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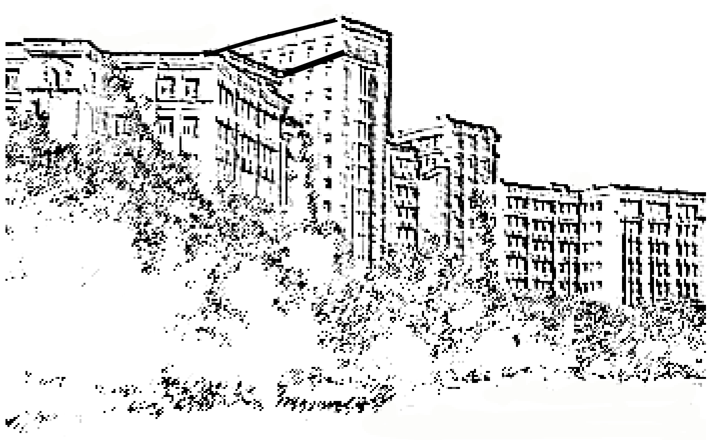

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До Віснику включено статті з математичного аналізу, математичної фізики, диференціальних рівнянь, математичної теорії керування та механіки, які містять нові теоретичні результати у зазначених галузях і мають прикладне значення.

Для викладачів, наукових працівників, аспірантів, працюючих у відповідних або суміжних сферах.

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Случайные блуждания на конечных группах с классовой вероятностью: алгебраический подход

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Хорошо известны необходимые и достаточные условия сходимости n-кратной свертки вероятности на конечной группе G к равномерной (тривиальной) вероятности на G при $n \to \infty$. Оценке скорости этой сходимости посвящено много работ.
Цель статьи — получение оценок скорости этой сходимости для вероятностей, постоянных на классах сопряженных элементов конечных групп.
Ключевые слова: вероятность, конечная группа, сходимость.

Вишневецький О. Л., **Випадкові блукання на скінченних групах із класовою ймовірністю: алгебраїчний підхід.** Добре відомі необхідні і достатні умови збіжності n-кратної згортки імовірності на скінченній групі G до рівномірної (тривіальної) імовірності на G при $n \to \infty$. Оцінці швидкості цієї збіжності присвячено багато робіт.
Ціль статті — одержання оцінок швидкості цієї збіжності для ймовірностей, постійних на класах спряжених елементів скінченних груп.
Ключові слова: імовірність, скінченна група, збіжність.

A. L. Vyshnevetskiy, **Random walks on finite groups with conjugate class probability: algebraic approach.** Under well known conditions an n-fold convolution of probability on finite group G converges to the uniform probability on G ($n \to \infty$). A lot of works estimate a rate of that convergence. The aim of the article is to obtain estimates of the rate for the probabilities that are constant on classes of conjugate elements of finite groups.
Keywords: probability, finite group, convergency.

2010 Mathematics Subject Classification: 20D99, 60B15, 60B10.

Пусть P – вероятность на конечной группе G порядка $|G|$, $U(g) = \dfrac{1}{|G|}$ – равномерная (тривиальная) вероятность на G, $P^{(n)}$ – n-кратная свертка функции P. Хорошо известны [1] необходимые и достаточные условия, при которых $P^{(n)} \to U$ $(n \to \infty)$. Оценке скорости этой сходимости посвящено много работ (см., например, обзор [2]).

Цель статьи – получение оценок скорости сходимости для конечных групп и вероятностей, постоянных на классах сопряженных элементов. Сходимость в пространстве функций $F(g)$ на группе G понимается относительно норм $\|F\|_1 = \sum\limits_g |F(g)|$ и $\|F\| = \left(|G| \sum\limits_g |F(g)|^2\right)^{1/2}$ (мы пишем $\sum\limits_g$ вместо $\sum\limits_{g \in G}$). В [1, 2] норма $\|\|_1$ имеет коэффициент $\dfrac{1}{2}$.

Пусть $\mathbf{C}G$ – групповая алгебра группы G над полем $\mathbf{C}$ комплексных чисел. Сопоставим вероятности $P(g)$ элемент $p = \sum\limits_g P(g)g$ алгебры $\mathbf{C}G$; этот элемент мы обозначаем той же (но малой) буквой, что и породившая его функция, и называем *вероятностью на* $\boldsymbol{C}G$. Свертке

$$(P * Q)(h) = \sum_g P(g)Q(g^{-1}h), \quad h \in G$$

функций P и Q соответствует произведение pq вероятностей на $\mathbf{C}G$.

