname{Re}\kappa_n/[2\operatorname{Im}\kappa_n]$$

denotes the Q-factor of the eigen-oscillations of the spectral problem (5) at the eigen-frequencies $\kappa_n \in \Omega_{0,n\kappa} \subset \mathrm{H}_{0,n\kappa}$, see [8, 9].

In the considered case of E-polarization, the type of the investigated field is classified by the notation $H_{m,l,p}$ (or $TE_{m,l,p}$). The indices indicate the number of local maxima of $|E_x|$ (or $|U|$ due to $|U| = |E_x|$, cf. (3), (6)) along the coordinate axes x, y and z (see Fig. 1). Since the waves under study are homogeneous along the x-axis and quasi-homogeneous along the y-axis, we examine fields of the type $H_{0,0,p}$ (or $TE_{0,0,p}$), where the index p is equal to the number of local maxima of the function $|U|$ with respect to the argument z on the segment $[-2\pi\delta, 2\pi\delta]$.

4.1. Structure with a nonlinear decanalizing medium

The Figs. 2-4 show the properties of the layered structure containing an absolutely transparent (in the linear approximation) *decanalizing* medium $\{\varepsilon^{(\mathrm{L})} = 1, \alpha = -0.01\}$, $z \in [-2\pi\delta/3, 2\pi\delta/3]$, see (8).

Fig. 2 shows the results of a qualitative analysis of the wave scattering and generation properties by a three-layered object with a nonlinear *decanalizing* medium obtained in solving problems (4), (5) for $\kappa \equiv \kappa^{\text{inc}} = \operatorname{Re}(\kappa_1)$. The branches of the eigen-frequencies $\kappa_n = \kappa_n^{(\mathrm{NL})}(a_\kappa^{\text{inc}})$ of the investigated layered structure are depicted by the curves 5.1, 5.2, 6.1, 6.2 in Fig. 2 (left). Ibid the curves 3.1, 3.2, 4.1, 4.2 show the values of the corresponding eigen-frequencies $\kappa_n = \kappa_n^{(\mathrm{L})}(a_\kappa^{\text{inc}}) = \text{const}$ of the linear problems (for $\alpha \equiv 0$). The eigen-frequencies $\kappa_n^{(\mathrm{L})}$ do not depend on the amplitude characteristics of the fields, in particular, the relation $\lim\limits_{a_\kappa^{\text{inc}} \to 0} \kappa_n^{(\mathrm{NL})}(a_\kappa^{\text{inc}}) = \kappa_n^{(\mathrm{L})}$ holds, see Fig. 2 (left). If a decanalizing layer is present in the layered object, the growth of the excitation amplitude a_κ^{inc} leads to an increase of $\operatorname{Re}\kappa_n^{(\text{NL})}$, $n = 1, 3$ (curves 5.1, 6.1), and a decrease of $\operatorname{Im}\kappa_n^{(\text{NL})}$, $n = 1, 3$ (curves 5.2, 6.2). Here the increase of $\operatorname{Re}\kappa_n^{(\text{NL})}$, $n = 1, 3$ (see Fig. 2 (left)), is due to a decrease of the values $\operatorname{Re}\varepsilon_{n\kappa}$, $n = 1, 3$ (see. (2) and Fig. 3).

