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Flow modelling in a straight rigid-walled duct with two rectangular axisymmetric narrowings. Part 2. An alternative approach

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A second-order technique is suggested to study fluid motion in a two-dimensional hard-walled duct with two abrupt constrictions. In this technique, the governing relationships are integrated via their rewriting in a non-dimensional form, deriving their integral analogues, performing a discretization of the derived integral relationships, simplifying the obtained (after making the discretization) coupled non-linear algebraic equations, and final solving the resulting (after making the simplification) uncoupled linear ones. The discretization consists of the spatial and temporal parts. The first of them is performed with the use of the total variation diminishing scheme and the two-point scheme of discretization of the spatial derivatives, whereas the second one is made on the basis of the implicit three-point non-symmetric backward differencing scheme. The noted uncoupled linear algebraic equations are solved by an appropriate iterative method.

Keywords: *fluid motion, plane duct, abrupt constriction, technique.*

Моделювання течії у прямому жорсткостінному каналі з двома прямокутними осесиметричними звуженнями. Частина 2. Альтернативний підхід

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Запропоновано аналітично-чисельний метод другого порядку точності, котрий дозволяє вивчати рух рідини у двовимірному жорсткостінному каналі з двома послідовними жорсткими осесиметричними обривними звуженнями. Він складається з п'яти основних етапів. На першому з них вибираються відповідні масштаби задачі, на основі яких проводиться безрозмірювання співвідношень, що описують рух рідини у досліджуваному каналі. Далі (другий етап) виводяться інтегральні аналоги цих безрозмірних співвідношень і виконується їх дискретизація (третій етап). На четвертому етапі зв'язані нелінійні алгебраїчні рівняння для швидкості і тиску, одержані у результаті проведення зазначеної дискретизації, зводяться до відповідних незалежних лінійних. Для цього приймаються фізично обґрунтовані припущення, виконуються відповідні математичні операції, а також застосовується процедура знаходження та узгодження між собою послідовних наближень шуканих величин. При цьому кількість наближень визначається необхідною точністю розв'язку. І на останньому (п'ятому) етапі вибирається метод розв'язування вказаних лінійних рівнянь. Зазначена вище дискретизація складається із просторової та часової частин. Перша частина виконується на основі використання total variation diminishing схеми, а також двоточкової схеми дискретизації просторових похідних. При проведенні ж другої частини дискретизації застосовується неявна триточкова несиметрична схема з різницями назад. Що стосується методу розв'язування вказаних вище лінійних рівнянь, то це – відповідний ітераційний метод, який послідовно використовує методи відкладеної корекції та спряжених градієнтів, а також солвери ICCG (для симетричних матриць) та Bi-CGSTAB (для асиметричних матриць).

Ключові слова: *рух рідини, плоский канал, обривне звуження, метод.*

Моделирование течения в прямом жесткостенном канале с двумя прямоугольными осесимметричными сужениями. Часть 2. Альтернативный подход

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Предложен метод второго порядка точности, позволяющий изучать движение жидкости в двухмерном жесткостенном канале с двумя обрывными сужениями. В этом методе соотношения, описывающие движение жидкости, интегрируются путем их обезразмеривания, получения их интегральных аналогов, дискретизации этих аналогов, упрощения полученных при этом связанных нелинейных алгебраических уравнений, и последующего решения независимых линейных алгебраических уравнений, полученных в результате указанного упрощения. Дискретизация состоит из пространственной и временной частей. Первая часть выполняется с использованием total variation diminishing схемы, а также двухточечной схемы дискретизации пространственных производных. При проведении же второй части дискретизации применяется неявная трехточечная несимметричная схема с разностями назад. Указанные линейные алгебраические уравнения решаются соответствующим итерационным методом.

Ключевые слова: *движение жидкости, плоский канал, обрывное сужение, метод.*

1. Introduction

Study of light and/or heavy fluid motions in ducts is an actual problem in gas and oil industry, chemical industry, aircraft and car industry, architecture, medicine, municipal economy, etc. Among others, here a significant interest is related to studying flow behavior in ducts with local constrictions, such as wall deposits, welding joints, stenoses, etc. That is explained by the fact that such irregularities in the duct geometry cause local changes in the flow structure and/or character, as well as changes in the flow local and integral characteristics, etc. Those changes can result in the corresponding consequences not only in the vicinity of, but also far from the irregularities (see, for instance, [1-8]).

As analysis of the scientific literature shows, study of fluid motions in ducts with local constrictions has been paid much attention to. In those studies, straight hard-walled ducts and their constrictions of the simplest geometries were considered. The basic flow (i.e., the flow upstream of a (first) constriction) was laminar, axisymmetric and steady. As for fluids, they were assumed to be homogeneous, incompressible and Newtonian[1]. These allowed one, on the one hand, to study (within the framework of appropriate models chosen and with acceptable accuracy) the influence of the basic parameters of duct, its constriction and the basic flow on the flow not only near but also far downstream of the constriction(s), and, on the other hand, simplify significantly solutions to the corresponding problems of interest (see, for example, [1-10]).

Among the results obtained in those studies, numerical methods, which have been developed to investigate flows around duct constrictions, are of a particular interest. One of the latest of them was presented in [11]. It has been devised to solve a problem of flow in a straight hard-walled two-dimensional duct with two rigid constrictions of a rectangular axisymmetric shape. That method allows one to study the fluid motion in the noted duct in the variables stream function-vorticity-pressure, has high stability of a solution and a second order of accuracy in the spatial co-ordinates. However, its first order of accuracy in the temporal coordinate should apparently stimulate researchers either to develop more accurate appropriate computational techniques or to improve the method in such a way to make its temporal accuracy higher.

In this study, an alternative technique is presented to solve the same problem. This technique uses the fluid velocity and pressure as the basic variables, has nearly the same stability of a solution, the same order of accuracy in the spatial coordinates and higher (i.e., the second) order of accuracy in the temporal coordinate. However, due to the large amount of mathematical operations used in this method, it needs more computational time to obtain a solution compared to the above one.

The paper consists of an introduction (Section 1), three main sections and a list of references. The formulation of the problem to be solved is made and the corresponding governing equations, as well as the boundary and initial conditions are given in Section 2. Then (in Section 3) the solution method to the formulated problem is described in detail. Finally, the conclusions of the investigation are formulated in Section 4 and a list of the literature cited is given.

2. Formulation of the problem

The formulation of the problem to be solved, as well as the corresponding governing equations and the initial and boundary conditions are given in [11]. Therefore, here we shall only briefly remind all of them.

[1] The other types of ducts, their constrictions, fluids and the basic flow are not considered in this paper, because they were studied much less intensively compared with the noted ones.

An infinite straight hard-walled plane channel of width D, having two consecutive rigid constrictions of a rectangular axisymmetric shape, is considered (Fig.1). The constrictions are situated at the distance l_{12} from one another, and have the diameters d_i and the lengths l_i $(i=1,2)$. In this channel, a viscous homogeneous Newtonian fluid moves. The fluid has mass density ρ and kinematic viscosity ν. Its flow is characterized by a small Mach number and the rate q per unit depth of the channel. In addition, the flow upstream of the first constriction (i.e., the basic flow) is steady and laminar. It is necessary to study the flow around the constrictions.

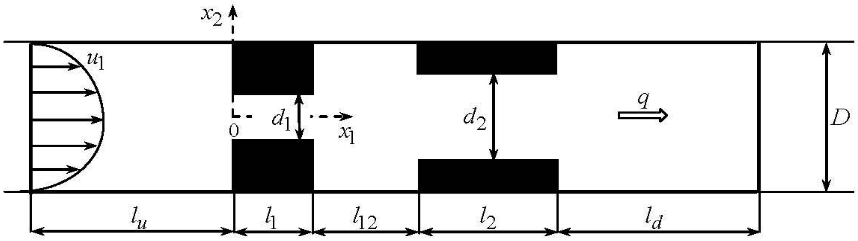

Fig. 1. Geometry of the problem and the corresponding computational domain.

The fluid motion in the duct is governed by the two-dimensional momentum equation, viz.

