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The hipersingular integral equation method in the problem of determination of vibration modes and frequencies of a circular plate immersed in a liquid

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The new approach for studying the vibration modes and frequencies of a circular plate immersed in a liquid has been developed. The approach is based on the use of hypersingular integral equations and the assumed mode method. It is assumed that a round thin elastic plate is immersed in an ideal incompressible fluid, and its motion is considered to be irrotational. Under these conditions, there is a velocity potential that satisfies the Laplace equation everywhere outside the plate, and the no-flow condition is satisfied for the plate surface. The fluid pressure has been determined by using the linearized Cauchy-Lagrange integral. For solving the boundary value problem with regard to the velocity potential, an integral representation in the form of a double layer potential has been used, the potential density being proportional to the fluid pressure drop. The assumed mode method makes it possible to reduce the problem of determining the added masses of a liquid to solving hypersingular equations on a circular domain. The two-dimensional hypersingular integral equations have been reduced to one-dimensional ones. Therefore, the inner integrals take the form of elliptic integrals of the second kind without singularities. To calculate the external integral, which exists only in the sense of Hadamard, the boundary element method is proposed. A procedure for calculating the matrix elements of a system of linear algebraic equations for determining the density of the double layer potential has been developed. A numerical solution of the hypersingular integral equation has been obtained, and a comparison of the numerical and analytical solutions has been carried out. The right-hand sides of hypersingular integral equations are the modes of vibrations of a rigidly fixed circular plate, which are obtained analytically. A technique for calculating the matrix of added masses has been developed, which makes it possible to reduce the considered problem to solving the problem of eigenvalues.

Key words: *thin plate, perfect incompressible fluid, vibration, hypersingular integral equation, boundary element method*

Метод гіперсингулярних інтегральних рівнянь в задачі визначення частот та форм коливань круглої пластинки, зануреної в рідину

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Для дослідження частот та форм коливань круглої пластинки, що занурена в рідину, розроблено новий підхід, заснований на застосуванні гіперсингулярних інтегральних рівнянь та методу заданих форм. Припускається, що кругла тонка пружна пластина занурена в ідеальну нестисливу рідину, рух якої вважається безвихровим. В цих умовах існує потенціал швидкостей, що задовольняє рівнянню Лапласа всюди за межами пластини, а на поверхні пластини виконується умова непротікання. Тиск рідини визначено з лінеаризованого інтегралу Коші-Лагранжа. При розв'язанні крайової задачі щодо потенціалу швидкостей використано інтегральне зображення у вигляді потенціалу подвійного шару, при цьому густина потенціалу пропорційна перепаду тиску рідини. Використання методу заданих форм дозволило звести задачу визначення приєднаних мас рідини до розв'язання гіперсингулярних рівнянь на круговій області. Здійснено зведення двовимірних гіперсингулярних інтегральних рівнянь до одновимірних. Внутрішні інтеграли при цьому набувають форму еліптичних інтегралів другого роду, що не мають особливостей. Для обчислення зовнішнього інтегралу, який існує лише в сенсі Адамара, запропоновано використати метод граничних елементів. Розроблено процедуру обчислення елементів матриці системи лінійних алгебраїчних рівнянь для знаходження невідомої густині потенціалу подвійного шару. Здійснено розв'язок гіперсингулярного рівняння та наведено порівняння числових та аналітичних розв'язків. Праві частини гіперсингулярних інтегральних рівнянь є формами коливань жорстко закріпленої круглої пластини, які отримані аналітичним шляхом. Розроблено методику обчислення матриці приєднаних мас, що дозволило звести задачу, що розглядається, до розв'язання проблеми власних значень.

Ключові слова:*тонка пластина, ідеальна нестислива рідина, коливання, гіперсингулярне інтегральне рівняння, метод граничних елементів.*

Метод гиперсингулярных интегральных уравнений в задаче определения частот и форм колебаний круглой пластины, погруженной в жидкость

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Для исследования частот и форм колебаний круглой пластинки, погруженной в жидкость, разработан новый подход, основанный на применении гиперсингулярных интегральных уравнений и метода заданных форм. Предполагается, что круглая тонкая упругая пластина погружена в идеальную несжимаемую жидкость, движение которой считается безвихревым. В этих условиях существует потенциал скоростей, который удовлетворяет уравнению Лапласа всюду вне пластины, а на поверхности пластины выполняется условие непротекания. Давление жидкости определено с помощью линеаризованного интеграла Коши-Лагранжа. При решении краевой задачи относительно потенциала скоростей использовано интегральное представление в виде потенциала двойного слоя, при этом плотность потенциала пропорциональна перепаду давления жидкости. Использование метода заданных форм позволило свести задачу определения присоединенных масс жидкости к решению гиперсингулярных уравнений на круговой области. Осуществлено сведение двумерных гиперсингулярных интегральных уравнений к одномерным. Внутренние интегралы при этом приобретают форму эллиптических интегралов второго рода, не имеющих особенностей. Для вычисления внешнего интеграла, который существует только в смысле Адамара, предложено использовать метод граничных элементов. Разработана процедура вычисления элементов матрицы системы линейных алгебраических уравнений для нахождения неизвестной плотности потенциала двойного слоя. Получено численное решение гиперсингулярного интегрального уравнения, и проведено сравнение численного и аналитического решений. Правые части гиперсингулярных интегральных уравнений являются формами колебаний жестко закрепленной круглой пластины, полученных аналитическим путем. Разработана методика вычисления матрицы присоединенных масс, что позволило свести рассматриваемую задачу к решению проблемы собственных значений.

Ключевые слова*: тонкая пластина, идеальная несжимаемая жидкость, колебания, гиперсингулярное интегральное уравнение, метод граничных элементов*

1. Introduction

Problems of hydroelastic interaction of structural elements with the environment have been of great interest to many scientists in recent decades. These problems are conventionally divided into two classes. The first includes the task of determining the dynamic characteristics of the structures containing a liquid; i.e. the surfaces of the structural element are in a unilateral contact with the liquid. The works [1-3] are devoted to the problems of liquid splashes for rigid shell systems. The vibrations of elastic reservoirs partially filled with a liquid are analyzed in the article [4]. The free vibrations of an elastic head cover of a hydro turbine during interaction with a liquid are considered in [5, 6].

