DOI: https://doi.org/10.26565/2304-6201-2022-56-02

UDC 517.95 : 517.988

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Using the two-sided approximations method for the numerical research of nanoelectromechanical systems under the action of the Casimir force

O. Konchakovska, M. Sidorov

Relevance. Developing the method of two-sided approximations for finding a positive solution to a nonlinear boundary value problem that models an electrostatic nanoelectromechanical system under external pressure has been considered. The presented mathematical model takes into account the influence of Casimir forces as an additional force of attraction between the components of nanosystems. Such systems feature the nonlinear phenomenon of pull-in instability, which occurs due to the interaction of conductive plates under a critical electric voltage. This phenomenon significantly limits the range of system's stable states and is typical of many nanodevices, in particular, accelerometers, switches, micromirrors, microresonators, etc. It is suggested to study the model parameters and estimate their values in order to analyze the stable states of nanoelectromechanical systems.

Goal. To develop a method of two-sided approximations for solving the given problem by using the methods of the nonlinear operator theory in semi-ordered Banach spaces.

Research methods. The nonlinear elliptic equation that models the operation of the electrostatic nanoelectromechanical system using the Green's function method is replaced by its Hammerstein integral equation equivalent. The specified integral equation is considered to be a nonlinear operator equation with a monotone operator in the space of continuous functions, semi-ordered by using a cone of non-negative functions. The conditions for the existence of a unique positive solution to the specified problem and the two-sided convergence of successive approximations to such a solution have been obtained.

The results. The developed method has been implemented and investigated by solving test problems. The results of computational experiment are shown in graphical and tabular form.

Conclusions. The performed computational experiments have confirmed the effectiveness of the developed method and can be used to solve the problems of mathematical modeling of nonlinear processes in micro- and nanoelectromechanical systems. The prospects for further research may lie in applying the method of two-sided approximations for models of nanoelectromechanical systems with repulsive Casimir forces.

Keywords: *method of two-sided approximations, Green's function, invariant conical segment, monotone operator, nanoelectromechanical system, external pressure, Casimir force.*

How to quote: O. Konchakovska, M. Sidorov, "Using the two-sided approximations method for the numerical research of nanoelectromechanical systems under the action of the Casimir force." *Bulletin of V.N. Karazin Kharkiv National University, series "Mathematical modelling. Information technology. Automated control systems*, vol. 56, pp. 21-34, 2022. https://doi.org/10.26565/2304-6201-2022-56-02

1. Introduction

The development of microelectromechanical systems (MEMS) has been driven by combining the principles of mechanics and electrostatics [1, 2]. Mathematical and numerical modeling of MEMS has been a key principle, initially focusing on stationary states [3-5] and later expanding to models including external pressure and non-stationary states governed by parabolic laws [4, 6].

Such systems in mathematical modeling are usually presented in partial differential equations with appropriate initial and boundary conditions. The most common method of solving such non-stationary problems is to reduce them (according to Rothe's method) to a system of ordinary differential equations

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in the time variable and then to solve the resulting equations numerically or analytically. Some discretization approaches are also used: the finite element method, the boundary element method, and the finite difference method [7]. For the numerical analysis of the corresponding stationary problems, it is convenient to use numerical methods with a two-sided nature of convergence. This approach allows a posteriori estimation of the error of the approximate solution at each step of the iterative process [8].

Therefore, developing and improving existing approaches to mathematical modeling and numerical analysis of the problems arising while studying electromechanical systems is a relevant field of research.

2. NEMS and the impact of Casimir's forces

When the scale of these systems decreases to the nanoscale, it is necessary to consider nanoelectromechanical systems (NEMS). Most typical models of electrostatic NEMS consist of two conductive plates: an elastic plate fixed along the boundary at the top and a fixed, rigid plate at the bottom. The applied electric voltage between the two plates leads to the deflection of the elastic plate and the following change in the system's capacitance [9]. The operation scheme of the simplest NEMS is shown in Figure 2.1.

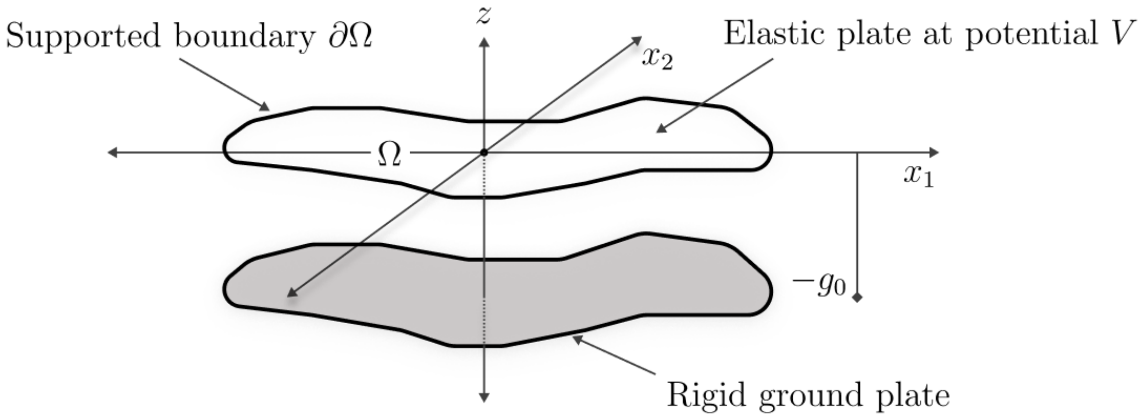

Fig. 2.1 Scheme of the electrostatic NEMS operation

The so-called pull-in instability is a distinguishing feature that limits the effectiveness of electrostatic NEMS. This effect occurs when the applied voltage exceeds a specific critical value. As a result, the plates snap together, limiting the range of stable operation of the devices. Many researchers have investigated pull-in instability [9-12]. Moreover, the miniaturization of devices requires considering the Casimir force along with the Coulomb force in a mathematical model.

An essential effect of quantum electrodynamics, namely, the effect of quantum fluctuations on non-quantum objects, was theoretically demonstrated by Hendrik Casimir in 1948 [13]. He considered the interaction of two conducting parallel plates in a vacuum and determined the force of gravity between them

$$F = -\frac{\pi^2 \eta c}{240 g_0^4},$$

where g_0 is the distance between the plates, η is a Planck constant, c is the speed of light in a vacuum.

The presence of Planck's constant indicates the purely quantum nature of the phenomenon.

The Casimir force for NEMS models is related to the interaction of plates. It is well-known that each wave presses the plate, and the closer the plates are, the fewer waves can exist between them, and the weaker their pressure is. At the same time, there is no limitation for the external space, so there can be much more waves outside. This creates a relatively sizeable converging force from the outside, which is illustrated in Figure 2.2. It turns out that the smaller the distance between the plates, the more they are attracted. This effect restricts the stability of micro- and nanoscale systems, as the parts of the mechanisms tend to snap together.

Consequently, to develop reliable, high-performance devices, it is necessary to determine stable modes of operation to prevent the occurrence of pull-in instability.

