DOI: 10.26565/2304-6201-2021-50-07

UDC 539.3

The method of integral equations in the problems of studying oscillations of shells partially filled with liquid

Maria Myronenko

Myronenko Maria *PhD student*
Institute of Mechanical Engineering Problems A. N. Podgorny NASU, st. Pozharsky, 2/10, Kharkiv, 61046, Ukraine
e-mail: mariamyronenko87@gmail.com
http://orcid.org/ 0000-0002-0266-4463

G-force reaching several g affect the stability of the launch vehicle in the launching phase. The mathematical modeling methods are used to study the longitudinal vibration stability of liquid-fueled launch vehicles in the launching phase. The paper presents the modeling of small oscillations of fluid motion in a rigid, partially filled shell of rotation. The modeling is based on the developed mathematical model: fluid is supposed to be ideal and incompressible, fluid motion being vortexless, velocity potential gradient being fluid velocity. The conditions for the velocity potential at the boundaries of the computational domain are determined. The kinematic boundary condition and dynamic boundary condition on the free surface and nonpermeability condition on the bottom and side surfaces of the tank are fulfilled. The solution of the differential equations system for the boundary conditions has been obtained. The liquid sloshing in a low gravity has been investigated and the boundary conditions have been generalized. In the dynamic boundary condition the surface tension is accounted for. The assumed mode method has been developed to solve problems of free and forced oscillations of shell structures with compartments filled with liquid. The system of differential equations relative to the elastic movements of the structure and the active liquid pressure is obtained. For its solution three sets of basic functions have been used. The gravitational component in the singular equation system in the problem of sloshing in a rigid shell is taken into account. The cases of control points being positioned on the liquid free surface, as well as on the shell surface are considered. The solution of the system of equations determines the velocity potential.

Key words: *mathematical modeling, g-force, low gravity, surface tension, free surface, boundary element method.*

Метод інтегральних рівнянь в задачах дослідження коливань оболонок, частково заповнених рідиною

Мироненко Марія Леонідівна *аспірант*
Інститут проблем машинобудування ім. А. Н. Підгорного НАНУ, вул. Пожарського, 2/10, Харків, 61046, Україна

Перевантаження на активних ділянках польоту впливають на стійкість ракети-носія та досягають величини у кілька g. Для дослідження стійкості рідинних ракет-носіїв щодо дії поздовжніх коливань на активній ділянці польоту застосовуються методи математичного моделювання. В статті наведено моделювання малих коливань руху рідини у жорсткій, частково заповненій оболонці обертання на основі розробленої математичної моделі: рідина ідеальна та нестислива, рух рідини є безвихровим, градієнт потенціалу швидкості є швидкістю рідини. Визначені умови для потенціалу швидкості на границях розрахункової області. Виконуються кінематична та динамічна умови на вільній поверхні та умова непротікання на днищі та бокових поверхнях резервуару. Отримано розв’язок системи диференційних рівнянь з виконанням граничних умов. Досліджені коливання рідини в умовах низької гравітації та узагальнено граничні умови. В динамічній граничній умові здійснено врахування поверхневого натягу. Для розв’язання задач про власні та вимушені коливання оболонкових конструкцій з відсіками, що містять рідину, розроблено метод заданих форм. Отримано систему диференційних рівнянь відносно пружних переміщень конструкції та діючого тиску рідини, для розв’язання якої використано три набори базисних функцій. Виконано урахування гравітаційної складової у системі сингулярних рівнянь в задачі коливань рідини в жорсткій оболонці. При цьому розглянуті випадки положення контрольних точок на вільній поверхні рідини та на поверхні оболонки. Розв’язок системи рівнянь визначає потенціал швидкостей. Досліджено умови, при яких вплив поверхневого натягу стає несуттєвим. Отримані результати свідчать про те, що при значних параметрах перевантаження вплив поверхневого натягу стає несуттєвим. Але зі зменшенням цього параметру (при низьких рівнях гравітації) поверхневий натяг стає домінуючим.

Ключові слова: *математичне моделювання, перевантаження, низька гравітація, поверхневий натяг, вільна поверхня, метод граничних елементів.*

© Myronenko M., 2021

Метод интегральных уравнений в задачах исследования колебаний оболочек, частично заполненных жидкостью

Мироненко Мария Леонидовна *аспирант*
Институт проблем машиностроения им. А. Н. Подгорного НАНУ, ул. Пожарского, 2/10, Харьков, 61046, Украина

Перегрузки на активных участках полета влияют на устойчивость ракеты-носителя и достигают величины в несколько g. Для исследования устойчивости жидкостных ракет-носителей к действию продольных колебаний на активном участке полета применяются методы математического моделирования. В статье приведены моделирования малых колебаний движения жидкости в жесткой, частично заполненной оболочке вращения на основе разработанной математической модели: жидкость идеальная и несжимаемая, движение жидкости является безвихревым, градиент потенциала скорости является скоростью жидкости. Определены условия для потенциала скорости на границах расчетной области. Выполняются кинематическое и динамическое условия на свободной поверхности, условие непротекания на днище и боковых поверхностях резервуара. Получено решение системы дифференциальных уравнений с выполнением граничных условий. Исследованы колебания жидкости в условиях низкой гравитации и обобщены граничные условия. В динамическом граничном условии осуществлен учет поверхностного натяжения. Для решения задач о собственных и вынужденные колебания оболочечных конструкций с отсеками, содержащими жидкость, разработан метод заданных форм. Получена система дифференциальных уравнений относительно упругих перемещений конструкции и действующего давления жидкости, для решения которой использовано три набора базисных функций. Выполнен учет гравитационной составляющей в системе сингулярных уравнений в задаче колебаний жидкости в жесткой оболочке. При этом рассмотрены случаи положения точек на свободной поверхности жидкости и на поверхности оболочки. Решение системы уравнений определяет потенциал скоростей.

Ключевые слова: *математическое моделирование, перегрузки, низкая гравитация, поверхностное натяжение, свободная поверхность, метод граничных элементов.*

1. Introduction

The mission plan of the launch vehicle includes several thrust and coast phases. The pulsations of the engine thrust in the active part of the flight influence the rocket as a closed system, and cause oscillations of the missile body and liquid in the fuel lines [1]. This can lead to loss of stability and cause serious consequences.