The second class incudes the determination of the dynamic characteristics of elastic thin structures where the bearing surfaces are in bilateral contact with a liquid, such as the blades of the impellers of

Francis and Kaplan turbines [7, 8], the blades of powerful air units [9, 10], the wings of aircraft [11], etc.

The powerful computational methods have been developed for solving such problems numerically. Among them there are the method of finite differences, [12], the methods of finite and boundary elements [13, 14], the method of R-functions [15].

But each new design, which is intended to work under intense power loads and to interact with the environment, requires careful analysis of its strength and dynamic characteristics. This leads to the need of developing new and improving existing computational methods that would take into account the specific features of the structure being analyzed.

The new effective method for solving hypersingular integral equations on a circular domain has been developed in the article. The method has been successfully used to determine the free vibration modes and frequencies of a circular plate immersed in a liquid.

2. Formulation of the problem

The vibration of a thin circular plate immersed in a liquid is analyzed. The Kirchhoff-Lev hypothesis has been used to model the motion of the plate [16].

The following notation is used: E is the Young's modulus, ν is the Poisson's ratio, and h is the plate thickness.

With a thickness of a homogeneous plate being constant, the equation of motion of the plate [16] has the form

$$D\Delta\Delta w+\rho_p h\frac{\partial^2 w}{\partial t^2}=q(x,y.t), \tag{2.1}$$

where $w(x,y,t)$ is a deflection of the plate;

ρ_p is density of the plate material;

$D=\dfrac{Eh^3}{12(1-\nu^2)}$ is cylindrical rigidity;

$q(x,y,t)$ is a force acting on the plate.

If the plate is immersed in a liquid, then

$$q(x,y,t)=p(x,y,t)\mathbf{n}+q_0(x,y,t), \tag{2.2}$$

where $p(x,y,t)$ is a liquid pressure drop on the plate;

$\mathbf{n}$ is a unit normal to the surface of the plate;

$q_0(x,y,t)$ is a disturbing force.

If in the equation (2.2) the function $q_0(x,y,t)$=0, then we have the case of free hydroelastic vibration.

To find the pressure $p(x,y,t)$, the following assumptions have been made: the fluid is an ideal and incompressible, and its motion is vortex-free. In this case, there is a velocity potential φ(*x*,*y*,*z*,*t*), and its gradient is the velocity of the liquid, namely:

$$V_x=\frac{\partial\varphi}{\partial x}, V_y=\frac{\partial\varphi}{\partial y}, V_z=\frac{\partial\varphi}{\partial z}.$$

This potential satisfies the Laplace equation

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=0. \tag{2.3}$$

The relationship between velocity potential and pressure is determined from the linearized Cauchy-Lagrange integral

$$p^{\pm}=-\rho_l\frac{\partial\varphi^{\pm}}{\partial t}+p_0$$

where ρ_l is density of the liquid;

p_0 is an atmospheric pressure.

The limit values $\varphi^{\pm}$ correspond to the transition to the plane of the plate from the positive and negative values of z coordinate, respectively.

Thus, for the pressure drop we have

$$p = p^+ - p^- = -\rho_l \left(\frac{\partial \varphi^+}{\partial t} - \frac{\partial \varphi^-}{\partial t} \right). \tag{2.4}$$

The boundary conditions for the differential equation (2.3) have been formulated. According to the condition of non-leakage [17] we could obtain the following equations for the surfaces σ of the plate:

$$\left. \frac{\partial \varphi^\pm}{\partial \mathbf{n}} \right|_\sigma = \frac{\partial w}{\partial t}. \tag{2.5}$$

The system of differential equations (2.1), (2.3) with boundary condition (2.5) must also be supplemented by the conditions of fixation, in other words by the conditions which respects to the function $w(x,y,t)$ on the plate contour.

Thus, we arrive at a related problem with respect to two unknown functions $w(x,y,t)$ and φ(x,y,z,t). Moreover, the right-hand side of the differential equation $w(x,y,t)$ includes time derivative of φ(x,y,z,t), while boundary conditions for the Laplace equation to φ(x,y,z,t) include time derivative of an unknown function $w(x,y,t)$.

3. The assumed mode method

To solve the problem of natural and forced vibrations of elastic plate immersed in a liquid, the assumed mode method [17] has been used. Generally, we consider the movement of the plate immersed in a liquid as:

$$w(x,y,t) = \sum_{k=1}^{N} c_k(t) w_k(x,y), \tag{3.1}$$

where $c_k(t)$ are unknown coefficients that depend only on time;

$w_k(x,y)$ are modes of plate vibration without taking into account the interactions with the liquid;

N is the number of modes retained in the calculations.

From (2.4), (3.1) we can see that for the velocity potential φ(x,y,z,t) it is necessary to choose the following

$$\varphi(x,y,z,t) = \sum_{k=1}^{N} \dot{c}_k(t) \varphi_k(x,y,z), \tag{3.2}$$

in which the functions $\varphi_k(x,y,z)$ are solutions of the following boundary value problems

$$\Delta \varphi_k = 0, \quad \frac{\partial \varphi_k}{\partial \mathbf{n}} = w_k. \tag{3.3}$$

It should be noted that in the boundary value problems (3.3) the right-hand sides in the boundary conditions are known.

In the next part we will describe the method of reducing the problem of determining the vibration modes and frequencies of a "plate-liquid" system to the hypersingular equations.

4. The hypersingular equations for determining the velocity potential and fluid pressure

To find the fluid pressure drop on the surfaces of the plate, the equation (2.4) can be used. It should be noted that the velocity potential is a harmonic function everywhere outside the circular plate, i.e. it is a continuous function in three-dimensional space with a section. In this case, according to the equation (2.5), this harmonic function has a continuous normal derivative at the intersection of the section in the form of a circle occupied by the plate. But according to the equation (2.4), this function has a finite gap in the specified area.

Such properties are known to be inherent in the potential of the double layer [18]. This potential has a form

$$\varphi(\mathbf{P}_0) = \frac{1}{4\pi} \iint_S \Gamma(\mathbf{P}) \frac{\partial}{\partial \mathbf{n}} \frac{1}{|\mathbf{P} - \mathbf{P}_0|} dS, \quad \mathbf{P} \in S, \tag{4.1}$$

where S is an area occupied by the circular plate;

n is a unit normal to the surface S;

P and $\mathbf{P}_0$ are the points of three-dimensional space with coordinates (x, y, z) and (x_0, y_0, z_0), respectively;

$|\mathbf{P} - \mathbf{P}_0| = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$ is a Cartesian distance between **P** and $\mathbf{P}_0$;

$\Gamma(\mathbf{P})$ is potential density, the function given on the surface S.