$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right),\qquad i=1,2, \tag{2.1}$$

and the continuity equation, viz.

$$\partial u_i/\partial x_i=0\,. \tag{2.2}$$

The boundary conditions consist in absence of the fluid motion at the channel wall, S_{ch}, and on both constrictions, S_j, ($j=1,2$), viz.

$$u_i\big|_{S_{ch}}=0,\qquad u_i\big|_{S_j}=0\,,\qquad i=1,2\,. \tag{2.3}$$

Apart from these, the flow rate q must be invariable along the channel axis, viz.

$$\partial q/\partial x_1=0\,,\qquad q=U_aD\,, \tag{2.4}$$

and the parabolic velocity profile is specified outside the disturbed flow region due to the constrictions[2], viz.

$$u_1\big|_{x_1=-l_u,l_1+l_{12}+l_2+l_d}=U_0\left(1-4x_2^2/D^2\right),\qquad u_2\big|_{x_1=-l_u,l_1+l_{12}+l_2+l_d}=0\,. \tag{2.5}$$

Relationships (2.4) are due to mass conservation in the channel, whereas conditions (2.5) are explained by the consideration of a laminar basic flow in the problem.

As for the pressure p, it is assumed to be constant both sufficiently far upstream of the first constriction, viz.

$$p\big|_{x_1=-l_u}=\mathrm{const}_u=p_u\,,$$

and far downstream of the second one, viz.

$$p\big|_{x_1=l_1+l_{12}+l_2+l_d}=\mathrm{const}_d=p_d\,.$$

In addition, the difference between p_u and p_d, $\Delta p=p_u-p_d=\mathrm{const}>0$, should ensure the existence of the given laminar regime of the basic flow. Also, without loss of generality, the pressure p_d is taken to be zero[3], and the magnitude p_u (which now is equal to Δp), like the pressure in the whole duct, needs to be found.

[2] This is the region before the constrictions, where the flow is still undisturbed by them, and far behind them, where the flow is already undisturbed (i.e., where the flow disturbances disappear, and it becomes like the basic one)).

[3] A choice of the value of p_d always can be compensated for by the choice of the corresponding value of p_u in such a way that the corresponding pressure drop Δp (which governs fluid motion in the duct) remains unchangeable.

Apart from these, the normal pressure derivative is zero on the rigid walls of the channel and both its constrictions, viz.

$$\left(\partial p / \partial \mathbf{n}\right)_{S_{ch},S_j} = 0 , \qquad j = 1,2 . \tag{2.6}$$

Regarding the initial conditions, they are in absence of fluid motion in the channel at the time instant $t = 0$ [11], viz.

$$u_i\big|_{t=0} = 0 ; \qquad p\big|_{t=0} = 0 . \tag{2.7}$$

In relationships (2.1)-(2.7) x_1, x_2, x_3 are the rectangular Cartesian coordinates shown in Fig. 1 (here the axis x_3 is normal to the plane x_1x_2 and directed to us); t the time; u_i the local fluid velocities in the directions x_i; U_0 and

$$U_a = \frac{1}{D}\int_{-D/2}^{D/2} u_1\big|_{x_1=-l_u, l_1+l_{12}+l_2+l_d}\, \mathrm{d}x_2 = \frac{2}{3}U_0$$

the maximum and averaged (over the duct cross-section) basic flow velocities, respectively; and the values of the distances l_u and l_d are given in Subsection 3.1. In addition, hereinafter the vector $\mathbf{n}$ denotes the outward unit normal to appropriate surface, and a summation on repeated indices is assumed throughout the paper.

3. Solution method

A solution to the problem formulated in the previous section consists of the six consecutive steps. More specifically, initially a computational domain is chosen and divided into appropriate small volumes. This is followed by rewriting the relationships of concern (presented in Section 2) in a non-dimensional form. Then integral analogues of the obtained non-dimensional relationships are derived by their integrating over the indicated small volumes. After that an appropriate discretization of the derived integral relationships is performed. Finally, the algebraic equations, which are obtained after making the discretization, are simplified in an appropriate manner and solved. Let one consider each of these steps separately.

3.1. Computational domain and non-dimensional relationships

The domain, in which a solution to the formulated problem should be found, is shown in Fig. 1. It is restricted by the duct sections $x_1 = -l_u$, $x_1 = l_1 + l_{12} + l_2 + l_d$ and $x_3 = x_{3a}$, $x_3 = x_{3a} + \mathrm{d}x_3$ (where $\mathrm{d}x_3 << 1$ and x_{3a} is the arbitrary value of the coordinate x_3). Herewith the left boundary of the domain, $x_1 = -l_u$, is taken upstream of the first constriction, where the flow is still undisturbed by it, and the right boundary, $x_1 = l_1 + l_{12} + l_2 + l_d$, behind the second constriction, where the flow disturbances already disappear, and the flow redevelops into the basic state at $x_1 = -l_u$. As for the distances l_u and l_d, for the basic flow velocities considered in this study[4], their values should vary in the ranges [2-4, 6, 7, 11]

$$l_u \le 0.5D , \qquad l_d \le 12D . \tag{3.1}$$

The chosen computational domain is divided into the small volumes V_{nm} by the duct cross and axial sections, $x_1 = x_{1n}$ and $x_2 = x_{2m}$ (where $x_{1n} = x_{1(n-1)} + \mathrm{d}x_1$, $\mathrm{d}x_1 << 1$ and $x_{2m} = x_{2(m-1)} + \mathrm{d}x_2$, $\mathrm{d}x_2 << 1$), as shown in Fig. 2. Herewith, in order to have a smooth velocity profile in an arbitrary duct cross-section, the steps $\mathrm{d}x_1$ and $\mathrm{d}x_2$ are reduced in an appropriate manner as one approaches either the duct or constrictions' walls.

Regarding scaling factors of this problem, these are the channel width D to be used as the length scale, the cross-sectionally averaged basic flow velocity, $U_a = q / D$, as the velocity scale, the ratio D / U_a as the time scale, and the double mean dynamic pressure of the basic flow, ρU_a^2, as the pressure scale.

[4] Since in this paper the question is about a laminar basic flow, its velocity should not exceed the value at which the Reynolds number (which is based on this velocity and the duct width) reaches the critical value of 2000 [1-4, 6-8, 11].

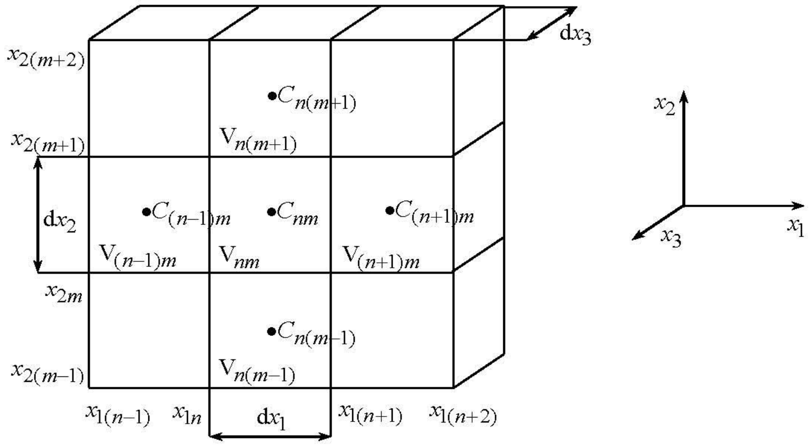

Fig. 2. A scheme of fragmentation of the computational domain into small volumes.

For these scaling factors, the non-dimensional forms of the momentum equations (2.1) and the continuity equation (2.2) are as follows, respectively

$$\frac{\partial U_i}{\partial T}+U_j\frac{\partial U_i}{\partial X_j}=-\frac{\partial P}{\partial X_i}+\frac{1}{\mathrm{Re}_D}\frac{\partial}{\partial X_j}\left(\frac{\partial U_i}{\partial X_j}\right), \qquad i=1,2, \tag{3.2}$$

$$\partial U_i/\partial X_i=0, \tag{3.3}$$

and the non-dimensional analogues of conditions (2.3)-(2.7) and (3.1) are written as

$$U_i\big|_{S_{ch},S_j}=0, \quad U_i\big|_{T=0}=0, \quad \partial Q/\partial X_1=0, \quad Q=1, \qquad i,j=1,2,$$

$$U_1\big|_{X_1=-L_u,L_1+L_{12}+L_2+L_d}=1.5\left(1-4X_2^2\right), \quad U_2\big|_{X_1=-L_u,L_1+L_{12}+L_2+L_d}=0, \tag{3.4}$$

$$P_d=P\big|_{X_1=L_1+L_{12}+L_2+L_d}=0, \quad \left(\partial P/\partial \mathbf{n}\right)_{S_{ch},S_j}=0, \quad P\big|_{T=0}=0, \quad L_u\le 0.5, \quad L_d\le 12.$$

In relationships (3.2)-(3.4) $U_i=u_i/U_a$ are the dimensionless fluid velocity components in the directions x_i; $X_i=x_i/D$ the dimensionless co-ordinates x_i $(i=1,2,3)$; $T=tU_a/D$ the dimensionless time; $P=p/(\rho U_a^2)$ the non-dimensional pressure; $\mathrm{Re}_D=U_aD/\nu$ the Reynolds number of the cross-sectionally averaged basic flow; $Q=q/(U_aD)$ the non-dimensional flow rate in the duct per its unit depth; $P_d=p_d/(\rho U_a^2)$ the non-dimensional pressure p_d; $L_u=l_u/D$ and $L_d=l_d/D$ the dimensionless distances l_u and l_d; and $L_1=l_1/D$, $L_{12}=l_{12}/D$ and $L_2=l_2/D$ the dimensionless lengths l_1, l_{12} and l_2, respectively.