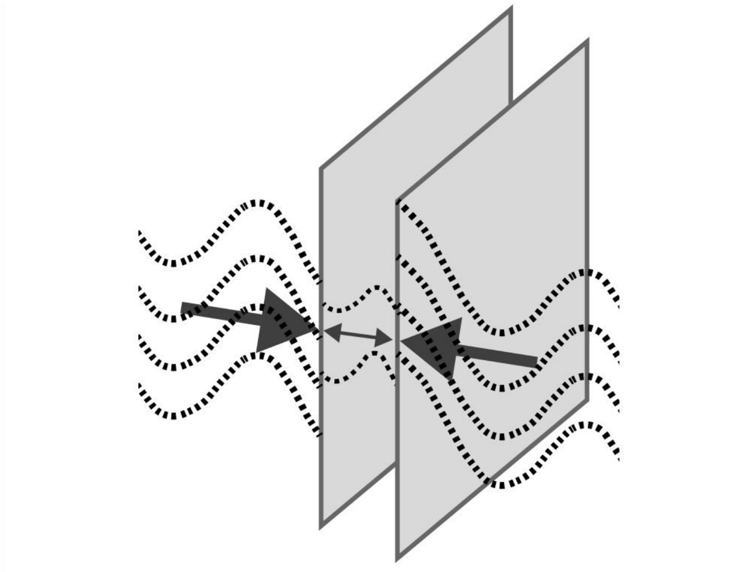

Fig. 2.2 Casimir forces acting on conductive plates

3. The research method

Let us consider a nonlinear boundary value problem modeling an electrostatic NEMS under the influence of external pressure [14, 15]:

$$-\Delta u = \frac{\lambda f(\mathbf{x})}{(1-u)^2} + \frac{\mu g(\mathbf{x})}{(1-u)^4} + P(\mathbf{x}),\ \mathbf{x} \in \Omega, \tag{3.1}$$

$$u(\mathbf{x}) > 0,\ \mathbf{x} \in \Omega, \tag{3.2}$$

$$u(\mathbf{x}) = 0,\ \mathbf{x} \in \partial\Omega, \tag{3.3}$$

where Ω is a plane domain with a piecewise smooth border $\partial\Omega$, $\mathbf{x} = (x_1, x_2)$, $P(\mathbf{x})$ is the external pressure, $P(\mathbf{x}) \geq 0$, the functions $f(\mathbf{x})$ and $g(\mathbf{x})$ describe the dielectric properties of the plate, the parameters λ and μ are the Coulomb and Casimir forces, respectively

$$\lambda = \frac{\varepsilon_0 V^2 L^2}{2\sigma_0 h g_0^3},\quad \mu = \frac{\eta c \pi^2 L^2}{240 \sigma_0 h g_0^5},$$

σ_0 is the tension in the plate, g_0 is the distance between the plates, h is the thickness of the deformed plate, ε_0 is the vacuum dielectric constant, L is the length of the plate, V is the applied voltage, η is a Planck constant, c is the speed of light in a vacuum.

From the physical content of the problem, it follows that the functions $f(\mathbf{x})$, $g(\mathbf{x})$ and $P(\mathbf{x})$ are continuous and non-negative at $\mathbf{x} \in \bar{\Omega}$.

The problem (3.1) – (3.3) is equivalent to the Hammerstein integral equation

$$u(\mathbf{x}) = \int_{\Omega} G(\mathbf{x},\mathbf{s}) \left[\frac{\lambda f(\mathbf{s})}{(1-u(\mathbf{s}))^2} + \frac{\mu g(\mathbf{s})}{(1-u(\mathbf{s}))^4} + P(\mathbf{s}) \right] d\mathbf{s}, \tag{3.4}$$

where $G(\mathbf{x},\mathbf{s})$ is the Green's function of this problem, $\mathbf{s} = (s_1, s_2)$.

Let $C(\bar{\Omega})$ be the Banach space of functions continuous in the domain $\bar{\Omega} = \Omega \cup \partial\Omega$, with the norm $\|u\| = \max_{\mathbf{x}\in\bar{\Omega}} |u(\mathbf{x})|$. Let us consider a normal (and even an acute) cone of non-negative functions $K_+ = \{u \in C(\bar{\Omega}) : u(\mathbf{x}) \geq 0,\ \mathbf{x} \in \bar{\Omega}\}$ in space $C(\bar{\Omega})$. And let us define semi-ordering in the $C(\bar{\Omega})$ with a cone K_+ by the rule: for $u, v \in C(\bar{\Omega})$ $u \preceq v$, if $v - u \in K_+$. It means that [16]

$$u \preceq v, \text{ if } u(\mathbf{x}) \leq v(\mathbf{x}) \text{ for all } \mathbf{x} \in \bar{\Omega}.$$

The nonlinear integral equation (3.4) semi-ordered by a cone K_+ will be considered in $C(\bar{\Omega})$ as an operator equation $u = T(u)$, where operator T acts in the $C(\bar{\Omega})$ according to the rule

$$T(u)(\mathbf{x}) = \int_{\Omega} G(\mathbf{x},\mathbf{s})\left[\frac{\lambda f(\mathbf{s})}{(1-u(\mathbf{s}))^2} + \frac{\mu g(\mathbf{s})}{(1-u(\mathbf{s}))^4} + P(\mathbf{s})\right]d\mathbf{s}. \tag{3.5}$$

It should be noted that if a classical solution of the problem (3.1) – (3.3) exists as a function $u^* \in C^2(\Omega) \cap C(\bar{\Omega})$ that satisfies equation (3.1) and conditions (3.2), (3.3), this function will also satisfy the integral equation (3.4). If the problem (3.1) – (3.3) does not have a classical solution, then the equation (3.4) will be taken as the definition of the generalized solution of the problem (3.1) – (3.3).

Definition. The generalized solution of the boundary value problem (3.1) – (3.3) will be the function $u^* \in K_+$, which is the solution of the integral equation (3.4).

The properties of the operator T (3.5) has been stated in the following lemma.

Lemma. The operator T (3.5) is:

a) a positive operator;

b) is a u_0-positive operator, where the function $u_0(\mathbf{x})$ is defined by the equality

$$u_0(\mathbf{x}) = \int_{\Omega} G(\mathbf{x},\mathbf{s})d\mathbf{s}; \tag{3.6}$$

c) is an isotonic operator;

d) has an invariant conical segment $<0, \beta>$, and the constant β, $0 < \beta < 1$, is a solution of the inequality

$$\lambda M_f (1-\beta)^2 + \mu M_g \le (\beta - M_P)(1-\beta)^4, \tag{3.7}$$

where

$$M_f = \max_{\mathbf{x}\in\bar{\Omega}} \int_{\Omega} G(\mathbf{x},\mathbf{s}) f(\mathbf{s})d\mathbf{s}, \quad M_g = \max_{\mathbf{x}\in\bar{\Omega}} \int_{\Omega} G(\mathbf{x},\mathbf{s}) g(\mathbf{s})d\mathbf{s}, \quad M_P = \max_{\mathbf{x}\in\bar{\Omega}} \int_{\Omega} G(\mathbf{x},\mathbf{s}) P(\mathbf{s})d\mathbf{s};$$

e) is a Lipschitz-continuous operator on $<0, \beta>$, i.e., for all $v, w \in <0, \beta>$ the following inequality

$$\|T(v) - T(w)\| \le \gamma \|v - w\|, \tag{3.8}$$

where $\gamma = \dfrac{2\lambda M_f}{(1-\beta)^3} + \dfrac{4\mu M_g}{(1-\beta)^5}$ is true.