The function defined by the formula (4.1) satisfies the Laplace equation (2.3), has a continuous normal derivative, and at the intersection of the surface *S* along the normal has a finite discontinuity, namely [18]

$$\varphi^+(\mathbf{P}_0) - \varphi^-(\mathbf{P}_0) = \Gamma(\mathbf{P}_0), \quad \mathbf{P}_0 \in S .$$

Therefore, we use the representation (4.1) for the velocity potential. To determine the pressure drop, it is necessary to find the density $\Gamma(\mathbf{P})$ on the surface of the plate. From (3.2) it follows that we need to obtain solutions of the boundary value problems (3.3). For the functions φ_k the expression (4.1) is used. The Laplace equation with (3.3) is satisfied, and the boundary condition leads to the integral equation in the form

$$\frac{\partial}{\partial \mathbf{n}_0} \frac{1}{4\pi} \iint_S \Gamma_k(\mathbf{P}) \frac{\partial}{\partial \mathbf{n}} \frac{1}{|\mathbf{P} - \mathbf{P}_0|} = w_k(\mathbf{P}_0), \quad \mathbf{P}, \mathbf{P}_0 \in S \tag{4.2}$$

As proved in [19, 20], the integral in (4.2) does not exist either as an ordinary improper one or in the sense of the Cauchy principal value. This integral can be interpreted only in the Hadamard sense [21, 22] as a limit value of the normal derivative of the double layer potential. For the case where the surface S is a part of a plane, we have

$$\frac{\partial}{\partial z_0} \frac{1}{4\pi} \iint_S \Gamma_k(\mathbf{P}) \frac{\partial}{\partial z} \frac{1}{|\mathbf{P} - \mathbf{P}_0|} = w_k(\mathbf{P}_0), \quad \mathbf{P}, \mathbf{P}_0 \in S ,$$

or

$$\lim_{z_0 \to 0} \frac{1}{4\pi} \iint_S \Gamma_k(x, y) \frac{\partial}{\partial z_0} \frac{-z_0}{\left(\sqrt{(x-x_0)^2 + (y-y_0)^2 + z_0^2}\right)^3} = w_k(x_0, y_0) . \tag{4.3}$$

The equations of the same type as (4.3) refer to hypersingular integral equations.

Some methods of numerical solution of such equations are investigated in [22].

5. Reduction of two-dimensional hypersingular equations to one-dimensional singular equations

If the domain *S* in the equation (4.3) is a circle, then it is possible to reduce a two-dimensional hypersingular equation to a one-dimensional one.

Let *S* be a circle on the *x*O*y* plane, namely

$$S = \{x, y : x^2 + y^2 \le R^2\}$$

Let us calculate the Cartesian distance $|\mathbf{P} - \mathbf{P}_0|$ for the points of the specified region *S*. In cylindrical coordinates we have

$$x = \rho\cos\theta, \quad y = \rho\sin\theta, \quad z = z, \quad x^2 + y^2 = \rho^2,$$
$$x_0 = \rho_0\cos\theta_0, \quad y_0 = \rho_0\sin\theta_0, \quad z_0 = z_0, \quad x_0^2 + y_0^2 = \rho_0^2.$$

as follows

$$|\mathbf{P} - \mathbf{P}_0| = \sqrt{\rho^2 + \rho_0^2 + (z-z_0)^2 - 2\rho\rho_0\cos(\theta - \theta_0)} .$$

Then we use standard notation [14]

$$a = \rho^2 + \rho_0^2 + (z-z_0)^2, \quad b = 2\rho\rho_0 .$$

Firstly, we suppose that the functions w_k, Γ_k do not depend on the circumferential coordinate θ, i.e. we consider the problem in an axially symmetric formulation.

Then by using (4.3) we obtain the following hypersingular integral equation $(z = z_0 = 0)$:

$$\frac{1}{4\pi} \iint_S \Gamma_k(x, y) \frac{-dxdy}{\left(\sqrt{(x-x_0)^2 + (y-y_0)^2}\right)^3} = w_k(x_0, y_0) .$$

With the transition to a cylindrical coordinate system and taking into account the assumptions about the axial symmetry of the problem, we find that

$$A\Gamma_k = \frac{1}{4\pi}\iint_S \Gamma_k(\rho)\frac{\rho d\theta d\rho}{\left(\sqrt{a-b\cos(\theta-\theta_0)}\right)^3} = -w_k(\rho_0).$$

Let us write this integral as a repeated one

$$A\Gamma_k = \frac{1}{4\pi}\int_0^R \rho\Gamma_k(\rho)\left[\int_0^{2\pi}\frac{d\theta}{\left(\sqrt{a-b\cos(\theta-\theta_0)}\right)^3}\right]d\rho = -w_k(\rho_0). \tag{5.1}$$

Let us make a variable replacement in the internal integral in (5.1)

$$\psi = \theta-\theta_0, \quad \theta = \psi+\theta_0, \quad d\theta = d\psi .$$

Since under the sign of the integral there is a periodic function that integrates by a period, the boundaries of integration will not change. Therefore, we take not $(0,2\pi)$ but $(-\pi\ ,\ \pi)$. For the internal integral we obtain

$$I_1(\rho,\rho_0) = \int_{-\pi}^{\pi}\frac{d\psi}{\left(\sqrt{a-b\cos\psi}\right)^3}.$$

Due to the parity of the subintegral function we have

$$I_1(\rho,\rho_0) = 2\int_0^{\pi}\frac{d\psi}{\left(\sqrt{a-b\cos\psi}\right)^3}.$$

Then we make the transformations in this integral

$$I_1(\rho,\rho_0) = 2\int_0^{\pi}\frac{d\psi}{\left(\sqrt{a-b\cos\psi}\right)^3} = 2\int_0^{\pi}\frac{d\psi}{\left(\sqrt{a+b-2b\cos^2(\psi/2)}\right)^3} = 4\int_0^{\pi/2}\frac{d\psi}{\left(\sqrt{a+b-2b\cos^2\psi}\right)^3}.$$