3.2. Integral relationships and their discretization

Integral equations and their discrete analogues

Integral analogues of equations (3.2) and (3.3) are obtained by their integrating over the control volumes[5] V_{nm}. It gives

$$\frac{\partial}{\partial T}\iiint\limits_{\mathrm{V}_{nm}}U_i\mathrm{d}V+\iiint\limits_{\mathrm{V}_{nm}}U_j\frac{\partial U_i}{\partial X_j}\mathrm{d}V=-\iiint\limits_{\mathrm{V}_{nm}}\frac{\partial P}{\partial X_i}\mathrm{d}V+\frac{1}{\mathrm{Re}_D}\iiint\limits_{\mathrm{V}_{nm}}\frac{\partial}{\partial X_j}\left(\frac{\partial U_i}{\partial X_j}\right)\mathrm{d}V, \tag{3.5}$$

$$\iiint\limits_{\mathrm{V}_{nm}}\partial U_i/\partial X_i\mathrm{d}V=0. \tag{3.6}$$

The application (wherever possible) of the Gauss theorem, viz.

$$\iiint\limits_{\mathrm{V}_{nm}}\nabla\cdot\mathbf{g}\mathrm{d}V=\iint\limits_{S_{nm}}\mathbf{g}\cdot\mathrm{d}\mathbf{S}, \qquad \iiint\limits_{\mathrm{V}_{nm}}\nabla\gamma\mathrm{d}V=\iint\limits_{S_{nm}}\gamma\mathrm{d}\mathbf{S}, \qquad \mathrm{d}\mathbf{S}=\mathbf{n}\mathrm{d}S \tag{3.7}$$

[5] In making this operation, the appropriate conservation laws take place in each volume V_{nm}.

to the terms of equations (3.5) and (3.6), and/or the expansion (wherever needed) of their integrands (which, for the convenience, are denoted by $\mathbf{f}(\mathbf{r})$) into the Taylor series around the mass center, C_{nm}, of the volume[6] V_{nm} (see Fig. 2), viz.

$$\mathbf{f}(\mathbf{r}) = \mathbf{f}(\mathbf{r}_{c_{nm}}) + \nabla(\mathbf{f})\big|_{\mathbf{r}=\mathbf{r}_{c_{nm}}} \cdot (\mathbf{r} - \mathbf{r}_{c_{nm}}) + O\Big((\mathbf{r} - \mathbf{r}_{c_{nm}})^2\Big) ,$$

$$\mathbf{r}_{c_{nm}} = \iiint\limits_{\mathrm{V}_{nm}} \mathbf{r}\mathrm{d}V / |\mathrm{V}_{nm}| , \ \left|\mathbf{r} - \mathbf{r}_{c_{nm}}\right| << 1 , \quad \mathbf{r} \in \mathrm{V}_{nm} , \ \mathbf{r} = X_i \mathbf{e}_i , \ \mathbf{r}_{c_{nm}} = X_{ic_{nm}} \mathbf{e}_i , \qquad (3.8)$$

$$X_{1c_{nm}} = X_{1n} + \mathrm{d}X_1 / 2 , \ \ X_{2c_{nm}} = X_{2m} + \mathrm{d}X_2 / 2 , \ \ X_{3c_{nm}} = X_{3a} + \mathrm{d}X_3 / 2 ,$$

further use of the first two terms of these series, viz.

$$\mathbf{f}(\mathbf{r}) = \mathbf{f}(\mathbf{r}_{c_{nm}}) + \nabla(\mathbf{f})\big|_{\mathbf{r}=\mathbf{r}_{c_{nm}}} \cdot (\mathbf{r} - \mathbf{r}_{c_{nm}}) \qquad (3.9)$$

and making appropriate discretization of the temporal and spatial derivatives, as well as the application (wherever necessary) of the TVD-scheme[7] [12, 13] allows one to proceed to considering the discrete analogues of relationships (3.5) and (3.6), which have the second order of accuracy (here ∇ is the gradient; S_{nm} the lateral surface of the volume V_{nm}; $\mathbf{r}$ and $\mathbf{r}_{c_{nm}}$ the position vectors of an arbitrary point in the region V_{nm} and its mass center C_{nm}, respectively; $|\mathrm{V}_{nm}| = \mathrm{d}X_1 \mathrm{d}X_2 \mathrm{d}X_3$ the volume of the region V_{nm}; the point in the Taylor series indicates scalar product of the corresponding magnitudes and $\mathbf{e}_i$ the unit directivity vector of the axis X_i).

Indeed, taking account of the linear representation (3.9) of the integrand U_i in the first (unsteady) term of equation (3.5), viz.

$$U_i(\mathbf{r}) = U_i(\mathbf{r}_{c_{nm}}) + \nabla(U_i)\big|_{\mathbf{r}=\mathbf{r}_{c_{nm}}} \cdot (\mathbf{r} - \mathbf{r}_{c_{nm}}) ,$$

as well as the use of the integral

$$\iiint\limits_{\mathrm{V}_{nm}} \left(\mathbf{r} - \mathbf{r}_{c_{nm}}\right) \mathrm{d}V = 0 \qquad (3.10)$$

(that follows from (3.8)) results in significant simplification of the term, viz.

$$\frac{\partial}{\partial T} \iiint\limits_{\mathrm{V}_{nm}} U_i \mathrm{d}V = \frac{\partial U_i(\mathbf{r}_{c_{nm}}, T)}{\partial T} |\mathrm{V}_{nm}| . \qquad (3.11)$$

After that the implicit three-point non-symmetric backward differencing scheme[8] [8, 12, 13], viz.

$$\frac{\partial \mathbf{f}(\mathbf{r}_{c_{nm}}, T)}{\partial T} = \frac{1.5\mathbf{f}^{k}_{c_{nm}} - 2\mathbf{f}^{k-1}_{c_{nm}} + 0.5\mathbf{f}^{k-2}_{c_{nm}}}{\Delta T} . \qquad (3.12)$$

is applied to make discretization of the temporal derivatives in (3.11). Here ΔT is a small fixed time step, $\mathbf{f}^{k}_{c_{nm}}$ a value of the function $\mathbf{f}$ at the point C_{nm} at the instant of time $T = k\Delta T$ to be found, and $\mathbf{f}^{k-1}_{c_{nm}}$ and $\mathbf{f}^{k-2}_{c_{nm}}$ its known values at the same point found at the previous time moments

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[6] Since the fluid in the duct is homogeneous (see the problem formulation), the mass center of the volume V_{nm} coincides with its geometrical center. The analogous situation is with the mass center of each side face of the volume V_{nm}.

[7] This is an abbreviation of the Total Variation Diminishing. The TVD-scheme provides satisfactory accuracy (that is higher than the first order) and finiteness of a solution [12, 13].

[8] This scheme has a second order of accuracy and is applied when the computational grid (see Fig.2) is immovable (this provides fixed positions of the mass centers of the corresponding small integration volumes) [12, 13].

$T=(k-1)\Delta T$ and $T=(k-2)\Delta T$, respectively[9].

