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Study of stress-strain state of the transversely isotropic plates using the refined theory

I. P. Bokov, N. S. Bondarenko, E. A. Strelnikova

A. N. Podgorny Institute for Mechanical Engineering Problems of the National Academy of Sciences of Ukraine, Ukraine

The fundamental solution of statics equations of transversely isotropic plate was obtained using the two-dimensional Fourier integral transform and the inversion technic based on the special G-function. Such method allows reducing the system of resolving differential equations, which describe static state of flat plate or shell, to a system of algebraic equations. Then, the inverse Fourier transform restores the fundamental solution. Numerical studies, which were carried out, have detected the behavior patterns of stress-strain state components in dependence on material elastic constants. This approach demonstrates development of refined theory of plates and shells based on the three-dimensional elasticity theory.

***Key words**: transversely isotropic plates, refined theory, the equations of statics, fundamental solution, special G-function.*

Фундаментальное решение уравнений статики трансверсально-изотропных пластин найдено с помощью двумерного интегрального преобразования Фурье и методики обращения, построенной с помощью специальной G-функции. Данный метод позволяет свести систему разрешающих дифференциальных уравнений статики пологих пластин и оболочек к системе алгебраических уравнений. После этого обратное преобразование Фурье восстанавливает фундаментальное решение. Проведены численные исследования, которые демонстрируют закономерности поведения компонент напряженно-деформированного состояния в зависимости от упругих констант трансверсально-изотропного материала. Данный подход демонстрирует развитие уточненных теорий пластин и оболочек на основе трехмерной теории упругости.

***Ключевые слова:** трансверсально-изотропные пластины, уточненная теория, уравнения статики, фундаментальное решение, специальная G-функция.*

Фундаментальне рішення рівнянь статики трансверсально-ізотропних пластин знайдено за допомогою двовимірного інтегрального перетворення Фур'є та методики звернення, побудованої за допомогою спеціальної G-функції. Цей метод дозволяє звести систему дозвільних диференціальних рівнянь статики пологих пластин та оболонок до системи алгебраїчних рівнянь. Після цього зворотне перетворення Фур'є відновлює фундаментальне рішення. Проведено численні дослідження, які демонструють закономірності поведінки компонент пружно-деформованого стану в залежності від пружних констант трансверсально-ізотропного матеріалу. Цей підхід демонструє розвиток уточнених теорій пластин і оболонок на основі тривимірної теорії пружності.

***Ключові слова:** трансверсально-ізотропні пластини, уточнена теорія, рівняння статики, фундаментальне рішення, спеціальна G-функція.*

Introduction

Engineering facilities consisting of thin-walled structural components, which are widely used in modern equipment, usually are submitted to the action of strong forces. Calculations related with such elements are never simple and become even more complicated when the forces have the concentrated nature.

This paper considers the transversely isotropic plates and proposes a technology that allows reducing the three-dimensional problem to two-dimensional one. To achieve this goal, we expand the sought functions in series of Legendre polynomials in the normal (transverse) coordinate. This approach allows us to take into account both the tangent and the normal transverse stresses. Applying it, we have derived the equations for transversely isotropic plates and developed a method to evaluate the stress-strain state (SSS) of these plates under the action of concentrated forces.

A lot of publications are dedicated to development of methods for construction of fundamental solutions of elasticity theory equations for thin plates and shells (particularly, in the case of concentrated forces). Problem statements, methods of their solution and a number of concrete results are presented in monographs and scientific articles of S.A. Ambartsumyan [1], A.L. Goldenveizer [2], S. Lukasiewicz [3], as well as in reviews of V.M. Darevsky [4], Y.P. Zhigalko [5] and others. Shear forces for isotropic plates were obtained in the paper of I.P. Bokov, E.A. Strelnikova [6].

Classical Kirchhoff-Love theory satisfactorily describes the SSS of relatively thin transversely isotropic plates, but it does not take into account the phenomena caused by shifts and compression. On the other hand, solving the elasticity theory problems in their three-dimensional formulation runs into considerable mathematical difficulties. One of the ways for further theory development proposes reduction of the three-dimensional problems to the two-dimensional ones.