Proof. a) The Green's function $G(\mathbf{x},\mathbf{s})$ of the first boundary value problem for the operator $-\Delta$ in the domain is continuous for $\mathbf{x}, \mathbf{s} \in \bar{\Omega}$, $\mathbf{x} \ne \mathbf{s}$ and

$$0 \le G(\mathbf{x},\mathbf{s}) \le k_0 \left|\ln \frac{1}{r_{\mathbf{xs}}}\right|,$$

where $r_{\mathbf{xs}} = |\mathbf{x} - \mathbf{s}| = \sqrt{(x_1 - s_1)^2 + (x_2 - s_2)^2}$ is the distance between the points $\mathbf{x}$ and $\mathbf{s}$ [17].

Taking into account the non-negativity and continuity of the functions $u(\mathbf{x})$, $f(\mathbf{x})$, $g(\mathbf{x})$ and $P(\mathbf{x})$ at $\mathbf{x} \in \bar{\Omega}$, the integral expression in (3.5) will also be non-negative and continuous at $\mathbf{x}, \mathbf{s} \in \bar{\Omega}$, $\mathbf{x} \ne \mathbf{s}$, and, therefore, the function $T(u)(\mathbf{x})$ is non-negative and continuous at $\mathbf{x} \in \bar{\Omega}$. This means that the operator T (3.5) acts in the space $C(\bar{\Omega})$ and leaves the cone K_+ invariant, i.e., converts the function from K_+ to the function from K_+. Therefore, T is a positive operator.

b) It is obvious that the function $u_0(\mathbf{x})$ of the form (3.6) belongs to $K_+ \setminus \{\theta\}$. If Ω_0 is a subdomain of the domain Ω, and $\mu(\Omega_0) > 0$, then there will be a such number $\gamma = \gamma(\Omega_0) > 0$ that the inequality holds [17]

$$\gamma \int_{\Omega} G(\mathbf{x},\mathbf{s})d\mathbf{s} \le \int_{\Omega_0} G(\mathbf{x},\mathbf{s})d\mathbf{s}.$$

On the other hand, if $u \in K_+ \setminus \{\theta\}$, then for some $\alpha_0 > 0$ there is a set $\Omega_0 \subset \Omega$ that $\mu(\Omega_0) > 0$ and $\dfrac{\lambda f(\mathbf{x})}{(1-u(\mathbf{x}))^2} + \dfrac{\mu g(\mathbf{x})}{(1-u(\mathbf{x}))^4} + P(\mathbf{x}) \ge \alpha_0$ for all $\mathbf{x} \in \Omega_0$. Then for all $\mathbf{x} \in \bar{\Omega}$

$$T(u)(\mathbf{x}) = \int\limits_{\Omega} G(\mathbf{x},\mathbf{s})\left[\frac{\lambda f(\mathbf{s})}{(1-u(\mathbf{s}))^2} + \frac{\mu g(\mathbf{s})}{(1-u(\mathbf{s}))^4} + P(\mathbf{s})\right] d\mathbf{s} \geq$$

$$\geq \int\limits_{\Omega_0} G(\mathbf{x},\mathbf{s})\left[\frac{\lambda f(\mathbf{s})}{(1-u(\mathbf{s}))^2} + \frac{\mu g(\mathbf{s})}{(1-u(\mathbf{s}))^4} + P(\mathbf{s})\right] d\mathbf{s} \geq \alpha_0 \int\limits_{\Omega_0} G(\mathbf{x},\mathbf{s}) d\mathbf{s} \geq \alpha_0\gamma \int\limits_{\Omega} G(\mathbf{x},\mathbf{s}) d\mathbf{s} = \alpha_0 \gamma u_0(\mathbf{x}).$$

is true.

Next, for all $\mathbf{x} \in \bar{\Omega}$

$$T(u)(\mathbf{x}) = \int\limits_{\Omega} G(\mathbf{x},\mathbf{s})\left[\frac{\lambda f(\mathbf{s})}{(1-u(\mathbf{s}))^2} + \frac{\mu g(\mathbf{s})}{(1-u(\mathbf{s}))^4} + P(\mathbf{s})\right] d\mathbf{s} \leq$$

$$\leq \max_{\mathbf{x}\in\bar{\Omega}}\left[\frac{\lambda f(\mathbf{x})}{(1-u(\mathbf{x}))^2} + \frac{\mu g(\mathbf{x})}{(1-u(\mathbf{x}))^4} + P(\mathbf{x})\right] \cdot \int\limits_{\Omega} G(\mathbf{x},\mathbf{s}) d\mathbf{s} = \max_{\mathbf{x}\in\bar{\Omega}}\left[\frac{\lambda f(\mathbf{x})}{(1-u(\mathbf{x}))^2} + \frac{\mu g(\mathbf{x})}{(1-u(\mathbf{x}))^4} + P(\mathbf{x})\right] \cdot u_0(\mathbf{x}).$$

Therefore, for all $\mathbf{x} \in \bar{\Omega}$ there will be a double inequality

$$\alpha u_0(\mathbf{x}) \leq \int\limits_{\Omega} G(\mathbf{x},\mathbf{s})\left[\frac{\lambda f(\mathbf{s})}{(1-u(\mathbf{s}))^2} + \frac{\mu g(\mathbf{s})}{(1-u(\mathbf{s}))^4} + P(\mathbf{s})\right] d\mathbf{s} \leq \beta u_0(\mathbf{x}),$$

where $\alpha = \alpha_0 \gamma > 0$, $\beta = \max\limits_{\mathbf{x}\in\bar{\Omega}}\left[\frac{\lambda f(\mathbf{x})}{(1-u(\mathbf{x}))^2} + \frac{\mu g(\mathbf{x})}{(1-u(\mathbf{x}))^4} + P(\mathbf{x})\right] > 0$, which means that the operator T is the u_0 -positive.