As for the second (convective) term of equation (3.5), initially its integrand is rewritten (based on the continuity equation (3.3)) in the equivalent form, viz.

$$U_j\frac{\partial U_i}{\partial X_j}=\frac{\partial U_iU_j}{\partial X_j}=\nabla\cdot(\mathbf{U}U_i),\qquad \mathbf{U}=U_i\mathbf{e}_i,\qquad U_3=0.$$

After that the first form of the Gauss theorem (3.7) is applied to the modified convective term, viz.

$$\iiint\limits_{V_{nm}}U_j\frac{\partial U_i}{\partial X_j}\mathrm{d}V=\iiint\limits_{V_{nm}}\left(\nabla\cdot(\mathbf{U}U_i)\right)\mathrm{d}V=\iint\limits_{S_{nm}}\left(\mathbf{U}U_i\right)\cdot\mathbf{n}\mathrm{d}S. \tag{3.13}$$

Since the side surface S_{nm} of the volume V_{nm} consists of the six flat faces $S_{nm}^{(i)}$, which have the outward unit normals $\mathbf{n}_i$ ($i=1,...,6$, Fig.3), viz.

$$S_{nm}=\sum_{i=1}^{6}S_{nm}^{(i)},\qquad \mathbf{n}_1=\mathbf{e}_1,\ \ \mathbf{n}_2=-\mathbf{e}_1,\ \ \mathbf{n}_3=\mathbf{e}_2,\ \ \mathbf{n}_4=-\mathbf{e}_2,\ \ \mathbf{n}_5=\mathbf{e}_3,\ \ \mathbf{n}_6=-\mathbf{e}_3,$$

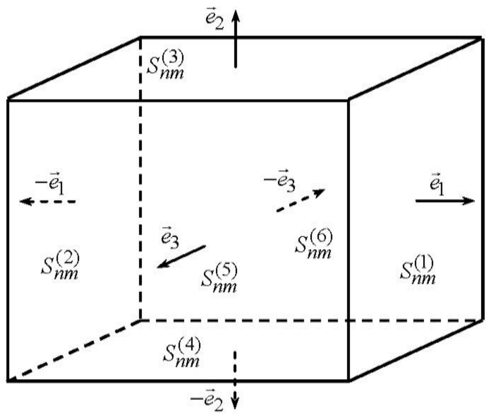

Fig. 3. The small volume V_{nm}*, its side faces* $S_{nm}^{(i)}$ *and their outward unit normals* $\mathbf{n}_i$ *(*$i=1,...,6$*).*

and the integrand in the surface integral in (3.13) can be represented (to small values of the second order of accuracy) by the linear part of its Taylor series around the mass center[6] $C_{nm}^{(i)}$ of the face $S_{nm}^{(i)}$, viz.

$$\mathbf{f}(\mathbf{r})=\mathbf{f}(\mathbf{r}_{c_{nm}^{(i)}})+\nabla(\mathbf{f})\big|_{\mathbf{r}=\mathbf{r}_{c_{nm}^{(i)}}}\cdot(\mathbf{r}-\mathbf{r}_{c_{nm}^{(i)}}),\quad \mathbf{r}_{c_{nm}^{(i)}}=\iint\limits_{S_{nm}^{(i)}}\mathbf{r}\mathrm{d}S\Big/\left|S_{nm}^{(i)}\right|,\quad \mathbf{r}=X_i\mathbf{e}_i,\quad \mathbf{r}_{c_{nm}^{(i)}}=X_{jc_{nm}^{(i)}}\vec{\mathbf{e}}_j, \tag{3.14}$$

this allows one to simplify relationship (3.13) significantly, viz.

$$\iiint\limits_{V_{nm}}U_j\frac{\partial U_i}{\partial X_j}\mathrm{d}V=\sum_{j=1}^{6}\iint\limits_{S_{nm}^{(j)}}(\mathbf{U}U_i)\cdot\mathbf{n}_j\mathrm{d}S=\sum_{j=1}^{4}(\mathbf{U}U_i)\big|_{\mathbf{r}=\mathbf{r}_{c_{nm}^{(j)}}}\cdot\mathbf{n}_j\left|S_{nm}^{(j)}\right|=\sum_{j=1}^{4}F_{nm}^{(j)}U_i(\mathbf{r}_{c_{nm}^{(j)}}) \tag{3.15}$$

(in (3.15) $\left|S_{nm}^{(j)}\right|$ is the area of the face $S_{nm}^{(j)}$ (see Fig. 3) and

$$F_{nm}^{(j)}=\mathbf{U}(\mathbf{r}_{c_{nm}^{(j)}})\cdot\mathbf{n}_j\left|S_{nm}^{(j)}\right|=U_i(\mathbf{r}_{c_{nm}^{(j)}})n_{ji}\left|S_{nm}^{(j)}\right| \tag{3.16}$$

the fluid flow across the face $S_{nm}^{(j)}$; in addition, here the integral

[9] They are computed at the indicated (previous) time stages, whereas at the initial time $T=0$ all the parameters of the problem are the known values.

$$\iint_{S_{nm}^{(j)}} \left(\mathbf{r} - \mathbf{r}_{c_{nm}^{(j)}}\right) \mathrm{d}S = 0$$

has been used that follows from (3.14) and is similar to (3.10)).

Further the values of the functions U_i $(i = 1, 2)$ at the mass centers $C_{nm}^{(j)}$ of the faces $S_{nm}^{(j)}$ in relationships (3.15), (3.16) are determined via the values of these functions at the mass centers of the control volumes having the joint faces $S_{nm}^{(j)}$ (see Fig. 2 and Fig. 3). For this purpose, the following TVD-scheme[7] is used [12, 13]

$$\mathbf{f}(\mathbf{r}_{c_{nm}^{(j)}}) = \mathbf{f}_1^{(j)} + \Phi\left(\mathbf{f}_2^{(j)} - \mathbf{f}_1^{(j)}\right), \tag{3.17}$$

Here $\mathbf{f}_1^{(j)}$ is the value of the function $\mathbf{f}$ at the point $C_{nm}^{(j)}$ found with the use of the backward differencing scheme which has the first order of accuracy, viz.

$$\mathbf{f}_1^{(j)} = \begin{cases} \mathbf{f}(\mathbf{r}_{c_{nm}}), & F_{nm}^{(j)} \geq 0, \\ \mathbf{f}(\mathbf{r}_{c_j}), & F_{nm}^{(j)} < 0, \end{cases}$$

the point C_j the mass center of the control volume having the joint face $S_{nm}^{(j)}$ with the volume V_{nm}, viz.

$$C_1 = C_{(n+1)m},\ C_2 = C_{(n-1)m},\ C_3 = C_{n(m+1)},\ C_4 = C_{n(m-1)},$$

$\mathbf{f}_2^{(j)}$ the value of the function $\mathbf{f}$ at the point $C_{nm}^{(j)}$ obtained on the basis of the central differencing scheme of the second order of accuracy, viz.

$$\mathbf{f}_2^{(j)} = \alpha_j \mathbf{f}(\mathbf{r}_{c_{nm}}) + (1 - \alpha_j)\mathbf{f}(\mathbf{r}_{c_j}),$$

with the coefficient α_j which is the ratio of the distances between the appropriate mass centers, viz.

$$\alpha_j = \left|\mathbf{r}_{c_{nm}^{(j)}} - \mathbf{r}_{c_j}\right| / \left|\mathbf{r}_{c_{nm}} - \mathbf{r}_{c_j}\right|,$$

and Φ the nonlinear flow restrictor, viz.

$$\Phi(\eta_j) = \max\left(0, \min\left(2\eta_j / \beta, 1\right)\right), \qquad \eta_j = \left|\mathbf{U}\left(\mathbf{r}_{c_{nm}^{(j)}}\right) - \mathbf{U}\left(\mathbf{r}_{c_{nm}}\right)\right| / \left|\mathbf{U}\left(\mathbf{r}_{c_j}\right) - \mathbf{U}\left(\mathbf{r}_{c_{nm}^{(j)}}\right)\right|,$$

where the coefficient β is chosen to be equal to[10] 0.5.

The simplification procedure for the third (gradient) term of equation (3.5) is based on application of the second form of the Gauss theorem (3.7) to it, viz.

$$\iiint_{\mathrm{V}_{nm}} \nabla P \mathrm{d}V = \iint_{S_{nm}} P\mathbf{n}\mathrm{d}S,$$

and further performing the operations with the surface integral which allowed one to proceed from (3.13) to (3.15). It gives

$$\iiint_{\mathrm{V}_{nm}} \frac{\partial P}{\partial X_i} \mathrm{d}V = \sum_{j=1}^{6} \iint_{S_{nm}^{(j)}} P n_{ji} \mathrm{d}S = \sum_{j=1}^{4} P(\mathbf{r}_{c_{nm}^{(j)}}) n_{ji} \left|S_{nm}^{(j)}\right|. \tag{3.18}$$

After that the pressure values at the points $C_{nm}^{(j)}$ in (3.18) are found with the use of the TVD-scheme (3.17).