After another variable replacement

$$\psi_1 = \pi/2-\psi; \quad d\psi_1 = -d\psi$$

we find that

$$I_1(\rho,\rho_0) = \frac{4}{\left(\sqrt{a+b}\right)^3}\int_0^{\pi/2}\frac{d\psi}{\left(\sqrt{1-k^2\sin^2\psi}\right)^3}, \quad k^2 = \frac{2b}{a+b}.$$

Next we use the formula from [23] and obtain

$$\int_0^{\pi/2}\frac{d\psi}{\left(\sqrt{1-k^2\sin^2\psi}\right)^3} = \frac{1}{k'^2}\mathsf{E}(k), \quad \mathsf{E}(k) = \int_0^{\pi/2}\sqrt{1-k^2\sin^2\psi}\,d\psi, \quad k'^2 = 1-k^2 .$$

Since $k'^2 = 1-k^2 = 1-\frac{2b}{a+b} = \frac{a-b}{a+b}$, then for the internal integral we have

$$I_1(\rho,\rho_0) = \frac{4}{\sqrt{a+b}(a-b)}\int_0^{\pi/2}\sqrt{1-k^2\sin^2\psi}\,d\psi = \frac{4}{(\rho+\rho_0)(\rho-\rho_0)^2}\mathsf{E}(k),$$

where $\mathsf{E}(k)$ is a complete elliptic integral of the second kind.

Thus, we obtain a one-dimensional hypersingular integral equation for determining the functions $\Gamma_m(\rho)$ by the known functions $w_m(\rho)$, $m = 1,2,...$

$$\frac{1}{\pi}\int_0^R \Gamma_m(\rho)\frac{\rho\mathsf{E}(k)d\rho}{(\rho+\rho_0)(\rho-\rho_0)^2} = -w_m(\rho_0). \tag{5.2}$$

Note that the kernel of the integral operator in (5.2) has the following form:

$$K(\rho,\rho_0) = \frac{\rho\mathsf{E}(k)}{(\rho+\rho_0)(\rho-\rho_0)^2}.$$

This kernel has a feature of type $(\rho-\rho_0)^{-2}$.

To clarify the nature of the function $K(\rho,\rho_0)$ near the singularity $(\rho-\rho_0=0)$ we use the asymptotic formula [23] for the function $\mathsf{E}(k)$ at k, close to 1

$$\mathsf{E}(k)=1+\frac{1}{2}k'^2\left(\ln\frac{4}{k'}-\frac{1}{2}\right)+\left(\frac{1}{2}\right)^2\frac{3}{4}k'^4\left(\ln\frac{4}{k'}-1-\frac{1}{3}\frac{1}{4}\right)+\ldots$$

Let l_0 be a special element, that is $l_0\in[0,R]$, $\rho_0\in l_0$. Let us introduce the notation

$$K_0(\rho,\rho_0)=\frac{\rho}{(\rho+\rho_0)(\rho-\rho_0)^2}\left[1+\frac{1}{2}k'^2\left(\ln\frac{4}{k'}-\frac{1}{2}\right)\right].$$

Then for calculating the integral of the element containing the feature, we use the formula

$$\int_{l_0}\Gamma_m(\rho)K(\rho,\rho_0)d\rho=\int_{l_0}\Gamma_m(\rho)K_0(\rho,\rho_0)d\rho+\int_{l_0}\Gamma_m(\rho)[K(\rho,\rho_0)-K_0(\rho,\rho_0)]d\rho\,. \tag{5.3}$$

The first part here contains hypersingular and logarithmic components, and the second is an integral of the function that does not contain features.

6. Determination of free vibration modes and frequencies of a circular plate without taking into account the added masses of a liquid

Let us assume that in the equation (2.1) the function $q(x,y,t)$ = 0, i.e we assume that the plate vibrates freely. Then we look for a solution of the equation (2.1) in the form

$$w(x,y,t)=\exp(i\Omega t)w(x,y)\,. \tag{6.1}$$

Then we substitute (6.1) into the homogeneous equation (2.1) and obtain the differential equation in partial derivatives with respect to the amplitude values $w(x,y)$

$$\Delta\Delta w-\alpha^4 w=0,\quad \alpha^4=\Omega^2\frac{12\rho_p\left(1-\nu^2\right)}{Eh^2}=\Omega^2\frac{\rho_p h}{D}\,. \tag{6.2}$$

The equation (6.2) is represented in the form

$$\left(\Delta-\alpha^2\right)\left(\Delta+\alpha^2\right)w=0$$

which is equivalent to the system

$$\begin{cases}\left(\Delta-\alpha^2\right)w=0\\ \left(\Delta+\alpha^2\right)w=0\end{cases} \tag{6.3}$$

For a circular plate in polar coordinates we have

$$\Delta=\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}$$

In this case, the system (6.3) takes the form

$$\frac{\partial^2 w}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial w}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial w}{\partial\theta}-\alpha^2 w=0\,, \tag{6.4}$$

$$\frac{\partial^2 w}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial w}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial w}{\partial\theta}+\alpha^2 w=0\,. \tag{6.5}$$

The solutions of the equation (6.4) are the Bessel functions of the first and second kind of zero order $\mathrm{J}_0(\alpha\rho)$ and $\mathrm{Y}_0(\alpha\rho)$, and the solutions of the equation (6.5) are the modified Bessel functions of the first and second kind of $\mathrm{I}_n(\alpha\rho)$and $\mathrm{K}_n(\alpha\rho)$. Thus, the general solution of the equation (6.2) has the form

$$w(\rho)=A\mathrm{J}_0(\alpha\rho)+B\,\mathrm{Y}_0(\alpha\rho)+C\,\mathrm{I}_0(\alpha\rho)+D_1\,\mathrm{K}_0(\alpha\rho)$$

where A,B,C,D_1 are constants.

Since the functions $\mathrm{Y}_n(\alpha\rho)$ and $\mathrm{K}_n(\alpha\rho)$ will infinitely increase if $\rho\to 0$, we assume that $B=0,\ D_1=0$ to avoid non-physical displacements.