From the point of view of refined theory development, study of SSS of transversely isotropic plates affected by concentrated forces is an actual scientific and technical problem. Such study is the main objective of the present paper.

1. The problem statement

In accordance with [7, p. 231], we call “the concentrated force” some abstraction, which actually is a force of finite value acting over a patch of the surface.

We consider the transversely isotropic plate of thickness $2h$ in the rectangular Cartesian coordinate system x, y, z. Let us suppose that the concentrated force $\vec{F}$ affects a plate and the coordinate origin coincides with the force application point (singular point). Solving the problems on effect of concentrated forces, we assume that sought SSS is local, that is, it does not rich edges of the plate. Therefore, we can consider the plate as infinite one and set that the components of the sought SSS tend to zero at infinity. Any obtained solution is to be validated checking whether it meets this assumption.

Stated mathematically, the problem represents a complete system of equations of the elasticity theory with no boundary conditions over the plate real edges. The sought functions tend to zero at infinity. The system of equations, which describes the bended plate SSS on the base of the theory of S.P. Timoshenko for transversely isotropic plates, consists of [8, p. 35-37]:

• the geometrical relations

$$e_{x1} = h\frac{\partial \gamma_x}{\partial x}, \quad e_{xy1} = h\left(\frac{\partial \gamma_x}{\partial y} + \frac{\partial \gamma_y}{\partial x}\right), \quad e_{xz0} - \frac{e_{xz2}}{5} = \gamma_x + \frac{\partial w_0}{\partial x} \quad (x \to y). \quad (1.1)$$

• the elasticity relations (Hooke's law)

$$M_x = D\left(e_{x1} + \nu e_{y1}\right), \quad M_y = D\left(e_{y1} + \nu e_{x1}\right), \quad H = \frac{1-\nu}{2} De_{xy1}, \tag{1.2}$$

$$Q_x = \Lambda\left(e_{xz0} - \frac{e_{xz2}}{5}\right) \quad (x \to y),$$

where $D = \frac{2h^2}{3}\frac{E}{1-\nu^2}$, $\Lambda = \frac{5hG}{3}$.

• the equilibrium equations

$$\frac{\partial M_x}{\partial x} + \frac{\partial H}{\partial y} - Q_x + m_x = 0, \quad \frac{\partial M_y}{\partial y} + \frac{\partial H}{\partial x} - Q_y + m_y = 0, \tag{1.3}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q_z = 0.$$

To find the fundamental solution of the system (1.1) – (1.3), the vector components of body forces in formulas (1.3) should be taken in the form

$$m_x(x,y) = h^2 m_x^* \delta(x,y), \quad m_y(x,y) = h^2 m_y^* \delta(x,y), \tag{1.4}$$

$$q_z(x,y) = h^2 q_z^* \delta(x,y) \quad (x \to y),$$

where m_x^*, m_y^*, q_z^* are *constants* and $\delta(x,y)$ is a two-dimensional Dirac delta function [9].

2. Definition of transforms of generalized displacements

Substituting the geometrical relations (1.1) into the elasticity relations (1.2) and passing to the dimensionless coordinate system $x_1 = x/h$, $x_2 = y/h$, $x_3 = z/h$, we obtain

$$M_1 = D_0\left(\frac{\partial\gamma_1}{\partial x_1} + \nu\frac{\partial\gamma_2}{\partial x_2}\right), \quad M_2 = D_0\left(\frac{\partial\gamma_2}{\partial x_2} + \nu\frac{\partial\gamma_1}{\partial x_1}\right),$$

$$H = \frac{1-\nu}{2} D_0\left(\frac{\partial\gamma_1}{\partial x_2} + \frac{\partial\gamma_2}{\partial x_1}\right), \tag{2.1}$$

$$Q_1 = \Lambda_0\left(\gamma_1 + \frac{\partial w_0}{\partial x_1}\right), \quad Q_2 = \Lambda_0\left(\gamma_2 + \frac{\partial w_0}{\partial x_2}\right),$$

where $D_0 = \frac{D}{Eh^2} = \frac{2}{3}\frac{1}{1-\nu^2}$, $\Lambda_0 = \frac{5G}{3E}$.