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[10] In general, the parameter β can vary in the ranges $0 < \beta \leq 1$. Herewith the decrease of β corresponds to higher accuracy and lower stability of the computation, and vice versa, the computation accuracy decreases and the computation stability increases as β increases [8, 12-16].

Finally, the fourth (diffusive) term of equation (3.5) can be simplified after rewriting its integrand as

$$\frac{\partial}{\partial X_j}\left(\frac{\partial U_i}{\partial X_j}\right)=\nabla^2 U_i=\nabla\cdot\nabla U_i,$$

further application of the first form of the Gauss theorem (3.7), viz.

$$\iiint\limits_{V_{nm}}\frac{\partial}{\partial X_j}\left(\frac{\partial U_i}{\partial X_j}\right)\mathrm{d}V=\iiint\limits_{V_{nm}}\nabla\cdot\nabla U_i\mathrm{d}V=\iint\limits_{S_{nm}}\nabla U_i\cdot\mathbf{n}\mathrm{d}S$$

and making the operations with the surface integral which have been used in proceeding from (3.13) to (3.15). These result in

$$\iiint\limits_{V_{nm}}\frac{\partial}{\partial X_j}\left(\frac{\partial U_i}{\partial X_j}\right)\mathrm{d}V=\sum_{j=1}^{6}\iint\limits_{S_{nm}^{(j)}}\nabla U_i\cdot\mathbf{n}_j\mathrm{d}S=\sum_{j=1}^{4}\nabla U_i\Big|_{\mathbf{r}=\mathbf{r}_{c_{nm}^{(j)}}}\cdot\mathbf{n}_j\left|S_{nm}^{(j)}\right|. \tag{3.19}$$

After that the following discretization of the spatial derivatives $\nabla U_i\big|_{\mathbf{r}=\mathbf{r}_{c_{nm}^{(j)}}}$ is performed in (3.19)[6]

$$\nabla f\Big|_{\mathbf{r}=\mathbf{r}_{c_{nm}^{(j)}}}=\mathbf{e}_i\left(\partial f/\partial X_i\right)\Big|_{\mathbf{r}=\mathbf{r}_{c_{nm}^{(j)}}}, \tag{3.20}$$

$$\left(\partial f/\partial X_1\right)\Big|_{\mathbf{r}=\mathbf{r}_{c_{nm}^{(1)}}}=\left(f(\mathbf{r}_{c_{(n+1)m}})-f(\mathbf{r}_{c_{nm}})\right)/\,\mathrm{d}X_1,\quad \left(\partial f/\partial X_1\right)\Big|_{\mathbf{r}=\mathbf{r}_{c_{nm}^{(2)}}}=\left(f(\mathbf{r}_{c_{nm}})-f(\mathbf{r}_{c_{(n-1)m}})\right)/\,\mathrm{d}X_1,$$

$$\left(\partial f/\partial X_2\right)\Big|_{\mathbf{r}=\mathbf{r}_{c_{nm}^{(3)}}}=\left(f(\mathbf{r}_{c_{n(m+1)}})-f(\mathbf{r}_{c_{nm}})\right)/\,\mathrm{d}X_2,\quad \left(\partial f/\partial X_2\right)\Big|_{\mathbf{r}=\mathbf{r}_{c_{nm}^{(4)}}}=\left(f(\mathbf{r}_{c_{nm}})-f(\mathbf{r}_{c_{n(m-1)}})\right)/\,\mathrm{d}X_2.$$

The availability of relationships (3.11), (3.12), (3.15), (3.17)-(3.20) allows one to write a discrete form of the integral momentum equation (3.5):

$$\frac{1.5U_{ic_{nm}}^{k}-2U_{ic_{nm}}^{k-1}+0.5U_{ic_{nm}}^{k-2}}{\Delta T}\left|V_{nm}\right|+\sum_{j=1}^{4}F_{nm}^{(j)k}U_{ic_{nm}^{(j)}}^{k}-\frac{1}{\mathrm{Re}_D}\sum_{j=1}^{4}\nabla U_{ic_{nm}^{(j)}}^{k}\cdot\mathbf{n}_j\left|S_{nm}^{(j)}\right|=$$

$$=-\sum_{j=1}^{4}P_{c_{nm}^{(j)}}^{k}n_{ji}\left|S_{nm}^{(j)}\right|. \tag{3.21}$$

In (3.21) $F_{nm}^{(j)k}$ is the fluid flow across the face $S_{nm}^{(j)}$ at the instant of time $T=k\Delta T$, $P_{c_{nm}^{(j)}}^{k}$ the pressure at the point $C_{nm}^{(j)}$ at the time $k\Delta T$, the fluid velocity components at $T=k\Delta T$, $U_{ic_{nm}^{(j)}}^{k}$, are found on the basis of the scheme (3.17), and relationships (3.20) are used to compute the velocities' gradients $\nabla U_{ic_{nm}^{(j)}}^{k}$.

The right part of equation (3.21) also can be represented in the equivalent form, viz.

$$-\left(\partial P/\partial X_i\right)_{C_{nm}}^{k}\left|V_{nm}\right|. \tag{3.22}$$

It is obtained after expansion of the integrand in the gradient term in (3.5) into the Taylor series around the mass center C_{nm} of the volume V_{nm}, further considering only the first two terms of the series and using formula (3.10), viz.

$$\iiint\limits_{V_{nm}}\frac{\partial P}{\partial X_i}\mathrm{d}V=\frac{\partial P}{\partial X_i}\bigg|_{\mathbf{r}=\mathbf{r}_{c_{nm}},T=k\Delta T}\left|V_{nm}\right|.$$

As for the discrete form of the integral continuity equation (3.6), it looks as follows

$$\sum_{j=1}^{4}\mathbf{U}^k_{c^{(j)}_{nm}}\cdot\mathbf{n}_j\left|S^{(j)}_{nm}\right|=\sum_{j=1}^{4}U^k_{ic^{(j)}_{nm}}n_{ji}\left|S^{(j)}_{nm}\right|=\sum_{j=1}^{4}F^{(j)k}_{nm}=0\,. \tag{3.23}$$

Here the fluid velocities $U^k_{ic^{(j)}_{nm}}$ are computed with the use of the TVD-scheme (3.17).

Relationship (3.23) is derived by rewriting (3.6) as

$$\iiint\limits_{\mathrm{V}_{nm}}\nabla\cdot\mathbf{U}\mathrm{d}V=0\,, \tag{3.24}$$

applying the first form of the Gauss theorem (3.7) to (3.24), viz.

$$\iiint\limits_{\mathrm{V}_{nm}}\nabla\cdot\mathbf{U}\mathrm{d}V=\iint\limits_{S_{nm}}\mathbf{U}\cdot\mathbf{n}\mathrm{d}S=0\,,$$

subsequent making the operations with the surface integral which allowed one to proceed from (3.13) to (3.15), viz.

$$\iint\limits_{S_{nm}}\mathbf{U}\cdot\mathbf{n}\mathrm{d}S=\sum_{j=1}^{6}\iint\limits_{S^{(j)}_{nm}}\mathbf{U}\cdot\mathbf{n}_j\mathrm{d}S=\sum_{j=1}^{4}\mathbf{U}(\mathbf{r}_{c^{(j)}_{nm}})\cdot\mathbf{n}_j\left|S^{(j)}_{nm}\right|=0\,, \tag{3.25}$$

applying scheme (3.17) to the velocity $\mathbf{U}(\mathbf{r}_{c^{(j)}_{nm}})$ in (3.25) and taking account of relationship (3.16).