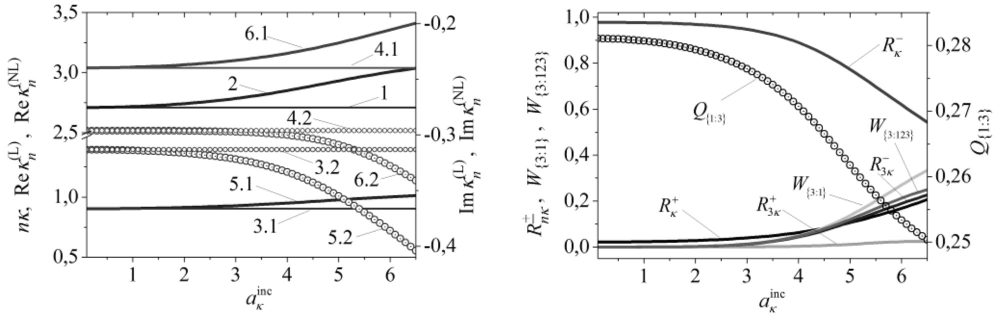

Fig. 2. Curves for $\kappa = \operatorname{Re}\kappa_1$. *Left: 1 -* $3\kappa \equiv 3\kappa^{(\text{L})}$ *at* $\alpha \equiv 0$, *2 -* $3\kappa \equiv 3\kappa^{(\text{NL})}$ *at* $\alpha = -0.01$; *3.1 -* $\kappa^{(\text{L})} \equiv \operatorname{Re}\kappa_1^{(\text{L})}$, *3.2 -* $\operatorname{Im}\kappa_1^{(\text{L})}$, *4.1 -* $\operatorname{Re}\kappa_3^{(\text{L})}$, *4.2 -* $\operatorname{Im}\kappa_3^{(\text{L})}$, *5.1 -* $\kappa^{(\text{NL})} \equiv \operatorname{Re}\kappa_1^{(\text{NL})}$, *5.2 -* $\operatorname{Im}\kappa_1^{(\text{NL})}$, *6.1 -* $\operatorname{Re}\kappa_3^{(\text{NL})}$, *6.2 -* $\operatorname{Im}\kappa_3^{(\text{NL})}$. *Right: The energetic properties of scattering and generation and the relative Q-factor.*

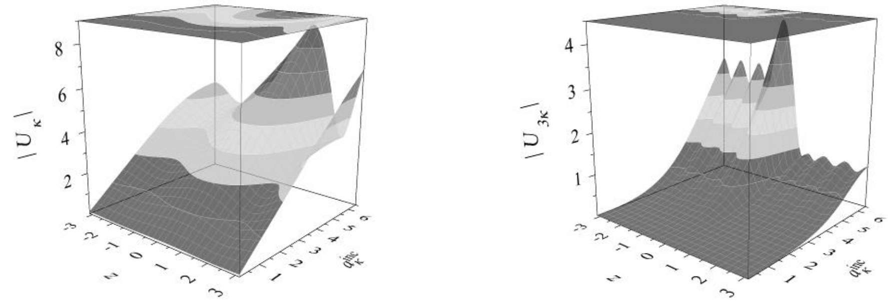

Fig. 3. The scattered (left) and generated (right) fields for $\alpha = -0.01$ *and* $\kappa = \operatorname{Re}\kappa_1$.

The spectral characteristics shown in Fig. 2 (left) allow us to analyze indirectly the energy exchange processes occurring during scattering and generation of waves by nonlinear objects within the framework of the developed self-consistent approach. In particular, the value of the relative quality factor (Q-factor) of the oscillations $Q_{\{1:3\}}$ is of interest. For example, comparing the values plotted on the curve $Q_{\{1:3\}}$ with the energy characteristics represented by the curves $W_{\{3:1\}}$ or $W_{\{3:123\}}$ in Fig. 2 (right), we see the following.

The spectral characteristics shown in Fig. 2 (left) allow us to analyze indirectly the energy exchange processes occurring during scattering and generation of waves by nonlinear objects within the framework of the developed self-consistent approach. In particular, the value of the relative quality factor (Q-factor) of the oscillations $Q_{\{1:3\}}$ is of interest. For example, comparing the values plotted on the curve $Q_{\{1:3\}}$ with the energy characteristics represented by the curves $W_{\{3:1\}}$ or $W_{\{3:123\}}$ in Fig. 2 (right), we see the following.

The local decrease in the relative Q-factor $Q_{\{1:3\}}(a_\kappa^{\mathrm{inc}})$ with increasing amplitude of the incident field a_κ^{inc} leads to a burst of energy $W_{\{3:1\}}$ and $W_{\{3:123\}}$ generated in the third harmonic, see Fig. 2 (right). In this case, the type of scattered and generated oscillations does not change. In the investigated range of amplitudes of the incident field, scattered fields of the type $H_{0,0,3}$ (see Fig. 3 (left)) and generated fields of the type $H_{0,0,8}$ (see Fig. 3 (right)) are observed.