Bending moments and torque are proportional to the value Eh^2, and shear forces – to the value Eh.

In the dimensionless coordinates, we obtain:

$$\frac{\partial M_1}{\partial x_1} + \frac{\partial H}{\partial x_2} - Q_1 + m_1 = 0, \quad \frac{\partial M_2}{\partial x_2} + \frac{\partial H}{\partial x_1} - Q_2 + m_2 = 0, \tag{2.2}$$

$$\frac{\partial Q_1}{\partial x_1}+\frac{\partial Q_2}{\partial x_2}+q_3=0,$$

where $m_1=m_1^*\delta(x_1,x_2)$, $\quad m_2=m_2^*\delta(x_1,x_2)$, $\quad q_3=q_3^*\delta(x_1,x_2)$.

Solving this system, we obtain the generalized displacement transforms:

$$\tilde{\gamma}_1=\frac{1}{2\pi}\left[\frac{m_1^*}{D_0}\frac{\xi_1^2}{p^4}+3(1+\nu)m_1^*\frac{\xi_2^2}{p^2\left(p^2+a^2\right)}+\frac{q_3^*}{D_0}\frac{i\xi_1}{p^4}+\frac{m_2^*}{D_0}\frac{\xi_1\xi_2}{p^4}-\right.$$

$$\left.-3(1+\nu)m_2^*\frac{\xi_1\xi_2}{p^2\left(p^2+a^2\right)}\right], \tag{2.3}$$

$$\tilde{\gamma}_2=\frac{1}{2\pi}\left[\frac{m_2^*}{D_0}\frac{\xi_2^2}{p^4}+3(1+\nu)m_2^*\frac{\xi_1^2}{p^2\left(p^2+a^2\right)}+\frac{q_3^*}{D_0}\frac{i\xi_2}{p^4}+\frac{m_1^*}{D_0}\frac{\xi_1\xi_2}{p^4}-\right.$$

$$\left.-3(1+\nu)m_1^*\frac{\xi_1\xi_2}{p^2\left(p^2+a^2\right)}\right],$$

$$\tilde{w}_0=\frac{1}{2\pi}\left[-\frac{m_1^*}{D_0}\frac{i\xi_1}{p^4}-\frac{m_2^*}{D_0}\frac{i\xi_2}{p^4}+\frac{q_3^*}{D_0}\frac{1}{p^4}+\frac{q_3^*}{\Lambda_0}\frac{1}{p^2}\right],$$

where $p^2=\xi_1^2+\xi_2^2$, $a^2=3(1+\nu)\Lambda_0$; (ξ_1,ξ_2) are the point coordinates in the space of transforms.

3. Finding of generalized moments in the space of transforms

The Fourier transform applied to the Hooke's law equations (2.1) gives:

$$\tilde{M}_1=-D_0\left(i\xi_1\tilde{\gamma}_1+i\nu\xi_2\tilde{\gamma}_2\right),\quad \tilde{M}_2=-D_0\left(i\xi_2\tilde{\gamma}_2+i\nu\xi_1\tilde{\gamma}_1\right),$$

$$\tilde{H}=\frac{1-\nu}{2}D_0\left(i\xi_2\tilde{\gamma}_1+i\xi_1\tilde{\gamma}_2\right), \tag{3.1}$$

$$\tilde{Q}_1=\Lambda_0\left(\tilde{\gamma}_1-i\xi_1\tilde{w}_0\right),\ \tilde{Q}_2=\Lambda_0\left(\tilde{\gamma}_1-i\xi_2\tilde{w}_0\right).$$