Discrete analogues of the boundary conditions and their application to equations (3.21), (3.23)

The discrete analogues of the boundary conditions for the velocity components and the flow rate from (3.4) have the following form

$$U^k_1\Big|_{X_1=-L_u,L_1+L_{12}+L_2+L_d}=1.5\left(1-4X_2^2\right),\qquad U^k_2\Big|_{X_1=-L_u,L_1+L_{12}+L_2+L_d}=0\,, \tag{3.26}$$

$$U^k_i\Big|_{S_{ch},S_j}=0\,,\quad \partial Q^k/\partial X_1=0\,,\quad Q^k=1\,,\quad L_u\le 0.5\,,\ L_d\le 12\,,\quad i,j=1,2\,.$$

They allow one to find the fluid flow, $F^{(j)k}_{nm}$, and the velocities' gradients, $\nabla U^k_{ic^{(j)}_{nm}}$, in equations (3.21) and (3.23) on the boundary of the computational domain. In fact, it follows from relationships (3.16) and (3.26) that (as expected) the fluid flow across the impenetrable walls of the duct and both constrictions is zero, viz.

$$F^{(j)k}_{nm}\Big|_{S_{ch},S_r}=\left(U^k_{ic^{(j)}_{nm}}n_{ji}\left|S^{(j)}_{nm}\right|\right)_{S_{ch},S_r}=0\,,\qquad r=1,2\,.$$

At the entrance and exit of the noted domain (which can be touched on by only the faces $S^{(2)}_{nm}$ and $S^{(1)}_{nm}$ of the volume V_{nm}, respectively (see Figs. 1 and 3)) one has

$$F^{(2)k}_{nm}\Big|_{X_1=-L_u}=-1.5\left(1-4X_2^2\right)\mathrm{d}X_2\mathrm{d}X_3\,,\qquad F^{(1)k}_{nm}\Big|_{X_1=L_1+L_{12}+L_2+L_d}=1.5\left(1-4X_2^2\right)\mathrm{d}X_2\mathrm{d}X_3\,,$$

$$F^{(3)k}_{nm}\Big|_{X_1=-L_u,L_1+L_{12}+L_2+L_d}=F^{(4)k}_{nm}\Big|_{X_1=-L_u,L_1+L_{12}+L_2+L_d}=0\,.$$

As for the gradients $\nabla U^k_{ic^{(j)}_{nm}}$, on the upper and lower walls of the duct (which can be touched on by only the faces $S^{(3)}_{nm}$ and $S^{(4)}_{nm}$, respectively) they are determined in the following way

$$\nabla U^k_{ic^{(3)}_{nm}}\Big|_{X_2=1/2}=-\mathbf{e}_2U^k_{ic_{nm}}/\mathrm{d}X_2\,,\qquad \nabla U^k_{ic^{(4)}_{nm}}\Big|_{X_2=-1/2}=\mathbf{e}_2U^k_{ic_{nm}}/\mathrm{d}X_2\,,$$

$$-L_u\le X_1\le 0\,,\quad L_1\le X_1\le L_1+L_{12}\,,\quad L_1+L_{12}+L_2\le X_1\le L_1+L_{12}+L_2+L_d\,.$$

At the entrance and exit of the computational domain the velocity gradients are as follows

$$\nabla U^k_{1c^{(2)}_{nm}}\Big|_{X_1=-L_u}=-12X_2\mathbf{e}_2\,,\qquad \nabla U^k_{1c^{(1)}_{nm}}\Big|_{X_1=L_1+L_{12}+L_2+L_d}=-12X_2\mathbf{e}_2\,,$$

$$\nabla U^k_{2c^{(2)}_{nm}}\Big|_{X_1=-L_u}=0\,,\qquad \nabla U^k_{2c^{(1)}_{nm}}\Big|_{X_1=L_1+L_{12}+L_2+L_d}=0\,.$$

Finally, on the constrictions' surfaces the magnitudes $\nabla U^k_{ic^{(j)}_{nm}}$ are written as

$$\nabla U^k_{ic^{(1)}_{nm}}\Big|_{\substack{X_1=0,D_1/2\le X_2\le 1/2,-1/2\le X_2\le -D_1/2\\ X_1=L_1+L_{12},D_2/2\le X_2\le 1/2,-1/2\le X_2\le -D_2/2}}=-\mathbf{e}_1U^k_{ic_{nm}}\,/\,\mathrm{d}X_1\,,$$

$$\nabla U^k_{ic^{(2)}_{nm}}\Big|_{\substack{X_1=L_1,D_1/2\le X_2\le 1/2,-1/2\le X_2\le -D_1/2\\ X_1=L_1+L_{12}+L_2,D_2/2\le X_2\le 1/2,-1/2\le X_2\le -D_2/2}}=\mathbf{e}_1U^k_{ic_{nm}}\,/\,\mathrm{d}X_1\,,$$

$$\nabla U^k_{ic^{(3)}_{nm}}\Big|_{0\le X_1\le L_1,X_2=D_1/2;L_1+L_{12}\le X_1\le L_1+L_{12}+L_2,X_2=D_2/2}=-\mathbf{e}_2U^k_{ic_{nm}}\,/\,\mathrm{d}X_2\,,$$

$$\nabla U^k_{ic^{(4)}_{nm}}\Big|_{0\le X_1\le L_1,X_2=-D_1/2;L_1+L_{12}\le X_1\le L_1+L_{12}+L_2,X_2=-D_2/2}=\mathbf{e}_2U^k_{ic_{nm}}\,/\,\mathrm{d}X_2$$

(here $D_1=d_1/D$, $D_2=d_2/D$, and also it was taken into account that the left sides of the constrictions can be touched on by only the face $S^{(1)}_{nm}$, the right sides – the face $S^{(2)}_{nm}$, the upper ones – the face $S^{(3)}_{nm}$, and the lower ones – the face $S^{(4)}_{nm}$ of the volume V_{nm}).

As for the discrete analogues of the boundary conditions for the pressure from relationships (3.4), they have the following form

$$P^k_{c^{(1)}_{nm}}\Big|_{X_1=L_1+L_{12}+L_2+L_d}=0\,,\qquad \left(\partial P^k/\partial X_2\right)_{S_{ch}}=0\,,\qquad \left(\partial P^k/\partial \mathbf{n}\right)_{S_j}=0\,,\qquad j=1,2\,.$$

Discrete analogues of the initial conditions and their application to equations (3.21), (3.23)

The discrete analogues of the initial conditions from relationships (3.4) are written as

$$U_i^{k=0}=0\,,\qquad P^{k=0}=0\,. \tag{3.27}$$

With their help one can determine the appropriate terms of equations (3.21), (3.23) at the initial instant of time in the computational domain. Indeed, one can see from relationships (3.16) and (3.27) that the fluid flow across each face $S^{(j)}_{nm}$ of the volume V_{nm} at the time instant $T=0$ is zero, viz.

$$F^{(j)k=0}_{nm}=0\,.$$

According to (3.20) and (3.27), the gradients $\nabla U^k_{ic^{(j)}_{nm}}$ at the indicated time are equal to zero too, viz.

$$\nabla U^{k=0}_{ic^{(j)}_{nm}}=0\,.$$

Also, based on conditions (3.27), the derivatives in (3.22) are zero at $T=0$, viz.

$$\left(\frac{\partial P}{\partial X_1}\right)^{k=0}_{C_{nm}}=\left(\frac{P^k_{c_{(n+1)m}}-P^k_{c_{(n-1)m}}}{2\mathrm{d}X_1}\right)_{k=0}=0\,,\qquad \left(\frac{\partial P}{\partial X_2}\right)^{k=0}_{C_{nm}}=\left(\frac{P^k_{c_{n(m+1)}}-P^k_{c_{n(m-1)}}}{2\mathrm{d}X_2}\right)_{k=0}=0\,.$$

3.3. Solution method to equations (3.21), (3.23)

The system of equations (3.21), (3.23) is solved numerically. In making this, one has to deal with the two significant problems. The first of them is connected with nonlinearity of the discrete momentum equation (3.21)[11] which is used to find the velocity components. The second one is due to absence of equation for the pressure which is available in the right part of equation (3.21)[12].

In order to solve the first problem, in this work the flow $F_{nm}^{(j)k}$ is modified in the appropriate way. More specifically, initially the velocity components in it are replaced by their values found at the previous time step. After that the components are replaced by their known previous approximations. These replacements allow one to proceed from solving the coupled systems of non-linear algebraic equations to the corresponding uncoupled linear ones.

The second problem is solved via introducing the pressure in the discrete continuity equation (3.23) and subsequent agreeing the velocity and the pressure with one another when making the noted modification of the flow $F_{nm}^{(j)k}$. The velocity and pressure values, which are obtained in making this, are corrected at each step by performing appropriate operations. Let one demonstrate the above-said in more detail.