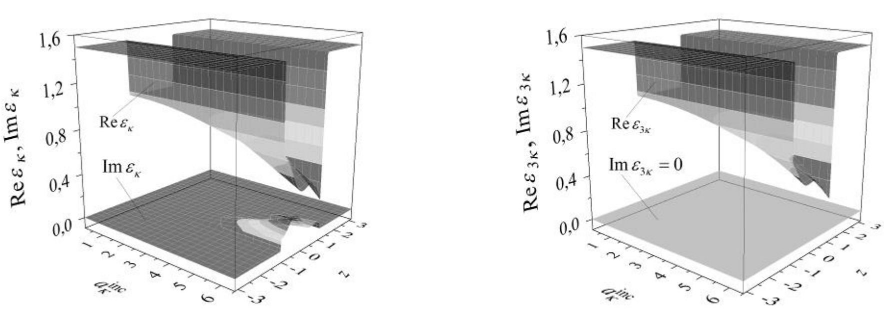

Fig. 4. The permittivity of the nonlinear layered structure for $\alpha = -0.01$ *and* $\kappa = \mathrm{Re}\,\kappa_1$.

The nonlinear components $\varepsilon_{n\kappa}^{(\mathrm{NL})}$ of the permittivity $\varepsilon_{n\kappa}$ at each of the frequencies κ and 3κ are determined by the values $U(\kappa;z)$ and $U(3\kappa;z)$. For non-absorbing media $\mathrm{Im}\,\varepsilon^{(\mathrm{L})}(z) \equiv 0$, taking into account the reality of the cubic susceptibility $\alpha(z)$, the equality $\mathrm{Im}\,\varepsilon_{n\kappa}(z) = \mathrm{Im}\,\varepsilon_{n\kappa}^{(\mathrm{NL})}(z)$ holds, see (2). The increase in the amplitude a_κ^{inc} of the incident field at the frequency κ leads to the generation of a third harmonic field $U(3\kappa;z)$. In the case under consideration, the quantity $\mathrm{Im}\,\varepsilon_\kappa^{(\mathrm{NL})}(z)$ (or $\mathrm{Im}\,\varepsilon_\kappa(z)$ if $\mathrm{Im}\,\varepsilon^{(\mathrm{L})}(z) \equiv 0$), that oscillates in general, takes *posititive* values along the height of the nonlinear layer, see Fig. 4 (left). The described situation characterizes the portion (loss) of energy in a nonlinear medium (at the excitation frequency κ) that went to the generation of the electromagnetic field of the third harmonic (at the frequency 3κ) [7, 8]. The generated fields at the frequency 3κ

are weak. They do not deliver energy for the generation of new harmonics. Here $\mathrm{Im}\,\varepsilon_{3\kappa}^{(\mathrm{NL})}(z) \equiv 0$ ($\mathrm{Im}\,\varepsilon_{3\kappa}(z) \equiv 0$ for $\mathrm{Im}\,\varepsilon^{(\mathrm{L})}(z) \equiv 0$), see Fig. 4 (right).

4.2. Structure with a nonlinear canalizing medium

The Figs. 5-7 show the properties of the layered structure containing an absolutely transparent *canalizing* medium $\{\varepsilon^{(\mathrm{L})} = 1, \alpha = +0.01\}$, $z \in [-2\pi\delta/3, 2\pi\delta/3]$, see (8).