To determine the constants A, C, we use the boundary conditions for fixing the plate along the contour. For the case of rigid fixation we have the following boundary conditions [16]:

$$w\big|_{\rho=R} = 0, \quad \frac{dw}{dr}\bigg|_{\rho=R}$$

From these conditions it follows that

$$\begin{cases} A\mathrm{J}_n(\alpha R) + C\,\mathrm{I}_n(\alpha R) = 0 \\ A\mathrm{J}'_n(\alpha R) + C\,\mathrm{I}'_n(\alpha R) = 0 \end{cases}. \tag{6.6}$$

For the system (6.6) to have a nonzero solution, it is necessary for the system determinant to be zero. Therefore, we obtain the equation for finding the unknown quantity α

$$\begin{vmatrix} \mathrm{J}_n(\alpha R) & \mathrm{I}_n(\alpha R) \\ \mathrm{J}'_n(\alpha R) & \mathrm{I}'_n(\alpha R) \end{vmatrix} = \mathrm{J}_n(\alpha R)[\mathrm{I}_{n-1}(\alpha R) + \mathrm{I}_{n+1}(\alpha R)] - \mathrm{I}_n(\alpha R)[\mathrm{J}_{n-1}(\alpha R) - \mathrm{J}_{n+1}(\alpha R)] = 0 \tag{6.7}$$

Let's denote $\lambda = \alpha R$. Table 6.1 shows the first six root values of the equation (6.7) for different values of n.

Table 6.1 The values of the roots of the characteristic equation

m	λ_m
1	3.196220616
2	6.306437050
3	9.439499140
4	12.57713064
5	15.71643853
6	18.85654552

The relation between the constants A and C for each α_m is obtained from the equality

$$A_{mn}\,\mathrm{J}_n(\alpha_{mn}R) + C_m\,\mathrm{I}_n(\alpha_{mn}R) = 0 \Rightarrow C_{mn} = -A_{mn}\frac{\mathrm{J}_n(\alpha_{mn}R)}{\mathrm{I}_n(\alpha_{mn}R)}$$

Note that we can choose $A_m = 1\ \forall m$.

Thus, the dependences of free vibration modes of the circular plate on ρ in the mode are obtained

$$w_{mn}(\rho) = \mathrm{J}_0(\alpha_{mn}\rho) - \frac{\mathrm{J}_n(\alpha_{mn}R)}{\mathrm{I}_n(\alpha_{mn}R)}\mathrm{I}_n(\alpha_m\rho). \tag{6.8}$$

These functions appear at right-hand sides of the hypersingular equations (5.2).

Fig. 6.1 shows the functions defined by the formulas (6.8) for $R = 1$.

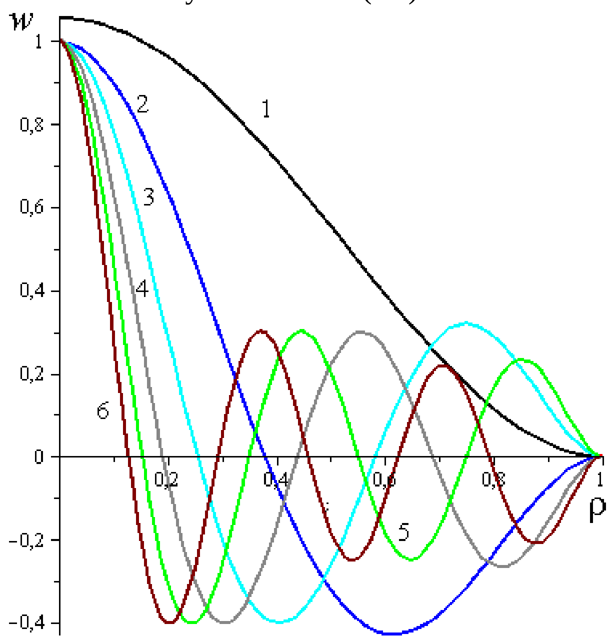

Figure 6.1 Dependences of vibration modes on the radial coordinate

Numbers 1-6 in Fig. 6.1 denote the modes corresponding to the values $\lambda_m = \alpha_m$, given in Table 6.1.

Fig. 6.2 shows the first six modes of axially symmetrical vibrations of a circular plate rigidly fixed along the contour.

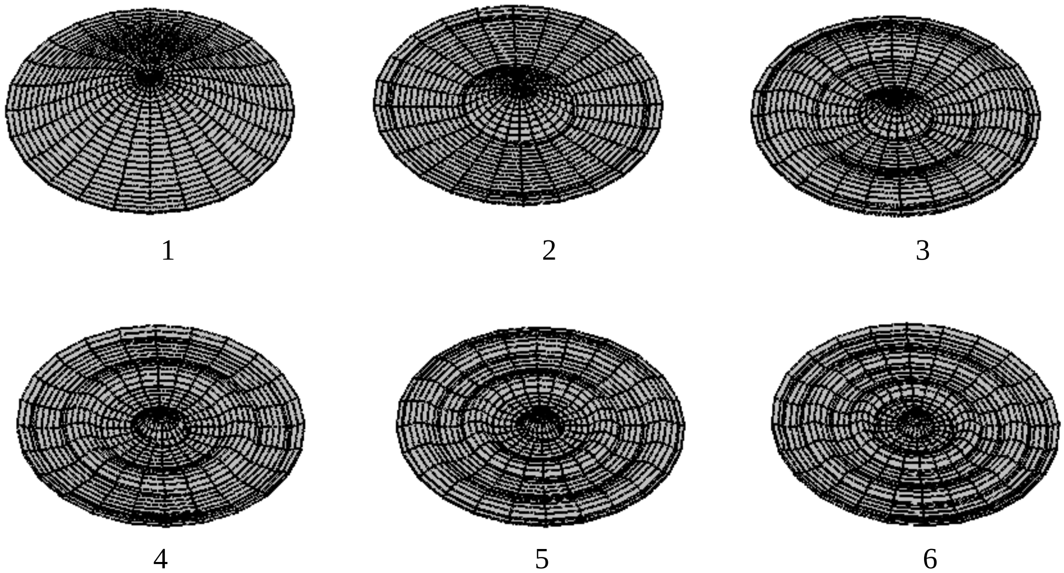

Figure 6.2 Modes of axially symmetric vibrations of a round plate

According to the formulas (6.8), for the increasing number *m*, vibration modes of $w_m(\rho)$ will asymptotically approach $J_0(\alpha_m \rho)$.

Checking the orthogonality of the natural vibration modes $w_m(\rho)$ we can find that

$$(w_m(\rho), w_l(\rho)) = A_m \delta_{ml} . \qquad (6.9)$$

The following values are obtained for the constants A_m

A_1 = 0.1018870979, A_2 =0.0506907858, A_3 = 0.0337792448, A_4 = 0.0253319976, A_5 =0.0202649244, A_6 = 0.0168871927.