c) Let $v, w \in K_+$ and v ,, w, that is $v(\mathbf{x}) \leq w(\mathbf{x})$ for all $\mathbf{x} \in \bar{\Omega}$. Then for all $\mathbf{x} \in \bar{\Omega}$

$$\frac{\lambda f(\mathbf{x})}{(1-v(\mathbf{x}))^2} + \frac{\mu g(\mathbf{x})}{(1-v(\mathbf{x}))^4} + P(\mathbf{x}) \leq \frac{\lambda f(\mathbf{x})}{(1-w(\mathbf{x}))^2} + \frac{\mu g(\mathbf{x})}{(1-w(\mathbf{x}))^4} + P(\mathbf{x}),$$

and hence, in view of the non-negativity of the Green's function $G(\mathbf{x},\mathbf{s})$,

$$T(v)(\mathbf{x}) = \int\limits_{\Omega} G(\mathbf{x},\mathbf{s})\left[\frac{\lambda f(\mathbf{s})}{(1-v(\mathbf{s}))^2} + \frac{\mu g(\mathbf{s})}{(1-v(\mathbf{s}))^4} + P(\mathbf{s})\right] d\mathbf{s} \leq$$

$$\leq \int\limits_{\Omega} G(\mathbf{x},\mathbf{s})\left[\frac{\lambda f(\mathbf{s})}{(1-w(\mathbf{s}))^2} + \frac{\mu g(\mathbf{s})}{(1-w(\mathbf{s}))^4} + P(\mathbf{s})\right] d\mathbf{s} = T(w)(\mathbf{x}).$$

Therefore, it follows from $v, w \in K_+$ and v ,, w that $T(v) \leq T(w)$. Therefore, the operator T (3.5) is isotonic.

d) The invariant conical segment $< v_0, w_0 >$ is defined by the inequalities $T(v_0) \ldots v_0$ and $T(w_0)$,, w_0. Let us take $v_0 = 0$ and $w_0 = \beta$. Then the specified inequalities will take form

$$\int\limits_{\Omega} G(\mathbf{x},\mathbf{s})[\lambda f(\mathbf{s}) + \mu g(\mathbf{s}) + P(\mathbf{s})] d\mathbf{s} \geq 0 \text{ for all } \mathbf{x} \in \Omega, \tag{3.9}$$

$$\int\limits_{\Omega} G(\mathbf{x},\mathbf{s})\left[\frac{\lambda f(\mathbf{s})}{(1-\beta)^2} + \frac{\mu g(\mathbf{s})}{(1-\beta)^4} + P(\mathbf{s})\right] d\mathbf{s} \leq \beta \text{ for all } \mathbf{x} \in \Omega. \tag{3.10}$$

The inequality (3.9) will always hold, since the Green's function $G(\mathbf{x},\mathbf{s})$ at $\mathbf{x}, \mathbf{s} \in \bar{\Omega}$, $\mathbf{x} \neq \mathbf{s}$, and functions $f(\mathbf{x})$, $g(\mathbf{x})$ and $P(\mathbf{x})$ at $\mathbf{x} \in \bar{\Omega}$ are non-negative, and parameters λ and μ are positive. The inequality (3.10) can be written as

$$\frac{\lambda}{(1-\beta)^2} \int\limits_{\Omega} G(\mathbf{x},\mathbf{s}) f(\mathbf{s}) d\mathbf{s} + \frac{\mu}{(1-\beta)^4} \int\limits_{\Omega} G(\mathbf{x},\mathbf{s}) g(\mathbf{s}) d\mathbf{s} + \int\limits_{\Omega} G(\mathbf{x},\mathbf{s}) P(\mathbf{s}) d\mathbf{s} \leq \beta \text{ for all } \mathbf{x} \in \Omega.$$

Taking the maximum in the last inequality leads to

$$\frac{\lambda M_f}{(1-\beta)^2} + \frac{\mu M_g}{(1-\beta)^4} + M_P \leq \beta, \tag{3.11}$$

where $M_f = \max\limits_{\mathbf{x}\in\bar\Omega} \int\limits_\Omega G(\mathbf{x},\mathbf{s}) f(\mathbf{s}) d\mathbf{s}$, $M_g = \max\limits_{\mathbf{x}\in\bar\Omega} \int\limits_\Omega G(\mathbf{x},\mathbf{s}) g(\mathbf{s}) d\mathbf{s}$, $M_P = \max\limits_{\mathbf{x}\in\bar\Omega} \int\limits_\Omega G(\mathbf{x},\mathbf{s}) P(\mathbf{s}) d\mathbf{s}$.

It follows from physical considerations that $0<\beta<1$. Then after multiplying by $(1-\beta)^4$ the inequality (3.11) takes the form (3.7).

e) Let us denote

$$F(\mathbf{x},u(\mathbf{x})) = \frac{\lambda f(\mathbf{x})}{(1-u(\mathbf{x}))^2} + \frac{\mu g(\mathbf{x})}{(1-u(\mathbf{x}))^4} + P(\mathbf{x}).$$

Let us choose $v, w \in <0, \beta>$ and consider the expression

$$|F(\mathbf{x},v(\mathbf{x})) - F(\mathbf{x},w(\mathbf{x}))| =$$

$$= \left| \left(\frac{\lambda f(\mathbf{x})}{(1-v(\mathbf{x}))^2} + \frac{\mu g(\mathbf{x})}{(1-v(\mathbf{x}))^4} + P(\mathbf{x}) \right) - \left(\frac{\lambda f(\mathbf{x})}{(1-w(\mathbf{x}))^2} + \frac{\mu g(\mathbf{x})}{(1-w(\mathbf{x}))^4} + P(\mathbf{x}) \right) \right| =$$

$$= \left| \lambda f(\mathbf{x}) \left(\frac{1}{(1-v(\mathbf{x}))^2} - \frac{1}{(1-w(\mathbf{x}))^2} \right) + \mu g(\mathbf{x}) \left(\frac{1}{(1-v(\mathbf{x}))^4} - \frac{1}{(1-w(\mathbf{x}))^4} \right) \right|.$$

The triangle inequality leads to

$$|F(\mathbf{x},v(\mathbf{x})) - F(\mathbf{x},w(\mathbf{x}))| \le \lambda f(\mathbf{x}) \left| \frac{1}{(1-v(\mathbf{x}))^2} - \frac{1}{(1-w(\mathbf{x}))^2} \right| + \mu g(\mathbf{x}) \left| \frac{1}{(1-v(\mathbf{x}))^4} - \frac{1}{(1-w(\mathbf{x}))^4} \right|.$$

Since at $0 < v,\ w < \beta$

$$\left| \frac{1}{(1-v)^2} - \frac{1}{(1-w)^2} \right| \le \frac{2}{(1-\beta)^3} |v-w|, \quad \left| \frac{1}{(1-v)^4} - \frac{1}{(1-w)^4} \right| \le \frac{4}{(1-\beta)^5} |v-w|,$$

then the inequality holds

$$|F(\mathbf{x},v(\mathbf{x})) - F(\mathbf{x},w(\mathbf{x}))| \le \left[\frac{2\lambda f(\mathbf{x})}{(1-\beta)^3} + \frac{4\mu g(\mathbf{x})}{(1-\beta)^5} \right] |v(\mathbf{x}) - w(\mathbf{x})|.$$

Therefore

$$\|T(v) - T(w)\| = \max_{\mathbf{x}\in\bar\Omega} |T(v)(\mathbf{x}) - T(w)(\mathbf{x})| = \max_{\mathbf{x}\in\bar\Omega} \left| \int_\Omega G(\mathbf{x},\mathbf{s})[F(\mathbf{s},v(\mathbf{s})) - F(\mathbf{s},w(\mathbf{s}))] d\mathbf{s} \right| \le$$

$$\le \left[\frac{2\lambda}{(1-\beta)^3} \max_{\mathbf{x}\in\bar\Omega} \int_\Omega G(\mathbf{x},\mathbf{s}) f(\mathbf{s}) d\mathbf{s} + \frac{4\mu}{(1-\beta)^5} \max_{\mathbf{x}\in\bar\Omega} \int_\Omega G(\mathbf{x},\mathbf{s}) g(\mathbf{s}) d\mathbf{s} \right] \cdot \max_{\mathbf{x}\in\bar\Omega} |v(\mathbf{x}) - w(\mathbf{x})| =$$

$$= \left[\frac{2\lambda M_f}{(1-\beta)^3} + \frac{4\mu M_g}{(1-\beta)^5} \right] \cdot \|v-w\| = \gamma \|v-w\|.$$

The lemma has been proven.