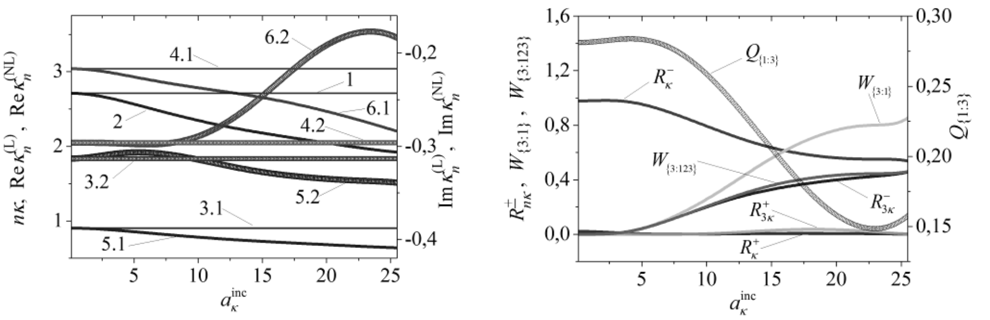

Fig. 5. Curves for $\kappa = \mathrm{Re}\,\kappa_1$. *Left: 1 -* $3\kappa \equiv 3\kappa^{(\mathrm{L})}$ *at* $\alpha \equiv 0$, *2 -* $3\kappa \equiv 3\kappa^{(\mathrm{NL})}$ *at* $\alpha = +0.01$*; 3.1 -* $\kappa^{(\mathrm{L})} \equiv \mathrm{Re}\,\kappa_1^{(\mathrm{L})}$, *3.2 -* $\mathrm{Im}\,\kappa_1^{(\mathrm{L})}$, *4.1 -* $\mathrm{Re}\,\kappa_3^{(\mathrm{L})}$, *4.2 -* $\mathrm{Im}\,\kappa_3^{(\mathrm{L})}$, *5.1 -* $\kappa^{(\mathrm{NL})} \equiv \mathrm{Re}\,\kappa_1^{(\mathrm{NL})}$, *5.2 -* $\mathrm{Im}\,\kappa_1^{(\mathrm{NL})}$, *6.1 -* $\mathrm{Re}\,\kappa_3^{(\mathrm{NL})}$, *6.2 -* $\mathrm{Im}\,\kappa_3^{(\mathrm{NL})}$. *Right: The energetic properties of scattering and generation and the relative Q-factor.*

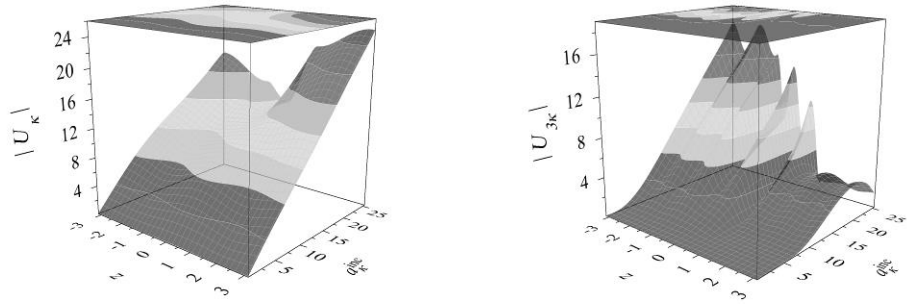

Fig. 6. The scattered (left) and generated (right) fields for $\alpha = +0.01$ *and* $\kappa = \mathrm{Re}\,\kappa_1$.

If a canalizing layer is present in the layered object, with an increase in the perturbation amplitude a_κ^{inc} a decrease of $\mathrm{Re}\,\kappa_n^{(\mathrm{NL})}$, $n = 1, 3$ (curves 5.1, 6.1), and piecewise monotonic variations of the quantities $\mathrm{Im}\,\kappa_n^{(\mathrm{NL})}$, $n = 1, 3$ (curves 5.2, 6.2) are observed. This behaviour of $\mathrm{Re}\,\kappa_n^{(\mathrm{NL})}$, $n = 1, 3$ (see curves 5.1, 6.1 in Fig. 5 (left)) is due to an increase of the values $\mathrm{Re}\,\varepsilon_{n\kappa}$, $n = 1, 3$ (see. (2) and Fig. 7).

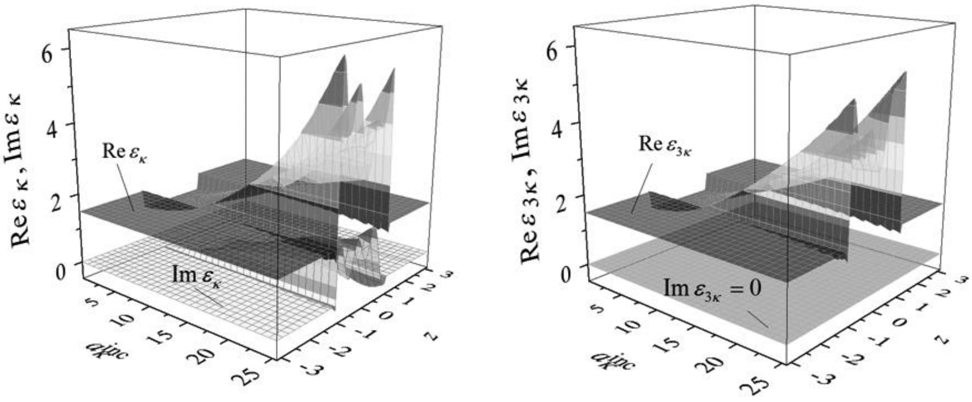

Fig. 7. The permittivity of the nonlinear layered structure for $\alpha = +0.01$ *and* $\kappa = \mathrm{Re}\,\kappa_1$.