Flight and control of the launch vehicle on the active part of the trajectory take place under the influence of such external forces as temperature changes and atmosphere density, wind disturbances, as well as liquid oscillations in fuel tanks which are caused by the full range of those external influences [2]. During the orbital shot the gravity and aerodynamic losses reduce the speed of the spacecraft by 2-3 km/s at the end of the active phase. Taking into account these factors, as well as other losses, the required launch speed will be exceeding the first cosmic velocity. The achievement of speed which exceeds 8000 m/s over a short period of time during the active part requires significant accelerations. For example, the acceleration of a 5000 kg rocket upon reaching orbit is approximately 6g [3]. Thus, the longitudinal acceleration acting on the launch vehicle in this flight path can reach several g and affect the stability.

A nonuniform rocket motion causes the appearance of inertial force, which is an additional load on structural elements. Accounting for the presence of rocket payload and assuming no angular oscillations of the rocket relative to the center of mass, the lateral-stability coefficient and longitudinal load factor, which show how the stress caused by the payload exceeds the payload weight, is determined in [4]. It is specified that the decrease in vehicle mass due to chemical combustion has the greatest effect on the longitudinal load coefficient, which changes unevenly on the active part of the trajectory and acquires the maximum value at the end of the engine working stage [4].

Researching stability of liquid-propellant missile concerning longitudinal oscillations on an active part of trajectory is pretty difficult on the ground. Verification tests in situ require significant material and financial costs, so the use of mathematical modeling techniques can help to avoid such expenses and contribute to solving the problem of longitudinal stability [5].

2. Problem statement

The elastic shell of revolution, which is partially filled with liquid, is considered. To simulate the flow of liquid, a computer model based on the postulates that the fluid is ideal and incompressible and the flow of fluid is vortex-free, has been developed. Only small-amplitude oscillation has been investigated (linear theory). There is a velocity potential $\Phi(x,y,z,t)$, the gradient of which is the fluid velocity, namely:

$$V_x = \frac{\partial \Phi}{\partial x}, V_y = \frac{\partial \Phi}{\partial y}, V_z = \frac{\partial \Phi}{\partial z} \tag{2.1}$$

The motion equation of the shell system is presented in an operator form

$$\boldsymbol{LU} + \boldsymbol{M}\ddot{\boldsymbol{U}} = p\boldsymbol{n} + \boldsymbol{Q}, \tag{2.2}$$

where $\boldsymbol{L}, \boldsymbol{M}$ are the operators of elastic and mass forces;

p is liquid pressure on a wetted surface of a shell structure, N/m^2;

$\boldsymbol{n}$ is an outer unit normal;

$\boldsymbol{Q}$ is exciting force, N.

Using the motion equations and the flow potentiality conditions, we arrive at the Cauchy-Lagrange integral in the form [6]:

$$p = -\rho_l \left[\frac{\partial \Phi}{\partial t} + a_x(t)x + a_z(t)z + gz + \frac{1}{2}\left|\nabla \Phi\right|^2 \right], \tag{2.3}$$

where ρ_l is liquid density, kg/m^3;

$a_x(t), a_z(t)$ are components of the acceleration of exciting force, m/sec^2;

g is a gravity acceleration, m/sec^2.

If we consider small liquid oscillations (linear formulation), then $\left|\nabla \Phi\right|^2 << 1$, , and from the formulas (2.3) we obtain

$$p = -\rho_l \left[\frac{\partial \Phi}{\partial t} + a_x(t)x + a_z(t)z + gz \right]. \tag{2.4}$$

Assuming that the flow is vortex-free, the motion of an ideal incompressible fluid is described by the Laplace equation for the velocity potential.

$$\nabla^2 \Phi = 0. \tag{2.5}$$

To determine the pressure, it is necessary to calculate the velocity potential. The boundary conditions are to be formulated for the velocity potential at the boundaries of the computational domain.

Nonpermeability condition is met on the bottom and the lateral surface. Kinematic and dynamic boundary conditions are fulfilled on the free surface. The defined conditions are as follows:

$$\left.\frac{\partial \Phi}{\partial t} + a_x(t)x + a_z(t)\zeta + g\zeta\right|_{S_0} = 0, \quad \left.\frac{\partial \Phi}{\partial \boldsymbol{n}}\right|_{S_0} = \frac{\partial \zeta}{\partial t}, \quad \left.\frac{\partial \Phi}{\partial \boldsymbol{n}}\right|_{\sigma} = \frac{\partial w}{\partial t}, \tag{2.6}$$

where $w = (\boldsymbol{U}, \boldsymbol{n})$ is the normal component of displacement of shell structure;

$\zeta = \zeta(x,y,t))$ is the free surface motion function.

Thus, we have the differential equation system for determining five unknown functions $U_1, U_2, U_3, \Phi, \zeta$:

$$\begin{cases} \boldsymbol{LU} + \boldsymbol{M\ddot{U}} = -\rho_l \left(\dfrac{\partial \Phi}{\partial t} + a_x(t)x + a_z(t)z + gz \right) + \boldsymbol{Q}(t), \\ \nabla^2 \Phi = 0. \end{cases} \tag{2.7}$$

We solve a system of differential equations (2.7) by fulfilling the boundary conditions (2.6).

To obtain an unambiguous solution of the equations system (2.7) with boundary conditions (2.6), we add the Von Neumann condition:

$$\iint_{S_0} \frac{\partial \Phi}{\partial \boldsymbol{n}} dS_0 = 0 \,. \tag{2.8}$$

The differential equations system (2.7) should also be supplemented by the conditions for securing the shell structure, i.e. the conditions with respect to the vector-function U.

2.1. Taking into account the surface tension in the study of liquid oscillations at low gravity

Note that the boundary conditions (2.6) in the case of low gravity must be generalized. Let σ_0 be the surface tension. According to the Laplace-Jung formula [7] we have

$$p_s = \sigma_0 \kappa \,, \tag{2.9}$$

where κ is the surface curvature.

The expression for the curvature can be linearized, as in [8]:

$$\kappa = -\Delta_s \zeta \,,$$

where Δ_s is the Laplacian.

Thus, the dynamic boundary condition on the free surface takes the form

$$\left. \frac{\partial \Phi}{\partial t} + a_x(t)x + a_z(t)z + g\zeta - \frac{\sigma_0}{\rho_l} \Delta_s \zeta \right|_{S_0} = 0. \tag{2.10}$$

The condition (2.10) allows us to account for the surface tension, which becomes a determining factor in the study of oscillations of the shell structure under conditions of low gravity.