Obviously, the operator T (3.5) is continuous and completely continuous.

Let us proceed to constructing the method of two-sided approximations for finding the positive solution of the integral equation (3.4) (and, therefore, the boundary value problem (3.1) – (3.3)).

Since the ends of the invariant conical segment $<0, \beta>$ will be chosen as the initial approximation, let us investigate the inequality (3.7) under the condition $0<\beta<1$ and denote $\varphi(\beta) = \lambda M_f (1-\beta)^2 + \mu M_g$, $\psi(\beta) = (\beta - M_P)(1-\beta)^4$. Then the inequality (3.7) takes the form $\varphi(\beta) \le \psi(\beta)$. It is obvious that $\varphi(\beta) > \mu M_g > 0$ if $0<\beta<1$. On the other hand $\psi(0) = -M_P \le 0$, $\psi(1) = \psi(M_P) = 0$ and if $M_P \ge 1$, then $\psi(\beta) < 0$ for $0<\beta<1$ and the inequality (3.7) will not hold. Therefore, the condition $0 \le M_P < 1$ must be fulfilled and $\psi(\beta) > 0$, if $M_P < \beta < 1$. In addition, the inequality (3.7) cannot be satisfied if $\sup\limits_{0<\beta<1} \psi(\beta) \le \inf\limits_{0<\beta<1} \varphi(\beta)$.

Accordingly, $\sup\limits_{0<\beta<1}\psi(\beta)=\psi\left(\dfrac{4M_P+1}{5}\right)=\dfrac{2^8(1-M_P)^5}{5^5}$, $\inf\limits_{0<\beta<1}\varphi(\beta)=\mu M_g$. If the inequality (3.7) has a solution, then the following inequality should be satisfied

$$\mu M_g<\frac{2^8(1-M_P)^5}{5^5}. \qquad (3.12)$$

It is obvious that the inequality (3.7) has the solution $\underline{\beta}\le\beta\le\overline{\beta}$ when

$$\sup_{0<\beta<1}\psi(\beta)=\psi\left(\frac{4M_P+1}{5}\right)=\frac{2^8(1-M_P)^5}{5^5}>\varphi\left(\frac{4M_P+1}{5}\right)=\frac{2^4}{5^2}\lambda M_f(1-M_P)^2+\mu M_g,$$

namely

$$2^4 5^3\lambda M_f(1-M_P)^2+5^5\mu M_g<2^8(1-M_P)^5. \qquad (3.13)$$

It is clear that the inequality (3.12) is a consequence of the inequality (3.13).

It should be noted that the inequality (3.7) will hold only at one point $\beta=\beta_0$ if

$$2^4 5^3\lambda M_f(1-M_P)^2+5^5\mu M_g=2^8(1-M_P)^5.$$

Therefore, the set of solutions of the inequality (3.7) for β forms the segment $[\underline{\beta},\overline{\beta}]$, only under the conditions:

$$0\le M_P<1,\ 2^4 5^3\lambda M_f(1-M_P)^2+5^5\mu M_g<2^8(1-M_P)^5.$$

Consequently,

$$\lambda<\frac{2^8(1-M_P)^5-5^5\mu M_g}{2^4 5^3 M_f(1-M_P)^2},$$

and the positivity of the parameter λ leads to

$$\mu<\frac{2^8(1-M_P)^5}{5^5 M_g},$$

or

$$\mu<\frac{2^8(1-M_P)^5-2^4 5^3\lambda M_f(1-M_P)^2}{5^5 M_g}\ \text{ and }\ \lambda<\frac{2^8(1-M_P)^5}{2^4 5^3 M_f(1-M_P)^2}. \qquad (3.14)$$

At the same time $M_P<\underline{\beta}<\overline{\beta}<1$.

Let us form an iterative process according to the scheme

$$v^{(k+1)}(\mathbf{x})=\int_\Omega G(\mathbf{x},\mathbf{s})\left[\frac{\lambda f(\mathbf{s})}{(1-v^{(k)}(\mathbf{s}))^2}+\frac{\mu g(\mathbf{s})}{(1-v^{(k)}(\mathbf{s}))^4}+P(\mathbf{s})\right]d\mathbf{s},\ k=0,1,2,\ldots; \qquad (3.15)$$

$$w^{(k+1)}(\mathbf{x})=\int_\Omega G(\mathbf{x},\mathbf{s})\left[\frac{\lambda f(\mathbf{s})}{(1-w^{(k)}(\mathbf{s}))^2}+\frac{\mu g(\mathbf{s})}{(1-w^{(k)}(\mathbf{s}))^4}+P(\mathbf{s})\right]d\mathbf{s},\ k=0,1,2,\ldots; \qquad (3.16)$$

$$v^{(0)}(\mathbf{x})=0,\ w^{(0)}(\mathbf{x})=\beta. \qquad (3.17)$$

The sequence $\{v^{(k)}(\mathbf{x})\}$ is non-decreasing under the cone K_+ and the sequence $\{w^{(k)}(\mathbf{x})\}$ is non-increasing under the cone K_+, since the conical segment $<0,\beta>$ is invariant for the isotonic operator T (3.5). The existence of boundaries $v^*(\mathbf{x})$ and $w^*(\mathbf{x})$ of these sequences follows from the normality of the cone K_+ and the complete continuity of the operator T. Therefore, the chain of inequalities holds

$$0=v^{(0)}\le v^{(1)}\le\ldots\le v^{(k)}\le\ldots\le v^*\le w^*\le\ldots\le w^{(k)}\le\ldots\le w^{(1)}\le w^{(0)}=\beta.$$

There are two possible cases: $v^*<w^*$ and $v^*=w^*$. In the second case $u^*:=v^*=w^*$ is unique fixed point of the operator T on the conical segment $<0,\beta>$, and therefore u^* is unique positive solution of the considered boundary value problem (3.1) – (3.3).