Пусть $L(G)$ – пространство функций над полем $\mathbf{C}$ на группе G, постоянных на ее классах сопряженных элементов (классовые или центральные функции). В дальнейшем, если не оговорено противное, все вероятности являются классовыми. На абелевой группе все вероятности являются классовыми. В пространстве $L(G)$ определено скалярное произведение: если $F_1, F_2 \in L(G)$, то

$$(F_1, F_2) = \frac{1}{|G|} \sum_g F_1(g)\overline{F}_2(g), \tag{1}$$

где черта означает комплексное сопряжение. Множество $Irr(G)$ неприводимых комплексных характеров группы G образует ортонормированный базис в $L(G)$ относительно скалярного произведения (1). Поэтому вероятность $P \in L(G)$ разлагается по базису $Irr(G) = \{\mathbf{1}_G, \chi_1, \dots, \chi_k\}$, причем ввиду

$$\sum_g P(g) = 1 \tag{2}$$

коэффициент при главном характере $\mathbf{1}_G$ равен $\frac{1}{|G|}$:

$$p = \frac{1}{|G|}\mathbf{1}_G + m_1\chi_1 + \ldots + m_k\chi_k \tag{3}$$

Положим $d_j = \deg \chi_j$, $b = \max\limits_j |b_j|$, где

$$b_j = \frac{|G| m_j}{d_j} \quad (j = 1, \dots, k). \tag{4}$$

Лемма 1 $b \le 1$

Доказательство. Так как $|\chi_j(g)| \le d_j$, то из (3)

$$|m_j| = |(P,\chi_j)| \le \frac{1}{|G|}\sum_g |P(g)\overline{\chi}_j(g)| =$$
$$= \frac{1}{|G|}\sum_g P(g)|\chi_j(g)| \le \frac{1}{|G|}\sum_g P(g)|\chi_j(g)| \le \frac{1}{|G|}\sum_g P(g)d_j = \frac{d_j}{|G|}.$$

Поэтому из (4) $|b_j| \le 1$ и, следовательно, $b \le 1$.

Пусть $\mathrm{supp}(P) = \{g \in G,\ P(g) \ne 0\}$ — носитель вероятности P. Оценку леммы 1 можно усилить для вероятностей, у которых носитель $\mathrm{supp}(P) = G$.

Теорема 1 $b \le 1 - \min\limits_g P(g)$.

Доказательство. Положим $l = \min\limits_g P(g)$, $l_0 = (1-l)^{-1}$.
Функция $P_1 = l_0(P - lU)$ является вероятностью на G. Для любого неглавного характера $\chi_j \in Irr(G)$ имеем

$$m_j(P_1) = (P_1,\chi_j) = l_0(P - lU,\chi_j) = l_0(P,\chi_j) = l_0 m_j,$$

где $m_j(P_1)$ – коэффициенты разложения (3) для P_1 ($j = 1,\dots,k$). Поэтому $b_j(P_1) = l_0 b_j(P)$ и

$$b(P_1) = l_0 b(P). \tag{5}$$

Так как $b(P_1) \le 1$ (лемма 1), то $b(P) \le l_0^{-1} = 1 - l$.

В дальнейшем будем считать, что числа b_j занумерованы так, что $b = |b_1| = \ldots = |b_t|$ и $|b_j| < b$ при $j > t$. Пусть $D = \left(\sum\limits_{j=1}^{t} d_j^2\right)^{1/2}$.