2.2. The assumed mode method

The assumed mode method has been developed to solve problems of proper oscillations and constrained oscillations of elastic shell structures with the liquid-filled compartment. A connected differential equations system with respect to the elastic displacements of the structure and the effective fluid pressure has been obtained. Three sets of basic functions are used to represent the solutions of this system. The first of them consists of normal oscillations modes of the structure in the absence of filler and is used to construct hydroelastic displacements. The second and third sets of basic functions are obtained by constructing the velocity potential and the function describing the time variation of the liquid free surface. The velocity potential is represented by the sum of two partial potentials. One of them corresponds to the liquid normal oscillations in the rigid reservoir, taking into account the gravity forces; the second refers to the normal oscillations of the elastic shell with the liquid without taking into account the gravity forces.

In general, we look for the displacements of the shell structure with compartments partially filled with liquid, in the form:

$$\boldsymbol{U} = \sum_{k=1}^{N} c_k(t) \boldsymbol{u}_k \,, \tag{2.11}$$

where $c_k(t)$ is the unknown coefficients which are time-dependent only;

$\boldsymbol{u}_k$ is the oscillation modes of the unfilled shell structure;

N is the number of modes obtained in the calculations.

The orthogonality relations are satisfied [9]

$$\boldsymbol{L}(\boldsymbol{u}_k) = \Omega_k^2 \boldsymbol{M}(\boldsymbol{u}_k), \quad (\boldsymbol{M}(\boldsymbol{u}_k), \boldsymbol{u}_j) = \delta_{kj}, \quad (\boldsymbol{L}(\boldsymbol{u}_k), \boldsymbol{u}_j) = \Omega_k^2 \delta_{kj}, \tag{2.12}$$

where Ω_k is the k-frequency of natural oscillation of the unfilled elastic structure.

We find the velocity potential as the sum of two potentials $\Phi = \Phi_1 + \Phi_2$. The potential Φ_1 is represented in the form

$$\Phi_1 = \sum_{k=1}^{N} \dot{c}_k(t)\phi_{1k} \tag{2.13}$$

where ϕ_{1k} is the basic functions.

In the formula (2.13), the time-dependent coefficients $c_k(t)$ are defined in the equation (2.11). For the functions ϕ_{1k} we have the following boundary value problems:

$$\nabla^2 \phi_{1k} = 0, \quad \left.\frac{\partial \phi_{1k}}{\partial \boldsymbol{n}}\right|_{\sigma} = w_k, \quad w_k = (\boldsymbol{u}_k, \boldsymbol{n}), \qquad \left.\phi_{1k}\right|_{S_0} = 0 \tag{2.14}$$

These functions are presented in [10].

Let us represent the potential Φ_2 in the form of a normal oscillation mode of liquid in a rigid tank

$$\Phi_2 = \sum_{k=1}^{M} \dot{d}_k(t)\phi_{2k}, \tag{2.15}$$

where $d_k(t)$ is the unknown time-dependent coefficients;

ϕ_{2k} is the basic functions;

M is the number of modes obtained in the calculations.

For the functions ϕ_{2k} we formulate boundary value problems as follows:

$$\nabla^2 \phi_{2k} = 0, \qquad \left.\frac{\partial \phi_{2k}}{\partial \boldsymbol{n}}\right|_{\sigma} = 0, \quad \left.\frac{\partial \phi_{2k}}{\partial \boldsymbol{n}}\right|_{S_0} = \frac{\partial \varsigma}{\partial t}; \quad \frac{\partial \phi_{2k}}{\partial t} + g\zeta = 0. \tag{2.16}$$

Thus on a free surface we have a ratio

$$\frac{\partial \phi_{2k}}{\partial \boldsymbol{n}} = \frac{\chi_k^2}{g}\phi_{2k}, \tag{2.17}$$

where χ_k is the frequency of normal oscillations of the free surface, Hz.

These basic functions are constructed in [11]. The free surface equation takes the form

$$\zeta = \zeta(x, y, t) = \sum_{k=1}^{N} c_k \frac{\partial \phi_{1k}}{\partial \boldsymbol{n}} + \sum_{k=1}^{M} d_k \frac{\partial \phi_{2k}}{\partial \boldsymbol{n}}, \tag{2.18}$$

and for the velocity potential we have

$$\Phi = \Phi\left(x,y,z,t\right) = \sum_{k=1}^{N} \dot{c}_k \phi_{1k} + \sum_{k=1}^{M} \dot{d}_k \phi_{2k} \,. \tag{2.19}$$

Note that the total potential (2.19) satisfies the following relations

$$\Delta\Phi = 0\,, \qquad \left.\frac{\partial\Phi}{\partial \boldsymbol{n}}\right|_{S_1} = \frac{\partial w}{\partial t}\,, \qquad \left.\frac{\partial\Phi}{\partial \boldsymbol{n}}\right|_{S_0} = \frac{\partial\zeta}{\partial t}\,. \tag{2.20}$$

Thus, for the final solution of the initial boundary value problem (2.7) with boundary conditions (2.6), (2.9) - (2.10) it is necessary to satisfy the differential equations system (2.6) and the dynamic boundary condition on the free surface described by the first of equations (2.6), and if the surface tension is taken into account, it is described by the equation (2.10).

In this case, the following relation is fulfilled on the free surface:

$$\sum_{k=1}^{N} \ddot{c}_k \phi_{1k} + \sum_{k=1}^{M} \ddot{d}_k \phi_{2k} + \left(g + a_z(t)\right)\left(\sum_{k=1}^{N} c_k \frac{\partial \phi_{1k}}{\partial \boldsymbol{n}} + \sum_{k=1}^{M} d_k \frac{\partial \phi_{2k}}{\partial \boldsymbol{n}}\right) + a_x(t)x = 0\,. \tag{2.21}$$

In addition, due to the equation (2.2) the relation is true

$$\boldsymbol{L}\left(\sum_{k=1}^{N} c_k(t)\boldsymbol{u}_k\right) + \boldsymbol{M}\left(\sum_{k=1}^{N} \ddot{c}_k(t)\boldsymbol{u}_k\right) =$$

$$= -\rho_l\left[\left(\sum_{k=1}^{N} \ddot{c}_k(t)\phi_{1k} + \sum_{k=1}^{M} \ddot{d}_k(t)\phi_{2k}\right) + a_x\left(t\right)x + a_z\left(t\right)z\right] + \boldsymbol{Q}\,. \tag{2.22}$$

In the equation (2.21) the surface tension is not taken into account.