In the region of monotonicity of the function $Q_{\{1:3\}}$, with an increase in the amplitude a_κ^{inc} of the incident field an increase of the energies $W_{\{3:1\}}$ and $W_{\{3:123\}}$ generated in the third harmonic is observed, see Fig. 5 (right). The main increase in the generated energy is observed in the region of monotonic decrease $a_\kappa^{\mathrm{inc}} \in (5, 23)$. The amplitude $a_\kappa^{\mathrm{inc}} \approx 23$ corresponding to the minimum of $Q_{\{1:3\}}$ is a critical inflection point of the functions $W_{\{3:1\}}$ and $W_{\{3:123\}}$. Therefore, in the range $a_\kappa^{\mathrm{inc}} \in [23, 25.5]$ there are an increase of $Q_{\{1:3\}}$ and a spike in the generated energies $W_{\{3:1\}}$, $W_{\{3:123\}}$.

Note that in the vicinity of critical points (such that $a_\kappa^{\mathrm{inc}} \approx 5$, $a_\kappa^{\mathrm{inc}} \approx 23$) and inflection points (for instance $a_\kappa^{\mathrm{inc}} \approx 12$) of the function $Q_{\{1:3\}}$, see Fig. 5, (right), a type conversion or a change in the configuration of the scattered and generated oscillations is observed, see. Fig. 6. For example, with increasing a_κ^{inc}, changes in the dynamics of types of the scattered oscillations $\mathrm{H}_{0,0,3}(a_\kappa^{\mathrm{inc}} \in [0.1, 12]) \to \mathrm{H}_{0,0,2}(a_\kappa^{\mathrm{inc}} \in [12, 23]) \to \mathrm{H}_{0,0,3}(a_\kappa^{\mathrm{inc}} \in [23, 25]) \to \mathrm{H}_{0,0,4}(a_\kappa^{\mathrm{inc}} \in [25, 25.5])$, see Fig. 6 (left), and generated oscillations $\mathrm{H}_{0,0,3}(a_\kappa^{\mathrm{inc}} = 5) \to \mathrm{H}_{0,0,4}(a_\kappa^{\mathrm{inc}} = 8) \to \mathrm{H}_{0,0,7}(a_\kappa^{\mathrm{inc}} \in [12, 25.5])$, see Fig. 6 (right), are observed.

The portion of energy in the nonlinear medium that went into the generation of the electromagnetic field of the third harmonic is described by the characteristic oscillating quantity $\mathrm{Im}\,\varepsilon_\kappa^{(\mathrm{NL})}(z)$ (or $\mathrm{Im}\,\varepsilon_\kappa(z)$, as here $\mathrm{Im}\,\varepsilon^{(\mathrm{L})}(z) \equiv 0$) along the height of the nonlinear layer [7, 8]. In the case under investigation, the values of the oscillating quantity $\mathrm{Im}\,\varepsilon_{\kappa}(z)$ are *positive* in the range $a_{\kappa}^{\mathrm{inc}} \in [0.1, 16.7)$, as well as *positive and negative* for $a_{\kappa}^{\mathrm{inc}} \in [16.7, 25.5]$, see Fig.7 (left).

The numerical results are obtained by means of the application of Simpson´s rule to the system of nonlinear integral equations (1). The resulting system of nonlinear algebraic equations (4) is solved using a self-consistent iterative algorithm that is based on a block method [8]. In the investigated range of problem parameters the dimension of the algebraic systems was 301. The relative error of the calculations did not exceed 10^{-7}.

5. Conclusion

The dynamics of resonant wave processes for layered structures containing nonlinear media has been investigated. In the framework of the developed self-consistent approach for the solution of systems of nonlinear boundary value problems, it is shown that, according to the spectral characteristics of the structures induced by the incident fields, it is possible to indirectly analyze the energy exchange processes that arise through the scattering and generation of waves by nonlinear objects. Thus, local critical processes of energy exchange correspond to a minimum of the relative Q-factor of the scattered and generated oscillations. The obtained results can be used, in particular, in problems of optimization of nonlinear electrodynamic devices possessing intense scattering and generation properties.

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