To obtain the conditions for the case $v^* = w^*$, let us estimate the norm $\left\| w^{(k+1)} - v^{(k+1)} \right\|$. Taking the inequality (3.8) into account leads to

$$\left\| w^{(k+1)} - v^{(k+1)} \right\| = \left\| T(w^{(k)}) - T(v^{(k)}) \right\| \le \gamma \left\| w^{(k)} - v^{(k)} \right\| = \gamma \left\| T(w^{(k-1)}) - T(v^{(k-1)}) \right\| \le$$
$$\le \gamma^2 \left\| w^{(k-1)} - v^{(k-1)} \right\| \le \ldots \le \gamma^{k+1} \left\| w^{(0)} - v^{(0)} \right\| = \gamma^{k+1}\beta .$$

It follows that $\lim_{k\to\infty} \left\| w^{(k+1)} - v^{(k+1)} \right\| = 0$, i.e. $v^* = w^*$, if $\gamma < 1$.

So, the following theorem is true.

Theorem 3.1. Let $<0, \beta>$ be an invariant conical segment for the operator T (3.5) and $\gamma < 1$. Then the iterative process (3.15) – (3.17) converges from two sides in the space $C(\bar{\Omega})$ norm to the continuous positive solution u^* of the boundary value problem (3.1) – (3.3), which is unique for the segment.

For the approximate solution of the boundary value problem (3.1) – (3.3) at the k -th iteration let us choose the function

$$u^{(k)}(\mathbf{x}) = \frac{v^{(k)}(\mathbf{x}) + w^{(k)}(\mathbf{x})}{2},$$

considering the mentioned above, the error of this approximation will be estimated by the inequality

$$\left\| u^* - u^{(k)} \right\| \le \frac{1}{2} \left\| w^{(k)} - v^{(k)} \right\| \le \frac{1}{2}\gamma^k \beta . \tag{3.18}$$

Therefore, if the accuracy $\varepsilon > 0$ is specified, the iterative process (3.15) – (3.17) should be carried out until the inequality

$$\left\| w^{(k)} - v^{(k)} \right\| = \max_{\mathbf{x}\in\bar{\Omega}} (w^{(k)}(\mathbf{x}) - v^{(k)}(\mathbf{x})) < 2\varepsilon$$

or inequalities $\gamma^k \beta < 2\varepsilon$ are satisfied, so with the accuracy ε it can be stated that $u^*(x) \approx u^{(k)}(x)$.

The number of iterations required to achieve the specified accuracy ε can be estimated from a priori estimation (3.18) and the inequality $\frac{1}{2}\gamma^k \beta < \varepsilon$, i.e.

$$k_0(\varepsilon) = \left[\frac{\ln \frac{\beta}{2\varepsilon}}{\ln \frac{1}{\gamma}} \right] + 1,$$

where the square brackets denote the whole part of the number.

Thus, it is necessary to choose $\beta = \underline{\beta}$ for the fastest convergence of the iterative process.

4. Numerical experiments

Let us consider the problem (3.1) – (3.3) in the domain $\Omega = \{\mathbf{x} = (x_1, x_2) \mid x_1^2 + x_2^2 < 1\} \subset \mathbf{R}^2$. Green's function in this case has the form

$$G(\mathbf{x}, \mathbf{s}) = \frac{1}{2\pi} \ln \frac{1}{r_{\mathbf{xs}}} - \frac{1}{2\pi} \ln \frac{1}{\rho r_{\mathbf{xs}^1}},$$

where $\mathbf{x} = (x_1, x_2)$, $\mathbf{s} = (s_1, s_2)$, $\rho = \sqrt{s_1^2 + s_2^2}$, points $\mathbf{s}$, $\mathbf{s}^1$ are symmetrical about the unit circle, $r_{\mathbf{xs}}$, $r_{\mathbf{xs}^1}$ is the distance between the points $\mathbf{x}$, $\mathbf{s}$ and $\mathbf{x}$, $\mathbf{s}^1$ respectively.

The researchers [4] suggest using functions of the plate's dielectric properties for the circle domain in the form

$$f(\mathbf{x}) = e^{\zeta(|\mathbf{x}|^2 - 1)} = e^{\zeta(x_1^2 + x_2^2 - 1)}, \quad g(\mathbf{x}) = e^{\zeta(|\mathbf{x}|^2 - 1)} = e^{\zeta(x_1^2 + x_2^2 - 1)},$$

where ζ is a non-negative constant.

To simulate the influence of external pressure, let us select a function

$$P(\mathbf{x}) = \kappa(1-|\mathbf{x}|^2)(e^{-\chi|\mathbf{x}-\mathbf{a}|^2} + e^{-\varpi|\mathbf{x}-\mathbf{b}|^2}),\ \kappa, \chi, \varpi > 0,\ \mathbf{a} = (0.75; 0.75),\ \mathbf{b} = (-0.15; -0.15)$$

($\kappa = 0$ is the case of no external pressure).

Let $\zeta = 3$, $\kappa = 0.5$, $\chi = 3$, $\varpi = 5$. Then $M_f = 0.0343$, $M_g = 0.0343$, $M_P = 0.0510$. The graph of the surface of the functions $f(\mathbf{x})$ and $g(\mathbf{x})$ is presented in Figure 4.1, and the graph of the function $P(\mathbf{x})$ surface is presented in Figure 4.2.