From the relations (2.21), (2.22) we find the unknown functions of time $c_k\left(t\right)$ and $d_k\left(t\right)$. For their unambiguous definition we use the initial conditions

$$c_k(0) = c_{k0}, \quad \dot{c}_k(0) = c_{k1}, \quad d_k(0) = d_{k0}, \quad \dot{d}_k(0) = d_{k1}\,. \tag{2.23}$$

This makes it possible to investigate the forced oscillations of the shell structure with compartments partially filled with liquid. Since it is usually assumed that at the initial time the "shell-liquid" system is at rest, zero initial conditions are used in the calculations.

To study free oscillations, we assume that

$$c_k(t) = C_k \exp(i\omega t), \quad d_k(t) = D_k \exp(i\omega t)\,. \tag{2.24}$$

Let us substitute these expressions into the relations (2.21), (2.22) and arrive at the eigenvalue problem, similar to that given in [11].

Next, we describe the method of reducing the task of determining the oscillation modes and frequencies of shell systems partially filled with liquid, to a set of singular equations.

3. The system of singular equations for velocity potential in the problem of liquid oscillations in a rigid shell taking into account a gravitational component

Let us represent the potential Φ_2 as follows

$$\varphi_2(x,y,z,t) = \sum_{k=1}^{n} \dot{d}_k(t)\phi_{2k}(x,y,z)$$

where the functions ϕ_{2k} are determined from the relations (2.16).

Hereinafter, for simplicity of notation $\phi_{2k} = \psi$.

Let us represent ψ as a sum of the potentials of a simple and double layer [12]

$$2\pi\psi = \iint\limits_{S} \frac{\partial\psi}{\partial n}\frac{1}{|P-P_0|}dS_0 - \iint\limits_{S} \psi\frac{\partial}{\partial n}\frac{1}{|P-P_0|}dS\,; S \in S_1 \cup S_0 \tag{3.1}$$

Here $r(P,P_0) = |P-P_0|$ is the Cartesian distance between P and P_0.
Let us consider different cases of the control point position. We leave the general integral solution

$$2\pi\psi = \iint\limits_{S_1} 0\frac{1}{r(P,\ P_0)}dS_1 + \iint\limits_{S_0} \frac{\partial\psi}{\partial n}\frac{1}{r(P,\ P_0)}dS_0 -$$

$$- \iint\limits_{S_1} \psi\frac{\partial}{\partial n}\frac{1}{r(P,\ P_0)}dS - \iint\limits_{S_1} \psi\frac{\partial}{\partial z}\frac{1}{r(P,\ P_0)}dS_0\,. \tag{3.2}$$

a) Let $P_0 \in S_0$ (the control point is on a free surface):

$$2\pi\psi_0 = \iint\limits_{S_0} \frac{k^2}{g}\psi_0\frac{1}{r(P,\ P_0)}dS_0 - \iint\limits_{S_1} \psi_1\frac{\partial}{\partial n}\frac{1}{r(P,\ P_0)}dS_1 - \iint\limits_{S_0} \psi_0\frac{\partial}{\partial z}\frac{1}{r(P,\ P_0)}dS_0\,. \tag{3.3}$$

In this case we have

$$\frac{\partial}{\partial z}\left(\frac{1}{r}\right) = \frac{z-z_0}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}; \tag{3.4}$$

In this case, on $S_0,\ z = z_0 \Rightarrow\ \left.\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right|_{S_0} = 0$

Then

$$2\pi\psi_0(P_0) = \frac{k^2}{g}\iint\limits_{S_0} \psi_0\frac{1}{r(P,\ P_0)}dS_0 - \iint\limits_{S_1} \psi_1\frac{\partial}{\partial n}\frac{1}{r(P,\ P_0)}dS_1. \tag{3.5}$$

b) Let $P_0 \in S_1$ (the control point is on the shell surface). We have the following integral equations

$$2\pi\psi_1(P_0) = \iint\limits_{S_0} \frac{\partial\psi}{\partial n}\frac{1}{r(P,\ P_0)}dS_0 - \iint\limits_{S_1} \psi\frac{\partial}{\partial n}\frac{1}{r(P,\ P_0)}dS_1 - \iint\limits_{S_0} \psi\frac{\partial}{\partial z}\frac{1}{r(P,\ P_0)}dS_0; \tag{3.6}$$

$$2\pi\psi_1(P_0) = \frac{k^2}{g}\iint\limits_{S_0} \psi_0\frac{1}{r(P,\ P_0)}dS_0 - \iint\limits_{S_1} \psi_1\frac{\partial}{\partial n}\frac{1}{r(P,\ P_0)}dS_1 - \iint\limits_{S_0} \psi_0\frac{\partial}{\partial z}\frac{1}{r(P,\ P_0)}dS_0. \tag{3.6}$$

Let us transform these equations by calculating the components with ψ_1 and ψ_0 . For control and integration points on the shell of revolution, we obtain

$$2\pi\psi_1 + \iint\limits_{S_1} \psi_1\frac{\partial}{\partial n}\frac{1}{r(P,\ P_0)}dS_1 = \mathrm{A}\psi_1. \tag{3.7}$$

For control points belonging to the shell and integration points on the free surface we have

$$\frac{k^2}{g}\iint\limits_{S_0} \psi_0\frac{1}{r(P,\ P_0)}dS_0 - \iint\limits_{S_0} \psi_0\frac{\partial}{\partial z}\frac{1}{r(P,\ P_0)}dS_0 = \frac{k^2}{g}B\psi_0 - C\psi_0, \tag{3.8}$$

where

$$B\psi_0 = \iint_{S_0} \psi_0 \left(\frac{1}{r}\right) dS_0;\ C\psi_0 = \iint_{S_0} \psi_0 \frac{\partial}{\partial z}\left(\frac{1}{r}\right) dS_0. \tag{3.9}$$

For the control points on the liquid free surface and the integration points belonging to the shell of revolution we obtain

$$-\iint_{S_1} \psi_1 \frac{\partial}{\partial n} \frac{1}{r(P,\ P_0)} dS_1 = D\psi_1. \tag{3.10}$$