7. The numerical method of solving a hypersingular integral equation

For a numerical solution of a hypersingular integral equation

$$\frac{1}{\pi}\int_0^R \Gamma(\rho)\frac{\rho \mathrm{E}(k)d\rho}{(\rho+\rho_0)(\rho-\rho_0)^2} = -w(\rho_0) \qquad (7.1)$$

with known right-hand sides, the method of boundary elements with a constant approximation of the density [22] has been used. Note that as compared with (5.2), the index *m* is omitted in the equation (7.1) to simplify the recording.

To do this, we divide the integration region [0, *R*] into N_1 boundary elements of an equal length. Let us denote

$$l_i = [\rho_{i-1}, \rho_i], \quad i = \overline{1,N}, \quad 2\tau = \rho_i - \rho_{i-1} = R/N_1, \quad \rho_0 = 0, \quad \rho_N = R .$$

Inside each element l_i, we select the points (collocation points) as follows:

$$\rho_{i0} = 0.5\cdot(\rho_i + \rho_{i-1}), \quad i = \overline{1,N_1} .$$

After we roughly replace the integral in (7.1) by the integral sum, assuming that for each element the unknown density is constant, that is $\Gamma(\rho) = \Gamma_i, \quad \rho \in [\rho_{i-1}, \rho_i]$, we obtain a system of linear algebraic equations with respect to Γ_l in the form

$$\frac{1}{\pi}\int_0^R \Gamma(\rho)K(\rho,\rho_{i0})d\rho = \sum_{j=1}^{N} A_{ij}\Gamma_j = -w(\rho_{i0}), \quad i = \overline{1,N_1} ,$$

where the system matrix elements are calculated by the formula

$$A_{ij} = \frac{1}{\pi}\int_{\rho_{j=1}}^{\rho_j} K(\rho, \rho_{0i})d\rho . \tag{7.2}$$

In the case of the boundary element not containing a collocation point, $i \neq j$, standard quadrature formulas can be used for the calculation of A_{ij}. If $i = j$, the following formula can be used

$$\Gamma_j \int_{\rho_{i-1}}^{\rho_i} K(\rho, \rho_{j0})d\rho = \Gamma_j \int_{\rho_{i-1}}^{\rho_i} K_0(\rho, \rho_{j0})d\rho + \Gamma_j \int_{\rho_{i-1}}^{\rho_i} \left[K(\rho, \rho_{j0}) - K_0(\rho, \rho_{j0})\right]d\rho . \tag{7.3}$$

According to the results obtained above, the second term in (7.3) is not singular, and standard quadrature formulas can also be used to calculate it. The first term contains hypersingular and logarithmic features. The hypersingular component has the form

$$K_h(\rho, \rho_0) = \frac{\rho}{(\rho+\rho_0)(\rho-\rho_0)^2} .$$

Let us make such transformations

$$K_h(\rho, \rho_0) = \frac{\rho+\rho_0}{(\rho+\rho_0)(\rho-\rho_0)^2} - \frac{\rho_0}{(\rho^2-\rho_0^2)(\rho-\rho_0)} = \frac{1}{(\rho-\rho_0)^2} - \frac{1}{2}\frac{1}{\rho-\rho_0}\left[\frac{1}{\rho-\rho_0} - \frac{1}{\rho+\rho_0}\right] =$$

$$= \frac{1}{2(\rho-\rho_0)^2} + \frac{1}{2}\frac{1}{(\rho^2-\rho_0^2)} = \frac{1}{2(\rho-\rho_0)^2} + \frac{1}{4\rho_0}\frac{1}{\rho-\rho_0} - \frac{1}{4\rho_0}\frac{1}{\rho+\rho_0}.$$

In this expression there is the component that does not contain features, namely

$$K_{h2}(\rho, \rho_0) = -\frac{1}{4\rho_0}\frac{1}{\rho+\rho_0}. \tag{7.4}$$

Let

$$K_{h1}(\rho, \rho_0) = \frac{1}{2(\rho-\rho_0)^2} + \frac{1}{4\rho_0}\frac{1}{\rho-\rho_0}.$$

We calculate the integral over a special domain from $K_{h1}(\rho, \rho_0)$

$$\int_{\rho_{i0}-\tau}^{\rho_{i0}-\tau}\left[\frac{1}{2(\rho-\rho_{i0})^2} + \frac{1}{4\rho_{i0}(\rho-\rho_{i0})}\right]d\rho = -\frac{1}{2(\rho-\rho_{i0})}\Bigg|_{\rho_{i0}-\tau}^{\rho_{i0}+\tau} + \frac{1}{4\rho_{i0}}\ln|\rho-\rho_{i0}|\Bigg|_{\rho_{i0}-\tau}^{\rho_{i0}+\tau} =$$

$$= -\frac{1}{2}\left(\frac{1}{\tau} + \frac{1}{\tau}\right) + \frac{1}{4\rho_{i0}}\ln\left|\frac{\tau}{\tau}\right| = -\frac{1}{\tau}$$

The logarithmic component has the form

$$K_l(\rho, \rho_0) = \frac{\rho k'^2}{2(\rho+\rho_0)(\rho-\rho_0)^2}\left(\ln\frac{4}{k'} - \frac{1}{2}\right) = \frac{\rho}{2(\rho+\rho_0)^3}\left[\ln 4 - \frac{1}{2} - \ln\left|\frac{\rho-\rho_0}{\rho+\rho_0}\right|\right].$$

After the transformations we have

$$K_l(\rho, \rho_0) = \left[\frac{1}{2(\rho+\rho_0)^2} - \frac{\rho_0}{2(\rho+\rho_0)^3}\right]\left[\ln 4 - \frac{1}{2} - \ln\left|\frac{\rho-\rho_0}{\rho+\rho_0}\right|\right].$$

Note that the component

$$K_{l2}(\rho, \rho_0) = \left[\frac{1}{2(\rho+\rho_0)^2} - \frac{\rho_0}{2(\rho+\rho_0)^3}\right]\left[\ln 4 - \frac{1}{2} - \ln|\rho+\rho_0|\right] \tag{7.5}$$

does not contain features, it is further referred to the regular part. Let

$$K_{l1}(\rho, \rho_0) = \frac{1}{2}\left[-\frac{1}{(\rho+\rho_0)^2} + \frac{\rho_0}{(\rho+\rho_0)^3}\right]\ln|\rho-\rho_0| .$$