Теорема 2 $Db^n \le \|P^{(n)} - U\| \le Db^n + a^n\left(|G| - 1 - D^2\right)^{1/2}$, *где* $0 \le a < b$, *a и b не зависят от n.*

Доказательство. Пусть p, u — вероятности на алгебре $\mathbf{C}G$, соответствующие вероятностям P, U. Если $e_j = \dfrac{d_j}{|G|}\sum\limits_g \chi_j(g)g \quad (j = 1,\dots,k)$, то из (3) и (4) следует

$$p = \tfrac{1}{|G|}\sum_g \mathbf{1}_G\, g + m_1\sum_g \chi_1(g)g + \ldots + m_k\sum_g \chi_k(g)g = u + \sum_{j=1}^{k}\tfrac{|G|m_j}{d_j}\, e_j =$$
$$= u + \sum_{j=1}^{k} b_j e_j.$$

Так как $u, e_1, \ldots, e_k$ – ортогональные идемпотенты центра алгебры $\mathbf{C}G$, то $p^n = u + \sum\limits_{j=1}^{k} b_j^n e_j$,

$$p^n - u = \sum_{j=1}^{k} b_j^n e_j = \sum_{j=1}^{k} b_j^n \sum_g \frac{d_j}{|G|}\chi_j(g)\, g = \sum_g \left(\sum_{j=1}^{k} \frac{d_j b_j^n}{|G|}\chi_j(g) \right) g,$$

или, возвращаясь к функциям на группе G,

$$p^n - U = \sum_{j=1}^{k} \frac{d_j b_j^n}{|G|}\chi_j.$$

Так как $\|\chi_j\|^2 = |G| \sum\limits_g |\chi_j(g)|^2 = |G|^2$, то функции $\dfrac{\chi_j}{|G|}$ $(j = 1, \ldots, k)$ образуют ортонормированное множество относительно нормы $\| \cdot \|$. Поэтому

$$\|P^{(n)} - U\|^2 = \sum_{j=1}^{k} |b_j^{2n}| d_j^2 = b^{2n} D^2 + \sum_{j>t} |b_j^{2n}| d_j^2 \le b^{2n} D^2 + a^{2n} \sum_{j>t} d_j^2,$$

где $a = \max\limits_{j>t} |b_j|, \;\; 0 \le a < b$.

Так как $0 \le \sum\limits_{j>t} d_j^2 = \left(\sum\limits_{j=1}^{k} d_j^2 - D^2 \right) = \left(|G| - 1 - D^2\right)$, то

$$D^2 b^{2n} \le \|P^{(n)} - U\|^2 \le D^2 b^{2n} + a^{2n} \left(|G| - 1 - D^2\right),$$

откуда следует утверждение теоремы.

Следствие 1.

1) $P^{(n)} \overrightarrow{\underset{n\to\infty}{\longrightarrow}} U$ по любой норме тогда и только тогда, когда $b < 1$.

2) Для достаточно больших n существует число $a_0 \in [0; 1)$, не зависящее от n, такое что

$$Db^n \le \|P^{(n)} - U\| \le (D + a_0^n)\, b^n \tag{6}$$

и

$$|G|^{-\frac{1}{2}} Db^n \le \|P^{(n)} - U\|_1 \le (D + a_0^n)\, b^n \tag{7}$$

Доказательство.

1) В конечномерном пространстве любая норма эквивалентна норме $\| \cdot \|$.

2) В доказательстве нуждается только (7). Если из $m \ge 2$ вещественных чисел $a_1 \ldots a_m$ хотя бы два не равны нулю, то в силу неравенств между средними

$$\sum_{i=1}^{m} a_i^2 < \left(\sum_{i=1}^{m} |a_i| \right)^2 \le m \sum_{i=1}^{m} a_i^2. \tag{8}$$

Ввиду (2) значения двух вероятностей на G не могут отличаться ровно на одном элементе. Поэтому при $m = |G|$ в качестве $a_1, \ldots, a_m$ можно взять значения функции $\left(P^{(n)} - U\right) g$, $g \in G$. Извлекая в (8) квадратный корень, получим

$$|G|^{-\frac{1}{2}} \|P^{(n)} - U\| < \|P^{(n)} - U\|_1 \leq \|P^{(n)} - U\|.$$

Поэтому (7) следует из (6).

Ввиду неравенств (6) и (7) число b играет важную роль в оценке скорости сходимости $P^{(n)}$ к U. Приведем оценки для величины b.

Теорема 3 $\left(\frac{\|P\|^2 - 1}{|G| - 1}\right)^{\frac{1}{2}} \leq b \leq \frac{\left(\|P\|^2 - 1\right)^{\frac{1}{2}}}{D}$.