Equations for the velocity and the pressure

If one formally solves relationship (3.21) with respect to the velocity components, one obtains the equation whose generalized form is as follows

$$U_{ic_{nm}}^{k} = A_{ic_{nm}}^{0} + A_{ic_{nm}}^{k} - \left(A_{ic_{nm}}^{p} / \left|\mathrm{V}_{nm}\right|\right) \sum_{j=1}^{4} P_{c_{nm}^{(j)}}^{k} n_{ji} \left|S_{nm}^{(j)}\right|. \qquad (3.28)$$

According to (3.22), equation (3.28) also can be rewritten as

$$U_{ic_{nm}}^{k} = A_{ic_{nm}}^{0} + A_{ic_{nm}}^{k} - A_{ic_{nm}}^{p} \left(\partial P / \partial X_i\right)_{C_{nm}}^{k}. \qquad (3.29)$$

In relationships (3.28) and (3.29) the term $A_{ic_{nm}}^{0}$ is a rational function whose numerator contains the known velocity values $U_{ic_{nm}}^{k-1}$ and $U_{ic_{nm}}^{k-2}$ found at the previous time steps at the point C_{nm}. Its denominator involves the flow $F_{nm}^{(j)k}$ that linearly depends on the unknown velocity components (see (3.16)). The term $A_{ic_{nm}}^{k}$ also is a rational function whose denominator only differs from that of the function $A_{ic_{nm}}^{0}$ in the multiplier $\left|\mathrm{V}_{nm}\right| / \Delta T$. Its numerator has both the unknown velocity components at the point C_j at the instant of time $T = k\Delta T$ and the unknown products $F_{nm}^{(j)k} U_{ic_j}^{k}$. As for the fractional multiplier $A_{ic_{nm}}^{p}$, its numerator only consists of the time step ΔT, whereas the denominator coincides with that of the function $A_{ic_{nm}}^{0}$.

From relationships (3.28), (3.29) one can obtain (by means of interpolation) an equation for the velocity components at the mass centers $C_{nm}^{(j)}$ of the side faces $S_{nm}^{(j)}$ of the volume V_{nm}, viz.

$$U_{ic_{nm}^{(j)}}^{k} = A_{ic_{nm}^{(j)}}^{0} + A_{ic_{nm}^{(j)}}^{k} - A_{ic_{nm}^{(j)}}^{p} \begin{cases} \left(1 / \left|\mathrm{V}_{nm}\right|\right) \sum_{r=1}^{4} P_{c_{nm}^{(r)}}^{k} n_{ri} \left|S_{nm}^{(r)}\right|, \\ \left(\partial P / \partial X_i\right)_{C_{nm}^{(j)}}^{k}, \end{cases} \quad j = 1, \dots, 4. \qquad (3.30)$$

If now (3.30) is substituted into (3.16) and then the obtained relationship into (3.23) this yields a desired equation for the pressure, viz.

[11] This nonlinearity is due to a dependence of the flow $F_{nm}^{(j)k}$ on the velocity components (see (3.16)).

[12] Within the framework of the incompressible fluid model, there is no equation for the pressure. Therefore, in case of necessity, one should find a way to derive it.

$$\sum_{j=1}^{4} A^{p}_{ic^{(j)}_{nm}} \left(\partial P / \partial X_i\right)^{k}_{C^{(j)}_{nm}} n_{ji} \left|S^{(j)}_{nm}\right| = \sum_{j=1}^{4} \left(A^{0}_{ic^{(j)}_{nm}} + A^{k}_{ic^{(j)}_{nm}} \right) n_{ji} \left|S^{(j)}_{nm}\right| . \qquad (3.31)$$

Further the coupled equations (3.28)/(3.29), (3.31)[13] are used to find the fluid velocity components and the pressure.

Gradual approximations of and agreeing between the velocity and the pressure

The system of equations (3.28)/(3.29), (3.31) is solved by means of finding gradual approximations of the velocity and the pressure, and their corresponding agreeing with one another. Herewith the number of the approximations is determined by the prescribed accuracy of the solution. The detailed description of this procedure is given in [14-17].

The first approximations of the velocity and the pressure

We begin to solve the system of equations (3.28)/(3.29), (3.31) with finding the first approximations of the fluid velocity components which are marked by the superscript asterisk[14]. For this purpose, equation (3.28) is modified in the appropriate way. More specifically, here the unknown pressure values $P^{k}_{c^{(j)}_{nm}}$ are replaced by the known ones $P^{k-1}_{c^{(j)}_{nm}}$ obtained at the previous time step $T = (k-1)\Delta T$. Also, all the functions $A^{\cdots}_{ic_{nm}}$ in (3.28) are modified by replacing the unknown velocity components in the flow $F^{(j)k}_{nm}$ (which is contained in $A^{\cdots}_{ic_{nm}}$) with their known magnitudes computed at $T = (k-1)\Delta T$. This results in the following system of linear algebraic equations for the first approximations of the velocity components at the points C_{nm} and C_j ($j = 1,...,4$):

$$U^{k*}_{ic_{nm}} = A^{0l}_{ic_{nm}} + A^{kl}_{ic_{nm}} - \left(A^{pl}_{ic_{nm}} / \left|\mathrm{V}_{nm}\right| \right) \sum_{j=1}^{4} P^{k-1}_{c^{(j)}_{nm}} n_{ji} \left|S^{(j)}_{nm}\right| \qquad (3.32)$$

(here $A^{\cdot\cdot l}_{ic_{nm}}$ are the functions $A^{\cdots}_{ic_{nm}}$ modified in accordance with the just noted; herewith $A^{0l}_{ic_{nm}}$ and $A^{pl}_{ic_{nm}}$ are independent of the unknown velocity components and $A^{kl}_{ic_{nm}}$ are linear functions of the velocities $U^{k*}_{ic_j}$). Relationships (3.32) are independent of the pressure at the time instant $T = k\Delta T$ to be found.

Once the first approximations of the velocity components are found from system (3.32)[15], they are further used to obtain the corresponding values of the operators $A^{\cdots}_{ic^{(j)}_{nm}}$, which are then substituted into (3.31). This yields the system of linear algebraic equations for the first approximation of the pressure[14,15] at the points C_{nm} and C_j ($j = 1,...,4$), viz.

$$\sum_{j=1}^{4} A^{p*}_{ic^{(j)}_{nm}} \left(\partial P / \partial X_i\right)^{k*}_{C^{(j)}_{nm}} n_{ji} \left|S^{(j)}_{nm}\right| = \sum_{j=1}^{4} \left(A^{0*}_{ic^{(j)}_{nm}} + A^{k*}_{ic^{(j)}_{nm}} \right) n_{ji} \left|S^{(j)}_{nm}\right| \qquad (3.33)$$

(here $A^{\cdot\cdot *}_{ic^{(j)}_{nm}}$ are the just noted values of the operators $A^{\cdots}_{ic^{(j)}_{nm}}$). System (3.33) is independent of the unknown velocity components at the instant of time $T = k\Delta T$.

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[13] Equations (3.28)/(3.29) for the velocity components depend on the unknown pressure, whereas equation (3.31) for the pressure depends on the velocity components to be found.

[14] Hereinafter the first approximations of the magnitudes to be found are marked by the upper index *, whereas the second and third ones by the upper indices ** and ***, respectively.

[15] The method of solution of this system is described in subsection 'Solution of equations for the gradual approximations of the velocity and the pressure'.

The second approximations of the velocity and the pressure

In this subsection, one applies a procedure which is similar to that described in the previous subsection. More specifically, the first approximations of the pressure found from (3.33)[15] are substituted into (3.28) instead of $P^{k}_{c^{(j)}_{nm}}$. Also, in the functions $A^{\cdot\cdot\cdot}_{ic_{nm}}$ in (3.28), the flow $F^{(j)k}_{nm}$ is modified by replacing the unknown velocity components in it with their first approximations obtained from (3.32). This results in the systems of linear algebraic equations for the second approximations (or the first corrections) of the velocity components[14] at the points C_{nm} and C_j ($j=1,...,4$), viz.

$$U^{k**}_{ic_{nm}} = A^{0l*}_{ic_{nm}} + A^{kl*}_{ic_{nm}} - \left(A^{pl*}_{ic_{nm}} / \left|\mathrm{V}_{nm}\right|\right) \sum_{j=1}^{4} P^{k*}_{c^{(j)}_{nm}} n_{ji} \left|S^{(j)}_{nm}\right| \quad (3.34)$$

(here $A^{\cdot\cdot l*}_{ic_{nm}}$ are the functions $A^{\cdot\cdot\cdot}_{ic_{nm}}$ in which the just noted flow modification has been performed; in addition, the magnitudes $A^{0l*}_{ic_{nm}}$ and $A^{pl*}_{ic_{nm}}$ are independent of the unknown second approximations of the velocity, and $A^{kl*}_{ic_{nm}}$ depends linearly on $U^{k**}_{ic_j}$). Relationships (3.34) are independent of the pressure to be found.