Having substitute the previously obtained expressions for transforms of generalized displacements (2.3) into the transforms of bending moments equations (3.1) we obtain:

$$\tilde{M}_1=-\frac{1}{2\pi}\left[m_1^*\frac{\xi_1^3}{p^4}+2m_1^*\frac{i\xi_1\xi_2^2}{p^2\left(p^2+a^2\right)}-q_3^*\frac{\xi_1^2}{p^4}+m_2^*\frac{i\xi_1^2\xi_2}{p^4}-\right.$$

$$\left.-2m_2^*\frac{i\xi_1^2\xi_2}{p^2\left(p^2+a^2\right)}+m_2^*\nu\frac{\xi_2^3}{p^4}-q_3^*\nu\frac{\xi_2^2}{p^4}+m_1^*\nu\frac{i\xi_1\xi_2^2}{p^4}\right], \tag{3.2}$$

$$\tilde{M}_2=-\frac{1}{2\pi}\left[m_2^*\frac{\xi_2^3}{p^4}+2m_2^*\frac{i\xi_1^2\xi_2}{p^2\left(p^2+a^2\right)}-q_3^*\frac{\xi_2^2}{p^4}+m_1^*\frac{i\xi_1\xi_2^2}{p^4}-\right.$$

$$\left.-2m_1^*\frac{i\xi_1\xi_2^2}{p^2\left(p^2+a^2\right)}+m_1^*\nu\frac{\xi_1^3}{p^4}-q_3^*\nu\frac{\xi_1^2}{p^4}+m_2^*\nu\frac{i\xi_1^2\xi_2}{p^4}\right].$$

Let us denote

$$\tilde{\Phi}_1(\xi_1,\xi_2)=\frac{i\xi_1^3}{p^4},\quad \tilde{\Phi}_2(\xi_1,\xi_2)=\frac{i\xi_1^2\xi_2}{p^4},$$

$$\tilde{\Phi}_3(\xi_1,\xi_2)=\frac{\xi_1^2}{p^4},\quad \tilde{\Phi}_4(\xi_1,\xi_2)=\frac{i\xi_1^2\xi_2}{p^2\left(p^2+a^2\right)}. \qquad (3.3)$$

Then the bending moments in the space of transforms take this form:

$$\tilde{M}_1=-\frac{1}{2\pi}\left[m_1^*\tilde{\Phi}_1(\xi_1,\xi_2)+2m_1^*\tilde{\Phi}_4(\xi_2,\xi_1)-q_3^*\tilde{\Phi}_3(\xi_1,\xi_2)+m_2^*\tilde{\Phi}_2(\xi_1,\xi_2)-\right.$$

$$\left.-2m_2^*\tilde{\Phi}_4(\xi_1,\xi_2)+m_2^*\nu\tilde{\Phi}_1(\xi_2,\xi_1)-q_3^*\nu\tilde{\Phi}_3(\xi_2,\xi_1)+m_1^*\nu\tilde{\Phi}_2(\xi_2,\xi_1)\right], \qquad (3.4)$$

$$\tilde{M}_2=-\frac{1}{2\pi}\left[m_2^*\tilde{\Phi}_1(\xi_2,\xi_1)+2m_2^*\tilde{\Phi}_4(\xi_1,\xi_2)-q_3^*\tilde{\Phi}_3(\xi_2,\xi_1)+m_1^*\tilde{\Phi}_2(\xi_2,\xi_1)-\right.$$

$$\left.-2m_1^*\tilde{\Phi}_4(\xi_2,\xi_1)+m_1^*\nu\tilde{\Phi}_1(\xi_1,\xi_2)-q_3^*\nu\tilde{\Phi}_3(\xi_1,\xi_2)+m_2^*\nu\tilde{\Phi}_2(\xi_1,\xi_2)\right].$$

4. Finding the originals of bending moments

Now we need to invert the expressions (3.4). First, to find the originals of functions (3.3), let us apply the Fourier integral transform [10, c. 58]

$$F^{-1}\left[\tilde{f}(\xi_1,\xi_2)\right]=f(x_1,x_2)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{f}(\xi_1,\xi_2)e^{-i(\xi_1x_1+\xi_2x_2)}d\xi_1d\xi_2. \qquad (4.1)$$