When calculating the integral of $K_{l1}(\rho, \rho_0)$ we carry out the integration of parts. Let us consider that

$$dV = \frac{1}{2}\left[-\frac{1}{(\rho+\rho_0)^2} + \frac{\rho_0}{(\rho+\rho_0)^3}\right], \quad U = \ln|\rho-\rho_0|$$

$$V = \frac{1}{2}\left[\frac{1}{(\rho+\rho_0)} - \frac{2\rho_0}{(\rho+\rho_0)^2}\right], \quad dU = \frac{1}{\rho-\rho_0}$$

At integration on a special element we will receive

$$\int_{\rho_{i0}-\tau}^{\rho_{i0}+\tau} K_{l1}(\rho,\rho_{i0})d\rho = \frac{1}{4}\ln|\rho-\rho_{i0}|\left[\frac{2\rho+\rho_{i0}}{(\rho+\rho_{i0})^2}\right]\Bigg|_{\rho_{i0}-\tau}^{\rho_{i0}+\tau} - \frac{1}{4}\int_{\rho_{i0}-\tau}^{\rho_{i0}+\tau}\frac{2\rho+\rho_{i0}}{(\rho+\rho_{i0})^2}\frac{1}{\rho-\rho_{i0}}d\rho .$$

Next we have

$$\int_{\rho_{i0}-\tau}^{\rho_{i0}+\tau} K_{l1}(\rho,\rho_{i0})d\rho = \frac{1}{4}\ln|\rho-\rho_{i0}|\left[\frac{2\rho+\rho_{i0}}{(\rho+\rho_{i0})^2}\right]\Bigg|_{\rho_{i0}-\tau}^{\rho_{i0}+\tau} + \frac{1}{8}\frac{1}{\rho+\rho_{i0}}\Bigg|_{\rho_{i0}-\tau}^{\rho_{i0}+\tau} - \frac{3}{16\rho_{i0}}\ln\left|\frac{\rho-\rho_{i0}}{\rho+\rho_{i0}}\right|\Bigg|_{\rho_{i0}-\tau}^{\rho_{i0}+\tau} =$$

$$= \frac{1}{4}\left[\frac{(\ln\tau)(3\rho_{i0}+\tau)}{(2\rho_{i0}+\tau)^2} - \frac{(\ln\tau)(3\rho_{i0}-\tau)}{(2\rho_{i0}-\tau)^2} + \frac{1}{2}\left(\frac{1}{2\rho_{i0}+\tau} - \frac{1}{2\rho_{i0}-\tau}\right) + \frac{3}{4\rho_{i0}}\ln\left|\frac{2\rho_{i0}+\tau}{2\rho_{i0}-\tau}\right|\right] =$$

$$= -\frac{\tau}{2(4\rho_{i0}^2-\tau^2)}\left(\frac{\rho_{i0}\ln\tau}{4\rho_{i0}^2-\tau^2} - \frac{1}{2}\right) + \frac{3}{16\rho_{i0}}\ln\left|\frac{2\rho_{i0}+\tau}{2\rho_{i0}-\tau}\right|$$

Thus, it is obtained that

$$\int_{\rho_{i-1}}^{\rho_i} K_0(\rho,\rho_{j0})d\rho = -\frac{1}{\tau} - \frac{\tau}{2(4\rho_{i0}^2-\tau^2)}\left(\frac{\rho_{i0}\ln\tau}{4\rho_{i0}^2-\tau^2} - \frac{1}{2}\right) + \frac{3}{16\rho_{i0}}\ln\left|\frac{2\rho_{i0}+\tau}{2\rho_{i0}-\tau}\right| + \int_{\rho_{i-1}}^{\rho_i} K_{h2}(\rho,\rho_{j0})d\rho + \int_{\rho_{i-1}}^{\rho_i} K_{l2}(\rho,\rho_{j0})d\rho .$$

The functions $K_{h2}(\rho,\rho_0)$ and $K_{l2}(\rho,\rho_0)$ are calculated by the formulas (7.4) and (7.5), respectively, and the integrals of these functions are calculated by the standard quadrature formulas.

To calculate the integrals with kernels in the form of elliptical integrals, we use the method from [24].

To test the proposed method, we use the analytical solution of the spatial hypersingular equation given in [22, 25]. For the hypersingular equation in the form

$$\frac{1}{\pi}\iint_S \frac{\Gamma(x,y)dxdy}{\sqrt{(x-x_0)^2+(y-y_0)^2}^3} = -\pi, \quad S = \{(x,y): x^2+y^2 \le R^2\}$$

in [22, 25] it is proved that

$$\Gamma(x,y) = \sqrt{R^2-x^2-y^2} .$$

If we consider the hypersingular equation in the form (7.1), then for the equation

$$\frac{1}{\pi}\int_0^R \Gamma(\rho)\frac{\rho E(k)d\rho}{(\rho+\rho_0)(\rho-\rho_0)^2} = -1 \qquad (7.6)$$

we will have the analytical solution

$$\Gamma(\rho) = \frac{4}{\pi}\sqrt{R^2-\rho^2} .$$

To numerically solve the equation (7.6), the method of boundary elements with a constant approximation of the density on the elements [22] has been used. A circle with the radius R = 1m is considered. 100 boundary elements have been selected. The comparison of numerical and analytical results is given in Fig. 7.1.