For control and integration points, which simultaneously belong to the liquid free surface, we have

$$2\pi\psi_0 - \frac{k^2}{g} \iint_{S_0} \psi_0 \frac{1}{r} dS_0 = 2\pi \mathrm{E}\psi_0 - \frac{k^2}{g} F\psi_0, \tag{3.11}$$

where

$$F\psi_0 = \iint_{S_0} \psi_0 \frac{1}{r} dS_0 \tag{3.12}$$

For control points on the surface $P_0 \in S_1$ we obtain

$$\mathrm{A}\psi_1 = \frac{k^2}{g} B\psi_0 - C\psi_0. \tag{3.13}$$

From (3.13) ψ_1 via ψ_0 is expressed as follows

$$\psi_1 = \frac{k^2}{g} A^{-1} B\psi_0 - A^{-1} C\psi_0. \tag{3.14}$$

For the control points from the liquid free surface $P_0 \in S_1$ we obtain

$$D\psi_1 = 2\pi E\psi_0 - \frac{k^2}{g} F\psi_0. \tag{3.15}$$

In (3.15) we substitute the expression for ψ_1 obtained above and arrive at the following integral relations

$$\frac{k^2}{g} DA^{-1} B\psi_0 - DA^{-1} C\psi_0 = 2\pi E\psi_0 - \frac{k^2}{g} F\psi_0;$$

$$\frac{k^2}{g} (DA^{-1} B\psi_0 + F\psi_0) = 2\pi E\psi_0 + DA^{-1} C\psi_0;$$

$$\frac{k^2}{g} \psi_0 = (DA^{-1} B + F)^{-1} (2\pi E + DA^{-1} C)\psi_0;$$

The last one takes the form

$$\left(\tilde{A} - \lambda E\right)\psi_0 = 0, \tag{3.16}$$

where

$$\tilde{A} = (DA^{-1} B + F)^{-1} (2\pi E + DA^{-1} C);$$

$$\lambda = \frac{k^2}{g}$$

That is, we came to the eigenvalue problem.

4. Reduction to one-dimensional singular equations

Let us construct integral equations to determine the potential in the form:

$$\begin{cases} 2\pi\psi_1 + \iint\limits_{S_1} \psi_1 \frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS_1 - \frac{k^2}{g} \iint\limits_{S_0} \psi_0 \frac{1}{r} dS_0 + \iint\limits_{S_0} \psi_0 \frac{\partial}{\partial z}\left(\frac{1}{r}\right) dS_0 = 0 \\ -\iint\limits_{S_1} \psi_1 \frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS_1 - 2\pi\psi_0 + \frac{k^2}{g} \iint\limits_{S_0} \psi_0 \frac{1}{r} dS_0 = 0 \end{cases} \tag{4.1}$$

From (3.16) we derive ψ_0 and κ.
The solution of the system (4.1) is sought in the form:

$$\psi = \psi(r,z)\cos\alpha\theta \tag{4.2}$$

where r, θ, z are cylindrical coordinates;

$$|r| = |P - P_0| = \sqrt{r^2 + r_0^2 + (z - z_0)^2 - 2rr_0\cos(\theta - \theta_0)}. \tag{4.3}$$

The normal derivative takes the form

$$\frac{\partial}{\partial n}\frac{1}{|P - P_0|} = -\frac{n_r[r - r_0\cos(\theta - \theta_0)] + n_z(z - z_0)}{\left(\sqrt{a - b\cos(\theta - \theta_0)}\right)^3}; \tag{4.4}$$

$$a = r^2 + r_0^2 + (z - z_0)^2; b = 2rr_0.$$

The transformation of kernels leads to the following formulas for calculating integrals in (4.1).

$$\iint\limits_{S_1} \psi \frac{\partial}{\partial n}\left(\frac{1}{r(P,\ P_0)}\right) dS_1 = \int\limits_{r} \psi(z)\Theta(z, z_0)r(z)d\Gamma, \tag{4.6}$$

$$\iint\limits_{S_0} \psi \left(\frac{1}{r(P,\ P_0)}\right) dS_0 = \int\limits_0^R \psi(\rho)\Phi(P, P_0)\rho d\rho \tag{4.7}$$

where

$$\Theta(z, z_0) = \frac{4}{\sqrt{a + b}}\left\{\frac{1}{2r}\left[\frac{r^2 - r_0^2 + (z - z_0)^2}{a - b}\mathrm{E}_\alpha(k) - F_\alpha(k)\right]n_r + \frac{z_0 - z}{a - b}\mathrm{E}_\alpha(k)n_z\right. \tag{4.8}$$

$$\Phi(P, P_0) = \frac{4}{\sqrt{a + b}}F_\alpha(k), \quad k^2 = \frac{2b}{a + b} \tag{4.9}$$

$$F_\alpha(k) = (-1)^\alpha \int\limits_0^{\pi/2} \frac{\cos 2\alpha\theta d\theta}{\sqrt{1 - k^2\sin^2\theta}} \tag{4.10}$$

$$\mathrm{E}_\alpha(k) = (-1)^\alpha(1 - 4\alpha^2)\int\limits_0^{\pi/2} \cos 2\alpha\theta\sqrt{1 - k^2\sin^2\theta}\, d\theta \tag{4.11}$$

In the given formulas $F_\alpha(k)$, $\mathrm{E}_\alpha(k)$ are integrals along the circumferential coordinate; Γ is the generating surface S_1; ρ is the polar coordinate of S_0.

Let us calculate the matrices of integrated equations of the eigenvalue problem (3.6). We obtain the following formulas:

$$\begin{cases} \mathrm{A}\psi_1 = \lambda\mathrm{B}\psi_0 - C\psi_0;\ \mathrm{A}\psi_1 + C\psi_0 - \lambda\mathrm{B}\psi_0 = 0 \\ \mathrm{D}\psi_1 = 2\pi\mathrm{E}\psi_0 - \lambda F\psi_0;\ \mathrm{D}\psi_1 - \mathrm{E}\psi_0 + \lambda\mathrm{F}\psi_0 = 0 \end{cases} \tag{4.12}$$

or in a matrix form

$$\begin{pmatrix} A & C \\ D & -E \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix} - \lambda \begin{pmatrix} B \\ -F \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix} = 0 \tag{4.13}$$

Note that the obtained integral operators contain logarithmic features. To numerically determine such integrals the method proposed in [13, 14] has been used.