We obtain

$$\Phi_1(x_1,x_2)=\frac{x_1\left(x_1^2+3x_2^2\right)}{2\left(x_1^2+x_2^2\right)^2},\qquad \Phi_2(x_1,x_2)=-\frac{x_2\left(x_1^2+3x_2^2\right)}{2\left(x_1^2+x_2^2\right)^2},$$

$$\Phi_3(x_1,x_2)=-\frac{1}{2}\ln\frac{\gamma\sqrt{x_1^2+x_2^2}}{2}-\frac{1}{4}\frac{x_1^2-x_2^2}{x_1^2+x_2^2}, \qquad (4.2)$$

$$\Phi_4(x_1,x_2)=\frac{x_2}{2\left(x_1^2+x_2^2\right)}G_{0,1}|a|\sqrt{x_1^2+x_2^2}+\frac{x_2\left(3x_1^2-x_2^2\right)}{2\left(x_1^2+x_2^2\right)^2}G_{1,2}|a|\sqrt{x_1^2+x_2^2}.$$

where $G_{n,\nu}(rz)$ is some special G-function [11].

Applying the formula of two-dimensional Fourier integral inversion (4.1) to the transforms of bending moments (3.4) and taking into account the expression (4.2), we obtain the expressions for M_1 and M_2 in the space of the originals

$$
\begin{aligned}
M_1 = -\frac{1}{2\pi}\Bigg[& m_1^* \frac{x_1\left(x_1^2+3x_2^2\right)}{2\left(x_1^2+x_2^2\right)^2} + 2m_1^*\left\{\frac{x_1}{2\left(x_1^2+x_2^2\right)} G_{0,1}|a|\sqrt{x_1^2+x_2^2} - \right. \\
& \left. - \frac{x_1\left(x_1^2-3x_2^2\right)}{2\left(x_1^2+x_2^2\right)^2} G_{1,2}|a|\sqrt{x_1^2+x_2^2}\right\} + q_3^*\left\{\frac{1}{2}\ln\frac{\gamma\sqrt{x_1^2+x_2^2}}{2} + \frac{1}{4}\frac{x_1^2-x_2^2}{x_1^2+x_2^2}\right\} - \\
& - m_2^* \frac{x_2\left(x_1^2+3x_2^2\right)}{2\left(x_1^2+x_2^2\right)^2} - 2m_2^*\left\{\frac{x_2}{2\left(x_1^2+x_2^2\right)} G_{0,1}|a|\sqrt{x_1^2+x_2^2} + \right. \\
& \left. + \frac{x_2\left(3x_1^2-x_2^2\right)}{2\left(x_1^2+x_2^2\right)^2} G_{1,2}|a|\sqrt{x_1^2+x_2^2}\right\} + m_2^*\nu \frac{x_2\left(3x_1^2+x_2^2\right)}{2\left(x_1^2+x_2^2\right)^2} + \\
& + q_3^*\nu\left\{\frac{1}{2}\ln\frac{\gamma\sqrt{x_1^2+x_2^2}}{2} + \frac{1}{4}\frac{x_2^2-x_1^2}{x_1^2+x_2^2}\right\} - m_1^*\nu \frac{x_1\left(3x_1^2+x_2^2\right)}{2\left(x_1^2+x_2^2\right)^2}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
M_2 = -\frac{1}{2\pi}\Bigg[& m_2^* \frac{x_2\left(3x_1^2+x_2^2\right)}{2\left(x_1^2+x_2^2\right)^2} + 2m_2^*\left\{\frac{x_2}{2\left(x_1^2+x_2^2\right)} G_{0,1}|a|\sqrt{x_1^2+x_2^2} + \right. \\
& \left. + \frac{x_2\left(3x_1^2-x_2^2\right)}{2\left(x_1^2+x_2^2\right)^2} G_{1,2}|a|\sqrt{x_1^2+x_2^2}\right\} + q_3^*\left\{\frac{1}{2}\ln\frac{\gamma\sqrt{x_1^2+x_2^2}}{2} + \frac{1}{4}\frac{x_2^2-x_1^2}{x_1^2+x_2^2}\right\} - \\
& - m_1^* \frac{x_1\left(3x_1^2+x_2^2\right)}{2\left(x_1^2+x_2^2\right)^2} - 2m_1^*\left\{\frac{x_1}{2\left(x_1^2+x_2^2\right)} G_{0,1}|a|\sqrt{x_1^2+x_2^2} - \right. \\
& \left. - \frac{x_1\left(x_1^2-3x_2^2\right)}{2\left(x_1^2+x_2^2\right)^2} G_{1,2}|a|\sqrt{x_1^2+x_2^2}\right\} + m_1^*\nu \frac{x_1\left(x_1^2+3x_2^2\right)}{2\left(x_1^2+x_2^2\right)^2} + \\
& + q_3^*\nu\left\{\frac{1}{2}\ln\frac{\gamma\sqrt{x_1^2+x_2^2}}{2} + \frac{1}{4}\frac{x_1^2-x_2^2}{x_1^2+x_2^2}\right\} - m_2^*\nu \frac{x_2\left(x_1^2+3x_2^2\right)}{2\left(x_1^2+x_2^2\right)^2}\Bigg].
\end{aligned}
\tag{4.3}
$$