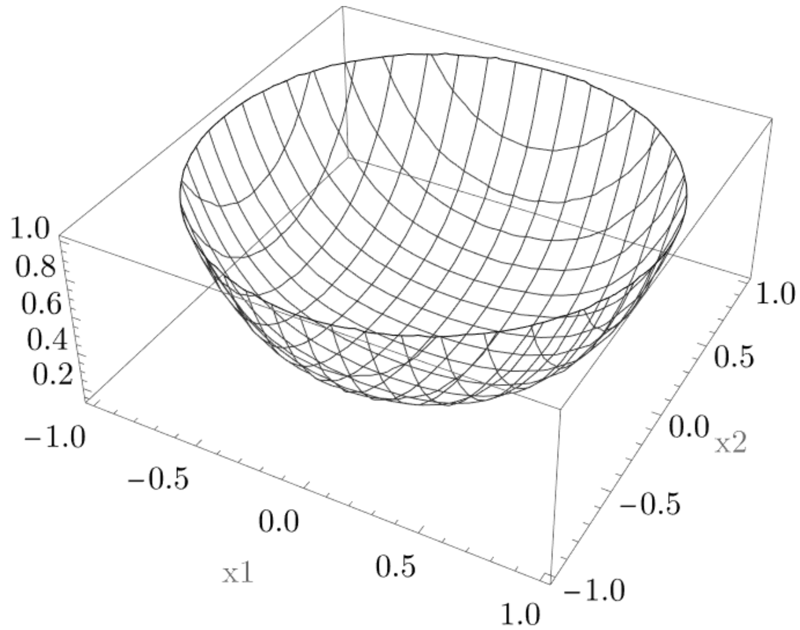

Fig. 4.1 The surface of the functions $f(\mathbf{x})$ *and* $g(\mathbf{x})$

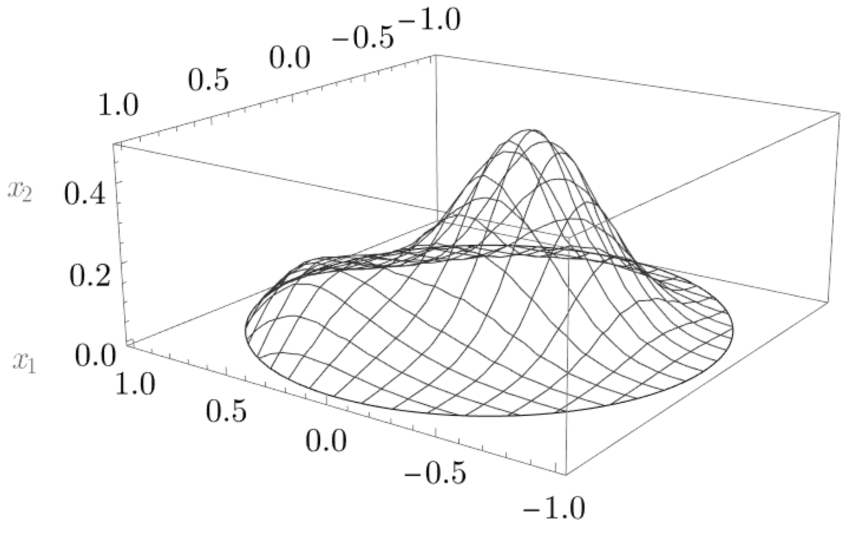

Fig. 4.2 The surface of the function $P(\mathbf{x})$

The condition (3.14) leads to a set of solutions for the parameters λ and μ, which is shown in Figure 4.3. According to (3.14): $\lambda < 3.1929$. Let $\lambda = 2$, so $\mu < 0.6875$. Let us choose $\mu = 0.5$. Then according to (3.7): $0.1998 \le \beta \le 0.4363$. Thus, the conical segment has the form $<0, \beta> = <0, \underline{\beta}> = <0,\ 0.1998>$. In this case $\gamma = 0.4764$. Since $\gamma < 1$, according to Theorem 3.1, the successive approximations formed by the scheme (3.15) – (3.17) converge from two sides to the positive solution of the problem, which is unique on the cone $<0, \beta>$.

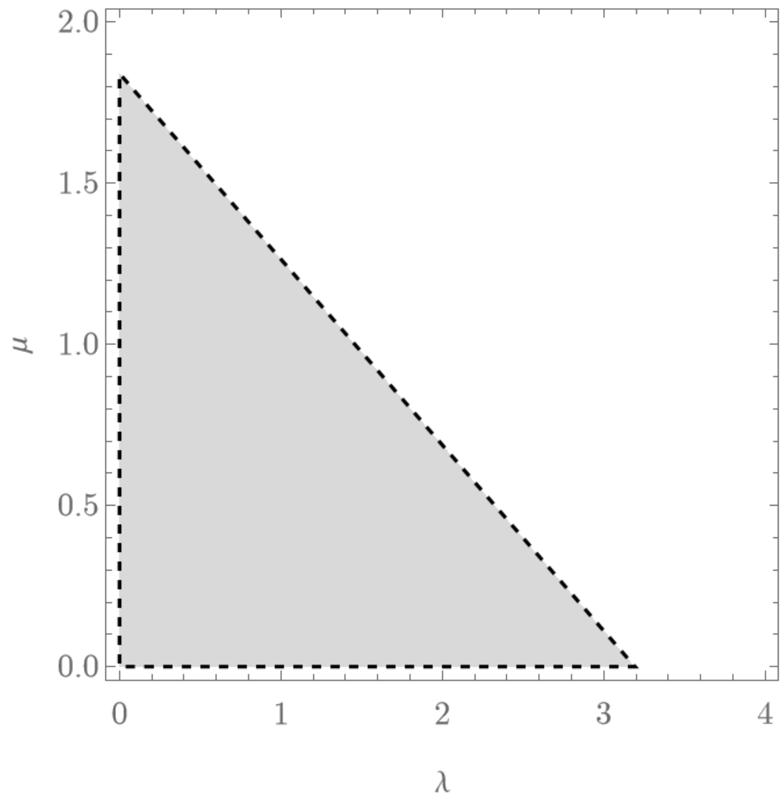

Fig. 4.3 The set of solutions for the parameters λ *and* μ

Let the accuracy be $\varepsilon = 10^{-4}$. Since the condition is fulfilled on the fifth iteration

$$\max_{\mathbf{x}\in\bar{\Omega}}(w^{(5)}(\mathbf{x}) - v^{(5)}(\mathbf{x})) = 0.285\cdot 10^{-3},$$

and with the accuracy $0.57\cdot 10^{-4}$, $u^*(\mathbf{x}) \approx u^{(5)}(\mathbf{x}) = \dfrac{w^{(5)}(\mathbf{x}) + v^{(5)}(\mathbf{x})}{2}$ can be chosen.

In this case $\left\|u^{(5)}\right\| = 0.1621$. The two-sided nature of the convergence of successive approximations is illustrated in Figure 4.4, where the graphs of the upper (solid line) and lower (dashed line) approximations to the solution at $x_2 = 0$ are shown. The contour lines and the surface of the approximate solution $u^{(5)}(\mathbf{x})$ are shown in Figures 4.5 and 4.6, respectively.

A computational experiment has been performed for different values of λ. Table 1 presents the value of the approximate solution norm of the problem (3.1) – (3.3) depending on the parameters λ and μ.

Table 1. The value of the approximate solution norm depending on the problem parameters

ζ	$\lambda_{\max}(\zeta)$	λ	$\mu_{\max}(\lambda)$	μ	$\|u\|$
3	3.1929	0	1.8403	1.80	0.1389
		1	1.2639	1.20	0.1545
		2	0.6875	0.50	0.1621
		3	0.1112	0.10	0.1896

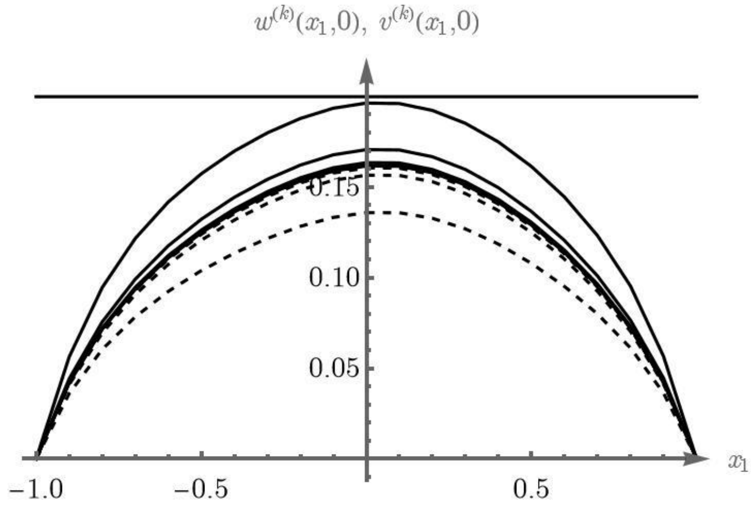

Fig. 4.4 Cross-sections of upper and lower approximations to the solution at $x_2 = 0$