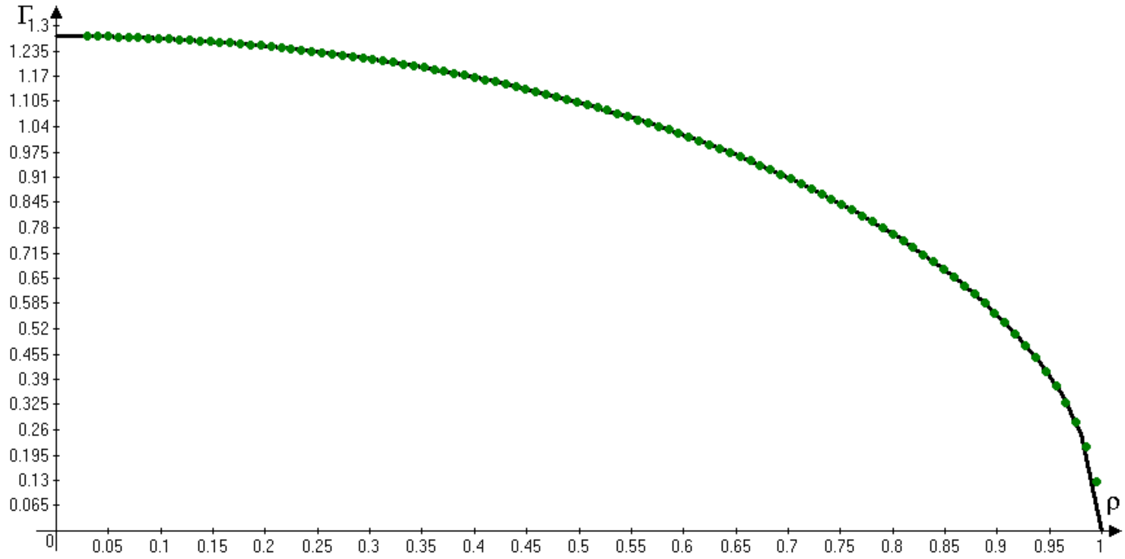

Figure 7.1 Numerical and analytical solutions of the hypersingular equation

The solid line corresponds to the analytical solution; the points indicate the numerical data obtained by the proposed method.

Let us note that at almost all points of the segment $[0, R]$ the accuracy is $\varepsilon = 10^{-4}$. The exception is points near the ends of the interval. Since at ρ, ρ_0, close to zero, an additional feature appears in the equation (7.6), it is proposed to replace the integration segment by $[\delta, R]$ [14]. For the above mentioned calculations, δ = 0.025/R is chosen. For ρ close to 1, the accuracy can be increased by taking into account the behavior of the solution at the ends of the integration interval [22].

For the method of boundary elements for the two-dimensional region in [22] we used 1200 boundary elements, and the obtained accuracy is $\varepsilon = 10^{-3}$. These data indicate the effectiveness of the proposed method.

8. Receiving the matrix of added masses

After determining the natural vibration modes $w_m(\rho)$ of a circular plate by the formulas (6.8) we find the solutions of hypersingular equations according to the method of boundary elements.

$$\frac{1}{\pi}\int_0^R \Gamma_m(\rho)\frac{\rho \mathrm{E}(k)d\rho}{(\rho+\rho_0)(\rho-\rho_0)^2} = -w_m(\rho_0)$$

We look for a solution to the problem of hydroelastic vibrations of the plate in the form of the series (3.1), (3.2).

Considering the equation (2.1) in the assumption that $q_0(x, y, t)$ = 0, and the pressure drop is obtained from the equation (2.4), we have the following relations

$$D\Delta\Delta w + \rho_p h\frac{\partial^2 w}{\partial t^2} = -\rho_l\left[\frac{\partial\varphi^+(x,y,t)}{\partial t} - \frac{\partial\varphi^-(x,y,t)}{\partial t}\right],$$

$$D\sum_{m=1}^N c_m(t)\Delta\Delta w_m + \rho_p h\sum_{m=1}^N \ddot{c}_m(t) w_m = -\rho_l\sum_{m=1}^N \ddot{c}_m(t)\Gamma_m. \quad (8.1)$$

Since free hydroelastic vibrations will be considered, we suppose that

$$c_m(t) = C_m \exp(i\omega t), \quad (8.2)$$

where ω is the frequency of natural hydroelastic vibrations; i is an imaginary unit; C_m is unknown.

According to the equations (6.2) and (8.2) we convert (8.1) to the form

$$D\sum_{m=1}^N C_m\alpha_m^4 w_m - \omega^2\left[\rho_p h\sum_{m=1}^N C_m w_m + \rho_l\sum_{m=1}^N C_m\Gamma_m\right] = 0. \quad (8.3)$$

The next step is fining a scalar product of the equation (8.3) on the functions $w_n(\rho)$, $n = 1,2,\ldots N$ and obtaining the system of linear homogeneous algebraic equations with respect to the constants

$$\sum_{m=1}^{N}\left\{\delta_{nm} A_m D\alpha_m^4 - \omega^2\left[\rho_p h A_m \delta_{nm} + \rho_l\left(\Gamma_m, w_n\right)\right]\right\}C_m = 0 \quad (8.4)$$

To find the nonzero solution of the equations (8.4) we obtain the characteristic equation with respect to ω

$$\det\left\{\rho_p h \Omega_m^2 A_n \delta_{nm} - \omega^2\left[\rho_p h A_m \delta_{nm} + \rho_l\left(\Gamma_m, w_n\right)\right]\right\} = 0 \quad (8.5)$$

Thus, the problem of determining the frequencies and modes of natural hydroelastic vibrations of a circular plate is reduced to solving the problem of natural values.

$$\left(\mathbf{A} - \omega^2\mathbf{B}\right)\mathbf{C} = 0\,,$$

where the elements of the matrices are determined by the formulas

$$a_{mn} = \Omega_m^2 \rho_p h A_m \delta_{mn}, \quad b_{mn} = \rho_p h A_m \delta_{mn} + \rho_l\left(\Gamma_m, w_n\right).$$

The elements of the matrix of the added masses of a liquid are determined by the formulas

$$g_{mn} = \left(\Gamma_m, w_n\right) = \tau\sum_{i=1}^{N_1}\rho_{i0}\Gamma_m\left(\rho_{i0}\right)w_n\left(\rho_{i0}\right),$$

that is, we use the formula of central rectangles.

Thus, for determining free vibration frequencies and modes of a circular plate immersed in a liquid the method based on the use of one-dimensional hypersingular equations has been developed.

9. Conclusions

The method for solving the problem of determining the frequencies and modes of vibrations of a circular elastic plate immersed in an ideal incompressible fluid, the motion of which is considered to be irrotational, has been developed. The method is based on using one-dimensional hypersinsular equations. The analysis of singularities of kernels of integral operators has been carried out, which allowed obtaining analytical formulas for calculating hypersingular terms and terms with logarithmic singularity. The test calculation of a hypersingular equation with a unit right-hand side has been performed. It has proved the effectiveness of the proposed method over the method of boundary elements in two-dimensional formulation. The algorithm for obtaining the matrix of added masses is proposed.

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