5. Free oscillations of shell structures at different gravity levels

To determine the conditions of stable movement of the shells partially filled with liquid under the action of external loads, at the first stage we find the frequencies and modes of oscillations of the "shell-liquid" system according to [9, 15, 16].

At first, we will limit ourselves to liquid oscillations in rigid shells. Note that in practice these oscillations correspond to the lowest oscillation frequencies which results in the detuning [9].

The cylindrical (C) and cylindrical-conical (CC) shells shown in Fig. 1 are considered. The frequencies and modes of liquid oscillations in these composite shells of revolution are obtained by the method proposed above.

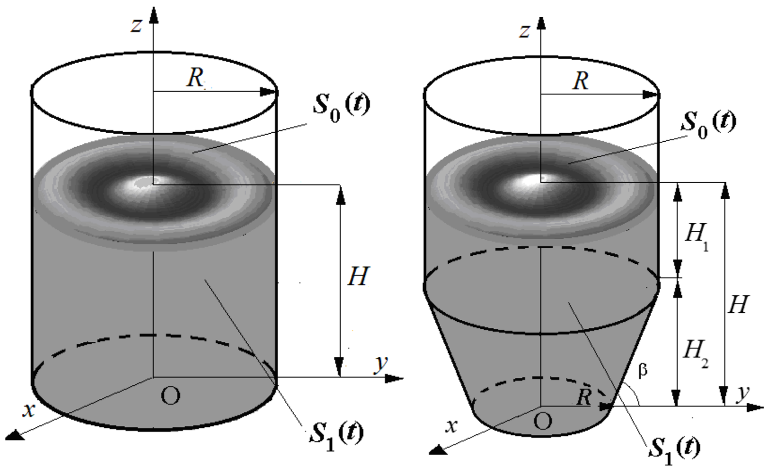

Fig. 1. Shells partially filled with liquid

Tab. 1 shows the frequencies of non-axisymmetric oscillations of the liquid for different shells with the following parameters: H = 2 м, $H_1 = H_2$ = 1 м, R = 1 м, β = 60°.

Table 1. Frequencies of non-axisymmetric oscillations of the liquid, Hz

α	Shell type	*n*						
		1	2	3	4	5	6	7
1	C	4.2474	7.2352	9.1573	10.726	12.089	13.312	14.433
	CC	4.2346	7.2352	9.1573	10.726	12.089	13.312	14.433
2	C	5.4733	8.1148	9.8966	11.377	12.678	13.855	14.939
	CC	5.4718	8.1148	9.8966	11.377	12.678	13.855	14.939
3	C	6.4197	8.8719	10.558	11.973	13.226	14.364	15.417
	C	6.4195	8.8719	10.558	11.973	13.226	14.364	15.417

Fig. 2 shows the first modes of neosesymmetric oscillations of the liquid at α=1.

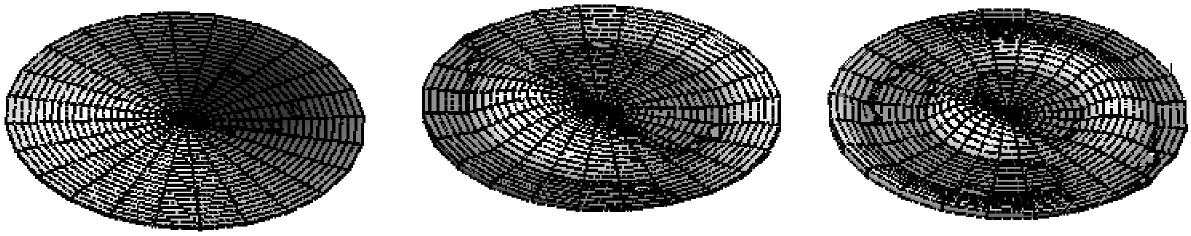

Fig. 2 The first modes of non-axisymmetric oscillations of the free surface, $\alpha = 1$

Note that for the selected parameters of the shells, the frequency and the modes of the oscillations are similar for both tanks considered. It indicates that for such parameters the shape of the bottom does not significantly affect the frequencies and modes of oscillations; the determining factor is the filling level of the shells *H*.

Next, we introduce the parameter [17], which characterizes the capillary length $\lambda_c = \sqrt{\dfrac{\sigma}{\rho_l g}}$. To take into account the surface tension, we use the formula [18]

$$\frac{\omega_{cl}^2}{g} = \frac{\omega_l^2}{g}\left[1 + \lambda_c^2 \frac{\omega_l^4}{g^2}\right], \quad l = 1,2,... \tag{5.1}$$

where ω_{cl}, ω_l are the oscillation frequencies with and without accounting for the surface tension, respectively.

Let us determine the effect of surface tension. To do this we calculate the frequency parameters ω_l^2 / g for a cylindrical shell, the parameters of which are given above. Next, we select $\lambda_c^2 = 10^{-3}$ and calculate the frequency parameters ω_{cl}^2 / g taking the surface tension into account.

The results are shown in Tab. 2. The g-force parameter n_g=1 is used here, which corresponds to the level of gravity on the Earth's surface.

Table 2. Frequency parameters with accounting for surface tension

Frequencies and frequency parameters	l						
	1	2	3	4	5	6	7
ω_l^2/g	1.8389	5.3361	8.5480	11.727	14.817	18.227	21.234
ω_{cl}^2/g	1.8451	5.4880	9.1725	13.339	18.069	24.282	30.808
ω_l,	4.2474	7.2352	9.1573	10.726	12.089	13.312	14.433
ω_{cl}	4.2544	7.3374	9.4859	11.439	13.314	15.434	17.384

Tab. 3 shows the data on the oscillation frequencies of the liquid at different values of the parameter n_g, with and without accounting for the surface tension respectively.

From the results above we see that even at $\lambda_c^2 = 10^{-3}$ there is a noticeable effect of surface tension, especially at high oscillation frequencies.

Table 3. Frequencies of oscillations at different values of the g-force parameter

The oscillation frequencies	G-force parameter n_g						
	0.1	0.25	0.5	1	2	3	4
ω_1	1.3431	2.1236	3.0032	4.2474	6.0065	7.3565	8.4946
ω_{c1}	1.5536	2.1803	3.0235	4.2544	6.0091	7.3579	8.4955

The obtained results indicate that for significant g-force parameters, the influence of surface tension becomes insignificant. But with the decrease of this parameter (at low levels of gravity) the influence of surface tension becomes dominant.