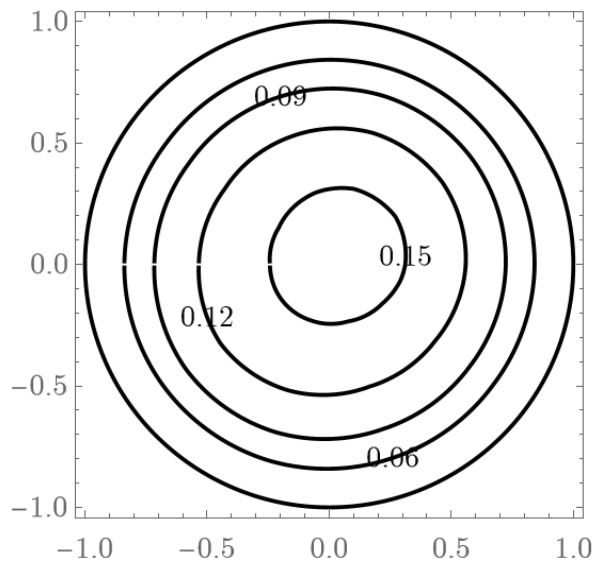

Fig. 4.5 The contour lines of the approximate solution

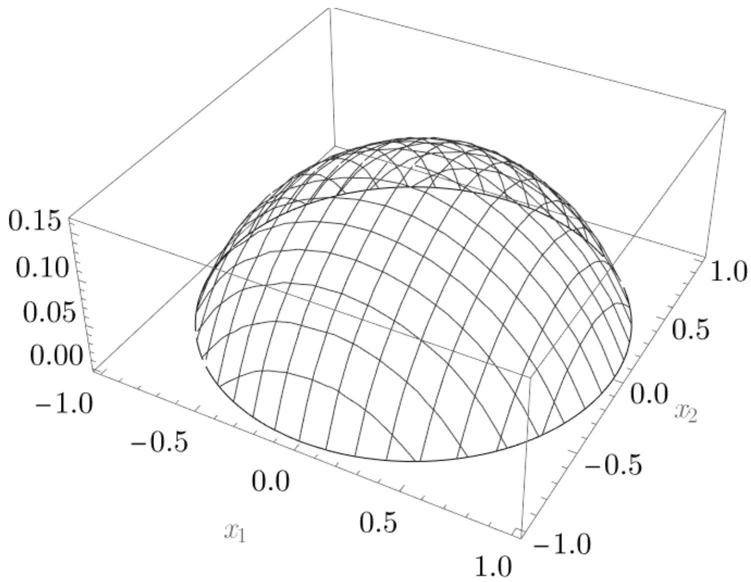

Fig. 4.6 The surface of the approximate solution

The analysis of the results of computational experiment shows that for a fixed value ζ, the parameter value $\mu_{\max}(\lambda)$ decreases with the increase of λ, and the norm $\|u\|$ increases. Those results define stable modes of operation for electrostatic NEMS. It should be noted that the choice of an unevenly and asymmetrically distributed external pressure leads to a violation of the radial symmetry of the solution.

5. Conclusions

The paper considers a generalized mathematical model of electrostatic nanoelectromechanical systems, which takes into account the Coulomb and Casimir forces and the external pressure. The method of two-sided approximations based on the usage of Green's function has been applied for the first time to the analysis of the proposed mathematical model, therefore, making it possible to obtain both the conditions for the existence of a unique positive solution of the problem and the two-sided convergence of successive approximations to it. The computational experiment conducted for the test values of the parameters has shown that this novel method is effective and can be used to study the parameters and modes of operation of real NEMS.

The research into Casimir's repulsive forces is a promising area of study due to the potential application in developing ultra-low friction and eliminating pull-in instability. The experiments have shown that repulsion occurs in one specific configuration; a plate and a sphere, instead of two flat plates, have been used in the model, as it is difficult to arrange the plates in parallel at such a small distance.

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Як цитувати: О.С. Кончаковська, М.В. Сидоров, “ Використання методу двобічних наближень для чисельного дослідження наноелектромеханічних систем під дією сили Казимира”. *Вісник Харківського національного університету імені В. Н. Каразіна, сер. «Математичне моделювання. Інформаційні технології. Автоматизовані системи управління»*. 2022. вип. 56. С. 21-34. https://doi.org/10.26565/2304-6201-2022-56-02

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Використання методу двобічних наближень для чисельного дослідження наноелектромеханічних систем під дією сили Казимира

О.С. Кончаковська, М.В. Сидоров

Актуальність. Розглянуто питання побудови методу двобічних наближень знаходження додатного розв'язку нелінійної крайової задачі, що моделює електростатичну наноелектромеханічну систему під дією зовнішнього тиску. Наведена математична модель враховує вплив сил Казимира як додаткову силу тяжіння між компонентами наносистем. Особливістю таких систем є нелінійне явище нестабільності відхиляння, яке виникає внаслідок взаємодії струмопровідних пластин під дією критичної електричної напруги. Це явище значно обмежує діапазон стійких станів системи та характерне для багатьох нанопристроїв, зокрема, акселерометрів, перемикачів, мікродзеркал та мікрорезонаторів тощо. Для дослідження стійких станів наноелектромеханічних систем запропоновано дослідити параметри моделі та отримати їх оцінки.

Мета. Користуючись методами теорії нелінійних операторів у напівупорядкованих банахових просторах розробити метод двобічних наближень розв'язання поставленої задачі.

Методи дослідження. Нелінійне еліптичне рівняння, що моделює роботу електростатичної наноелектромеханічної системи за допомогою методу функцій Гріна замінюється еквівалентним інтегральним рівнянням Гаммерштейна. Зазначене інтегральне рівняння розглядається як нелінійне операторне рівняння з монотонним оператором у просторі неперервних функцій, напівупорядкованому за допомогою конуса невід'ємних функцій. Отримано умови існування єдиного додатного розв'язку розглядуваної задачі та двобічної збіжності до нього послідовних наближень.

Результати. Розроблений метод програмно реалізовано та досліджено при розв'язанні тестових задач. Результати обчислювального експерименту наведено у вигляді графічної та табличної інформації.

Висновки. Проведені обчислювальні експерименти підтвердили ефективність розробленого метода і можуть бути використанні на практиці при розв'язання задач математичного моделювання нелінійних процесів у мікро- та наноелектромеханічних системах. Перспективи подальших досліджень можуть полягати у застосуванні методу двобічних наближень для моделей наноелектромеханічних системах з відштовхуючими силами Казимира.

Ключові слова: *метод двобічних наближень, функція Гріна, інваріантний конусний відрізок, монотонний оператор, наноелектромеханічна система, зовнішній тиск, сили Казимира.*

Надійшла у першій редакції 18.01.2022, в останній - 11.09.2022.