After that the second approximations of the velocity, obtained from (3.34)[15], are used to obtain the values $A^{\cdot\cdot**}_{ic^{(j)}_{nm}}$ of the operators $A^{\cdot\cdot\cdot}_{ic^{(j)}_{nm}}$. Subsequent replacement of the magnitudes $A^{\cdot\cdot\cdot}_{ic^{(j)}_{nm}}$ in (3.31) with these values allows one to write a system of linear algebraic equations for the second approximation (or the first correction) of the pressure[14,15] at the points C_{nm} and C_j ($j=1,...,4$) which is similar to (3.33), viz.

$$\sum_{j=1}^{4} A^{p**}_{ic^{(j)}_{nm}} \left(\partial P / \partial X_i\right)^{k**}_{C^{(j)}_{nm}} n_{ji} \left|S^{(j)}_{nm}\right| = \sum_{j=1}^{4} \left(A^{0**}_{ic^{(j)}_{nm}} + A^{k**}_{ic^{(j)}_{nm}}\right) n_{ji} \left|S^{(j)}_{nm}\right|. \quad (3.35)$$

System (3.35) is independent of the unknown velocity.

The third approximations of the velocity and the pressure

The third approximations of the velocity and the pressure can be found with the use of the procedure described in the previous subsection. More specifically, the unknown pressure values $P^{k}_{c^{(j)}_{nm}}$ in (3.28) are replaced with their second approximations obtained from the system of equations (3.35)[15]. Also, the unknown velocities in the flow $F^{(j)k}_{nm}$ (which is available in all the operators $A^{\cdot\cdot\cdot}_{ic_{nm}}$) in system (3.28) are replaced with their second approximations found from (3.34). This yields the systems of linear algebraic equations for the third approximations (or the second corrections) of the velocity components[14,15] at the points C_{nm} and C_j ($j=1,...,4$), viz.

$$U^{k***}_{ic_{nm}} = A^{0l**}_{ic_{nm}} + A^{kl**}_{ic_{nm}} - \left(A^{pl**}_{ic_{nm}} / \left|\mathrm{V}_{nm}\right|\right) \sum_{j=1}^{4} P^{k**}_{c^{(j)}_{nm}} n_{ji} \left|S^{(j)}_{nm}\right| \quad (3.36)$$

(here $A^{\cdot\cdot l**}_{ic_{nm}}$ denotes the corresponding operator $A^{\cdot\cdot\cdot}_{ic_{nm}}$ in which the flow $F^{(j)k}_{nm}$ has been modified in the above-noted manner; in addition, $A^{kl**}_{ic_{nm}}$ is a linear function of the unknown velocities $U^{k***}_{ic_j}$, whereas $A^{0l**}_{ic_{nm}}$ and $A^{pl**}_{ic_{nm}}$ are independent of $U^{k***}_{ic_j}$). Relationships (3.36) are independent of the unknown pressure.

When the third approximations of the velocity components are found from (3.36)[15], they allow one to determine the corresponding values of the operators $A^{\cdots}_{ic^{(j)}_{nm}}$ in (3.31). Subsequent substitution of these values into (3.31) gives one the system of linear algebraic equations for the third approximation (or the second correction) of the pressure[14,15] at the points C_{nm} and C_j ($j=1,...,4$) which is similar to (3.35), viz.

$$\sum_{j=1}^{4} A^{p***}_{ic^{(j)}_{nm}} \left(\partial P / \partial X_i\right)^{k***}_{C^{(j)}_{nm}} n_{ji} \left|S^{(j)}_{nm}\right| = \sum_{j=1}^{4} \left(A^{0***}_{ic^{(j)}_{nm}} + A^{k***}_{ic^{(j)}_{nm}} \right) n_{ji} \left|S^{(j)}_{nm}\right| \tag{3.37}$$

(here $A^{\cdots***}_{ic^{(j)}_{nm}}$ are the indicated values of the operators $A^{\cdots}_{ic^{(j)}_{nm}}$). System (3.37) is independent of the unknown velocities.

If the accuracy of the third approximations of the velocity components and the pressure is not satisfactory, then the just-described procedure must be carried out until the accuracy becomes as desired.

Solution of equations for the gradual approximations of the velocity and the pressure

The systems of linear algebraic equations (SLAEs) for the gradual approximations of the velocity components and the pressure, which have been obtained above, can be rewritten in the following generalized form:

$$a^k_{c_{nm}} \xi^k_{c_{nm}} + \sum_{i=1}^{4} a^k_{c_i} \xi^k_{c_i} = b^k_{c_{nm}}, \tag{3.38}$$

where $\xi^k_{c_{nm}}$ and $\xi^k_{c_i}$ are the magnitudes to be found, and $a^k_{c_{nm}}$, $a^k_{c_i}$ and $b^k_{c_{nm}}$ the known coefficients. In a scientific literature, such systems are solved by methods which, in general, can be divided into the two main groups. The first of them is formed by the direct methods, whereas the second one by the iterative methods. Usually, the direct methods are applied to small systems of equations and give good results [12, 13, 16-19]. However, when one deals with big SLAEs (especially with systems whose matrices are rarified), the direct methods need a huge amount of time[16] to obtain their solutions, and therefore here their application is unreasonable. The iterative methods, when applied to big SLAEs, need much less computational memory and time, save the rarefaction degree of their matrices (when the matrices are rarified) and give satisfactory results [12, 13, 16-19].

Proceed from the just-said, as well as from the dimension and the rarefaction degree of the matrix of system (3.38), in this paper an iterative method is chosen to solve the system. Within its framework, initially an initial approximation of the solution is chosen, which is then improved by making iterations until its accuracy reaches the prescribed value. Herewith the attention is paid to the following two features. The first of them concerns with the necessity of providing domination of the diagonal terms in the matrix of system (3.38). In this study, it is realized by applying the deferred correction implementation method [12, 13, 18, 19] to the convective term. In accordance with this method, the part of the convective term, which corresponds to the backward differencing scheme, is inserted into the matrix, whereas its remainder is placed into the right part of SLAE (3.38).

The second feature is related to a desire to have as minimal as possible number of the iterations. In this work, it is made by the use of the method of conjugate gradients [12, 13, 18, 19], which belongs to the most effective methods of solving SLAEs of big dimension. This method allows one to solve a SLAE via the iterations' number that does not exceed the number of its unknown values. Herewith, if a successful choice of the initial approximation is made, the number of iterations sharply decreases. Also, the preconditioning results in a significant reduction of the iterations' number. For this purpose, in this research the solvers ICCG (for symmetric matrices [18-20]) and Bi-CGSTAB (for asymmetric matrices [18-20]) are used.

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[16] In the direct methods, the number of operations needed to obtain a solution grows as the square of the number of the unknown values.

4. Conclusions

1. A second order analytical and numerical technique is suggested to study fluid motion in a two-dimensional straight hard-walled duct with two axisymmetric abrupt constrictions.

2. In this technique, the governing relationships are integrated via their rewriting in a non-dimensional form, deriving their integral analogues, performing a discretization of the derived integral relationships, simplifying the obtained (after making the discretization) coupled non-linear algebraic equations, and final solving the resulting (after making the simplification) uncoupled linear ones.

3. The discretization consists of the spatial and temporal parts. The first of them is performed with the use of the total variation diminishing scheme and the two-point scheme of discretization of the spatial derivatives, whereas the second one is made on the basis of the implicit three-point non-symmetric backward differencing scheme.

4. The above-noted uncoupled linear algebraic equations for the velocity and the pressure are solved by an appropriate iterative method, which uses the deferred correction implementation technique and the technique of conjugate gradients, as well as the solvers ICCG (for the symmetric matrices) and Bi-CGSTAB (for the asymmetric matrices).

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