6. Conclusions

The integral equations method is generalized for the research of free liquid oscillations in shells at different values of the g-force parameter. A connected system of differential equations with respect to the elastic displacements of the structure and the effective fluid pressure has been obtained. Liquid oscillations in cylindrical and cylindrical-conical shells have been considered. It has been established that for the selected parameters of shells frequencies and forms of oscillations are similar for both shells considered. Therefore, for the selected parameters the shape of the bottom has a negligible effect on the frequencies and modes of oscillations; the determining factor is the level of filling of the shells. The results characterizing the influence of the parameters of g-force and surface tension have been obtained. The conditions under which the effect of surface tension becomes insignificant have been investigated.

ЛІТЕРАТУРА

1. O'Neil, W.J. Project Galileo - The Jupiter Mission. In: Barbieri C., Rahe J.H., Johnson T.V. & Sohus A.M. (eds) The Three Galileos: The Man, the Spacecraft, the Telescope. Astrophysics and Space Science Library, Springer, Dordrecht. 1997, 220. P. 17-23. URL: https://link.springer.com/chapter/10.1007/978-94-015-8790-7_8 (дата звернення: 22.11.2020).
2. Silveira, A.M., Stephens, D.G., Leonard, H.W. An Experimental Investigation of Liquid Oscillation in Tanks with Various Baffles, NASA Technical Note D-715, Washington: National Aeronautics And Space Administration, 1961. P. 5-24. URL: https://apps.dtic.mil/sti/citations/AD0256026 (дата звернення: 22.11.2020).
3. Abramson, H.N. *Slosh Suppression*, NASA Technical Report SP-8031, Washington: National Aeronautics And Space Administration, 1969. P. 1-23. URL: http://www.dept.aoe.vt.edu/~cdhall/courses/aoe4065/NASADesignSPs/sp8031.pdf (дата звернення: 22.11.2020).
4. Nayfeh, Ali Hasan. Third-harmonic resonance in the interaction of capillary and gravity weaves, *Journal of Fluid Mechanics*, 1971, 48 (2). P. 385–395.
5. Goldrick, L. F. Mc. An experiment on second-order capillary gravity resonant wave interaction, *Journal of Fluid Mechanics*, 1970, 40(2). P. 251–271. URL: https://doi.org/10.1017/s0022112071001630 (дата звернення: 22.11.2020).
6. Strelnikova E., Kriutchenko D., Gnitko V., Degtyarev K., Boundary element method in nonlinear sloshing analysis for shells of revolution under longitudinal excitations. *Engineering Analysis with Boundary Elements*. 2020. Vol. 111. P. 78-87. DOI: doi.org/10.1016/j.enganabound.2019.10.008. URL: http://www.sciencedirect.com/science/article/pii/S0955799719306149 (дата звернення: 22.11.2020).
7. Ibrahim R. A. *Liquid sloshing dynamics: theory and applications*. Cambridge: Cambridge University Press, 2005. DOI:10.1017/CBO9780511536656.
8. Gnitko V., Degtyariov K., Karaiev A., Strelnikova E. Singular boundary method in free vibration analysis of compound liquid-filled shells. *WIT Transactions on Engineering Sciences*. 2019. Vol. 126. P.189-200. DOI: 10.2495/BE420171. URL: https://www.witpress.com/elibrary/wit-transactions-on-engineering-sciences/126/37317 (дата звернення: 22.11.2020)

9. Strelnikova A., Choudhary N., Kriutchenko D., Gnitko V., Tonkonozhenko A. Liquid vibrations in circular cylindrical tanks with and without baffles under horizontal and vertical excitations *Engineering Analysis with Boundary Elements.* 2020. Vol. 120. P. 13-27. DOI: doi.org/10.1016/j.enganabound.2020.07.024. URL: http://www.sciencedirect.com/science/article/pii/S0955799720302009 (дата звернення: 22.11.2020).
10. Еселева Е.В. Гнитько В.И., Стрельникова Е.А. Собственные колебания сосудов высокого давления при взаимодействии с жидкостью. *Пробл. машиностроения.* 2006. Т.9. №1. С.105 - 118.
11. Strelnikova E., Gnitko V., Krutchenko D., Naumemko Y. Free and forced vibrations of liquid storage tanks with baffles *J. Modern Technology & Engineering*, 2018, Vol.3, No.1. P.15-52.
12. Brebbia C.A, Telles J.C.F, Wrobel L.C. Boundary element techniques: theory and applications in engineering. Springer-Verlag Berlin, Heidelberg, 1984. P.15-52. DOI: https://doi.org/10.1007/978-3-642-48860-3.
13. Karaiev A., Strelnikova E. Singular integrals in axisymmetric problems of elastostatics. *International Journal of Modeling, Simulation, and Scientific Computing*. 2020. Vol. 11, № 1. DOI: https://doi.org/10.1142/S1793962320500038. URL: https://www.researchgate.net/publication/338174729_Singular_Integrals_in_Axisymmetric_Problems_of_Elastostatics (дата звернення: 25.11.2020).
14. Strelnikova E., Medvedovskaya T. Free Hydroelastic Vibrations of Hydroturbine Head Covers Intern. J. *Eng. and Advanced Research Technology (IJEART)*. 2015. Vol. 1, No 1. P. 45–50. URL: https://doi.org/10.13140/RG.2.1.3527.4961.
15. Butikov E. Analytical expressions for stability regions in the Ince–Strutt diagram of Mathieu equation. *American Journal of Physics*, 2018, Vol.86. P. 257–267. DOI: https://doi.org/10.1119/1.5021895. URL: https://aapt.scitation.org/doi/pdf/10.1119/1.5021895 (дата звернення: 25.11.2020).
16. Kolukula S. S., Chellapandi P. Dynamic stability of plane free surface of liquid in axisymmetric tanks, *Advances in Acoustics and Vibration*, 2013. P. 16. DOI: 10.1155 / 2013/298458. URL: https://www.researchgate.net/publication/258391808_Dynamic_Stability_of_Plane_Free_SurfaSu_of_Liquid_in_Axisymmetric_Tanks (дата звернення: 25.11.2020).
17. Landau L. D., Lifshitz E. M. Fiuid Mechanics, 2nd ed., Pergamon, New York, 1987. P. 238-250. URL: https://users-phys.au.dk/srf/hydro/Landau+Lifschitz.pdf (дата звернення: 26.11.2020).
18. Мироненко М. Л. Вимушені коливання паливних баків при різних числах Бонда. *Прикладні питання математичного моделювання*, 2019, том 2, №2. С. 50-57. URL: https://doi.org/10.32782/2618-0340/2019.2-2.4 (дата звернення: 25.11.2020).