5. Some numerical results

To study the features of the SSS of transversely isotropic plates under concentrated force action we set: $m_1^* = m_2^* = q_3^* = 1$.

The calculation results are shown in the dimensionless Cartesian coordinate system x_1, x_2.

Numerical evaluations were carried out for lead plates and zinc ones. Poisson's ratios (ν) for these materials are 0.446 and 0.212, respectively [12, p. 200]. The sliding compliance is $\frac{E}{G} = 2{,}6$.

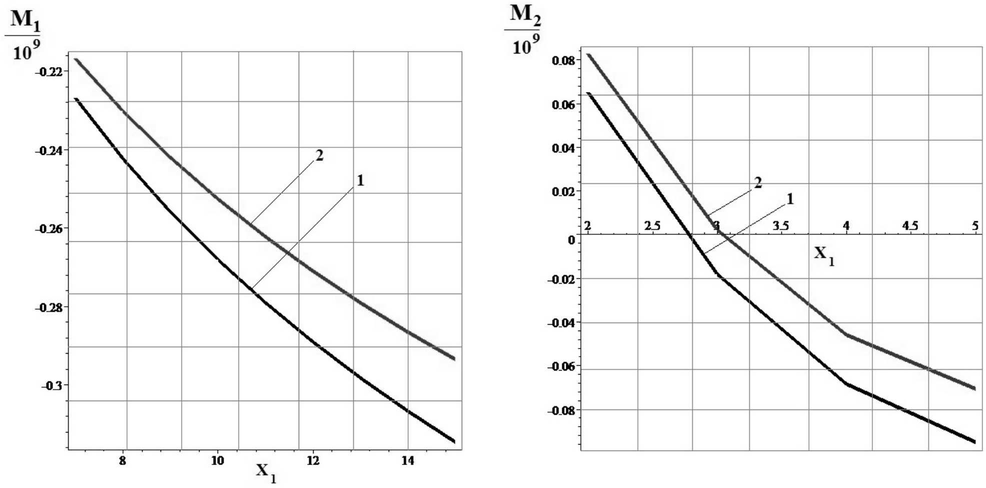

Fig. 1. The bending moments M_1, M_2 for plates of: 1 – lead, 2 - zinc

Fig. 1 contains the graphs of generalized moments M_1, M_2 along the abscissa axis ($x_2 = 0$). These graphs show that with decreasing of the Poisson's ratio the values of generalized moments increase.

6. Conclusions

The developed method allows calculating the internal force components for plates subjected to the action of a concentrated force. This makes it possible to consider the shells and plates with thickness of about 1/5 with respect to their characteristic size.

The resulting fundamental solution provides a tool for solving a number of new problems related to medium thickness plate bending. In presence of concentrated dislocations, the fundamental solutions, which are the Green's functions, serve as foundation for building the integral representations of displacement discontinuities distributed with unknown density. Such integral representations can be used for solving the problems of plates bending when the plates have different defects, cuts and incisions.

Numerical studies of the SSS of transversely isotropic plates allowed revealing the behavior patterns of SSS components, depending on the material elastic constants.

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