REFERENCES

1. O'Neil, W.J. Project Galileo - The Jupiter Mission. In: Barbieri C., Rahe J.H., Johnson T.V. & Sohus A.M. (eds) The Three Galileos: The Man, the Spacecraft, the Telescope. Astrophysics and Space Science Library, Springer, Dordrecht. 1997, 220, P. 17-23. URL: https://link.springer.com/chapter/10.1007/978-94-015-8790-7_8 (Last accessed: 22.11.2020).
2. Silveira, A.M., Stephens, D.G., Leonard, H.W. An Experimental Investigation of Liquid Oscillation in Tanks with Various Baffles, NASA Technical Note D-715, Washington: National Aeronautics And Space Administration, 1961, P. 5-24. URL: https://apps.dtic.mil/sti/citations/AD0256026 (Last accessed: 22.11.2020).
3. Abramson, H.N. *Slosh Suppression*, NASA Technical Report SP-8031, Washington: National Aeronautics And Space Administration, 1969, P. 1-23. URL: http://www.dept.aoe.vt.edu/~cdhall/courses/aoe4065/NASADesignSPs/sp8031.pdf (Last accessed: 22.11.2020).
4. Nayfeh, Ali Hasan. Third-harmonic resonance in the interaction of capillary and gravity weaves, *Journal of Fluid Mechanics*, 1971, 48 (2), P. 385–395.
5. Goldrick, L. F. Mc. An experiment on second-order capillary gravity resonant wave interaction, *Journal of Fluid Mechanics*, 1970, 40(2), P. 251–271. URL: https://doi.org/10.1017/s0022112071001630 (Last accessed: 22.11.2020).
6. Strelnikova E., Kriutchenko D., Gnitko V., Degtyarev K., Boundary element method in nonlinear

sloshing analysis for shells of revolution under longitudinal excitations. *Engineering Analysis with Boundary Elements*. 2020. Vol. 111, P. 78-87. DOI: doi.org/10.1016/j.enganabound.2019.10.008. URL: http://www.sciencedirect.com/science/article/pii/S0955799719306149 (Last accessed: 22.11.2020).
7. Ibrahim R. A. *Liquid sloshing dynamics: theory and applications*. Cambridge: Cambridge University Press, 2005. DOI:10.1017/CBO9780511536656.
8. Gnitko V., Degtyariov K., Karaiev A., Strelnikova E. Singular boundary method in free vibration analysis of compound liquid-filled shells. *WIT Transactions on Engineering Sciences.* 2019. Vol. 126, P.189-200. DOI: 10.2495/BE420171. URL: https://www.witpress.com/elibrary/wit-transactions-on-engineering-sciences/126/37317 (Last accessed: 22.11.2020)
9. Strelnikova A., Choudhary N., Kriutchenko D., Gnitko V., Tonkonozhenko A. Liquid vibrations in circular cylindrical tanks with and without baffles under horizontal and vertical excitations *Engineering Analysis with Boundary Elements.* 2020. Vol. 120, P. 13-27. DOI: doi.org/10.1016/j.enganabound.2020.07.024. URL: http://www.sciencedirect.com/science/article/pii/S0955799720302009 (Last accessed: 22.11.2020).
10. Eseleva E.V. Gnitko V.I. Strelnikova E.A. Natural oscillations pressure vessel interacting with the liquid. *Journal of Mechanical Engineering.* 2006. Vol.9. №1, P.105 - 118. [in Russian]
11. Strelnikova E., Gnitko V., Krutchenko D., Naumemko Y. Free and forced vibrations of liquid storage tanks with baffles *J. Modern Technology & Engineering*, 2018, Vol.3, No.1, P.15-52.
12. Brebbia C.A, Telles J.C.F, Wrobel L.C. Boundary element techniques: theory and applications in engineering. Springer-Verlag Berlin, Heidelberg, 1984, P.15-52. DOI: https://doi.org/10.1007/978-3-642-48860-3.
13. Karaiev A., Strelnikova E. Singular integrals in axisymmetric problems of elastostatics. *International Journal of Modeling, Simulation, and Scientific Computing*. 2020. Vol. 11, № 1. DOI: https://doi.org/10.1142/S1793962320500038. URL: https://www.researchgate.net/publication/338174729_Singular_Integrals_in_Axisymmetric_Problems_of_Elastostatics (Last accessed: 25.11.2020).
14. Strelnikova E., Medvedovskaya T. Free Hydroelastic Vibrations of Hydroturbine Head Covers Intern. J. *Eng. and Advanced Research Technology (IJEART)*. 2015. Vol. 1, No 1, P. 45–50. DOI: 10.13140/RG.2.1.3527.4961.
15. Butikov E. Analytical expressions for stability regions in the Ince–Strutt diagram of Mathieu equation. *American Journal of Physics*, 2018, Vol.86, P. 257–267. DOI: https://doi.org/10.1119/1.5021895. URL: https://aapt.scitation.org/doi/pdf/10.1119/1.5021895 (Last accessed: 25.11.2020).
16. Kolukula S. S., Chellapandi P. Dynamic stability of plane free surface of liquid in axisymmetric tanks, *Advances in Acoustics and Vibration*, 2013, P. 16. DOI: 10.1155 / 2013/298458. URL: https://www.researchgate.net/publication/258391808_Dynamic_Stability_of_Plane_Free_SurfaSu _of_Liquid_in_Axisymmetric_Tanks (Last accessed: 25.11.2020).
17. Landau L. D., Lifshitz E. M. Fiuid Mechanics, 2nd ed., Pergamon, New York, 1987, P. 238-250. URL: https://users-phys.au.dk/srf/hydro/Landau+Lifschitz.pdf (Last accessed: 26.11.2020).
18. Myronenko M.L. Forced Vibrations of Fuel Tanks at Different Bond's Numbers. *Applied Questions of Mathematical Modelling*, 2019, Vol. 2, №2, P. 50-57. [in Ukrainian] URL: https://doi.org/10.32782/2618-0340/2019.2-2.4 (Last accessed: 25.11.2020).

Надійшла у першій редакції 01.02.2021, в останній - 03.06.2021.