Доказательство. Возьмем скалярный квадрат равенства (3) относительно скалярного произведения (1):

$$\frac{1}{|G|} \sum_g P^2(g) = \frac{1}{|G|^2} + \sum_{j=1}^k m_j^2.$$

Умножая на $|G|^2$ и используя (4), получаем $\|P\|^2 - 1 = \sum_{j=1}^k b_j^2 d_j^2$. Так как

$$b^2 \sum_{j=1}^t d_j^2 \leq \sum_{j=1}^k b_j^2 d_j^2 \leq b^2 \sum_{j=1}^k d_j^2 = b^2(|G| - 1),$$

и $D^2 = \sum_{j=1}^k d_j^2$, то $b^2 D^2 \leq \|P\|^2 - 1 \leq b^2(|G| - 1)$. Поэтому $\|P\|^2 - 1 > 0$ и $\frac{\|P\|^2 - 1}{|G| - 1} \leq b^2 \leq \frac{\|P\|^2 - 1}{D^2}$, что завершает доказательство.

Можно оценить b снизу с помощью числа $s = |\mathrm{supp(P)}|$.

Следствие 2.

$$b \geq \left(\frac{1}{s} - \frac{1}{|G|}\right)^{\frac{1}{2}} \tag{9}$$

. Доказательство. Так как $s^{-1} \geq |G|^{-1}$, то из (2) и (8) получаем, что $\|P\|^2 |G|^{-1} \geq s^{-1}$. Поэтому

$$\frac{\|P\|^2 - 1}{|G| - 1} \geq \frac{\|P\|^2 - 1}{|G|} \geq \frac{1}{s} - \frac{1}{|G|}.$$

Замечания.
1. В силу (9) вероятности с малым носителем не могут быстро сходиться к U.
2. С помощью (9) можно получить нетривиальную оценку для b даже при $s = |G|$. Для этого нужно применить (9) к вероятности P_1 (см. доказательство теоремы 1), у которой $\mathrm{supp}(P) \neq G$.

Следствие 3. Если вероятность P равномерно распределена на нормальном s-элементном множестве, то $\frac{|G|s^{-1} - 1}{|G| - 1} \leq b^2 \leq \frac{|G|s^{-1} - 1}{D^2}$.
Доказательство. Для равномерного распределения $P(g) = s^{-1}$ $(g \in \mathrm{supp}P)$, поэтому

$$\|P\|^2 = |G| \sum_g P^2(g) = |G|s \cdot s^{-2} = |G|s^{-1}.$$

ЛИТЕРАТУРА

1. P. Diaconis, Group Representations in Probability and Statistics. Institute of Mathematical Statistics, 1988. – 198 p.

2. L. Saloff-Coste, Random walks on finite groups. In Probability on Discrete Structures. / H. Kesten, editor, Springer, 2004. – P. 263–340.

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Условия единственности положения равновесия задачи Коши для линейных матричных дифференциально-алгебраических уравнений

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Получены достаточные условия существования единственного положения равновесия задачи Коши для дифференциально-алгебраических уравнений. Предложена конструктивная схема построения положения равновесия задачи Коши в общем случае, когда линейный оператор L, соответствующий однородной части уравнения, не имеет обратного.
Ключевые слова: дифференциально-алгебраические матричные уравнения; псевдообратные матрицы.

Сысоев Д. В. **Умови існування єдиного положення рівноваги задачі Коші для лінійних матричних диференціально-алгебраїчних рівнянь.** Встановлено достатні умови існування єдиного положення рівноваги задачі Коші для диференціально-алгебраїчних рівнянь. Запропонована конструктивна схема побудови положення рівноваги задачі Коші у випадку, коли лінійний оператор L, відповідний однорідної частини рівняння, не має оберненого.
Ключові слова: диференціально-алгебраїчні матричні рівняння; псевдообернена матриця.

D.V. Sysoev. **A condition for the existence of a unique equilibrium position of the Cauchy problem for linear matrix differential-algebraic equations.** Sufficient conditions for the existence of a unique equilibrium position of the Cauchy problem for differential-algebraic equations are proposed. The paper proposes a constructive scheme of the equilibrium position in the Cauchy problem in the general case, when a linear operator L, corresponding to homogeneous of the equation, has no inverse.
Keywords: differential-algebraic matrix equation; pseudoinverse matrix.

2010 Mathematics Subject Classification: 15A24; 34B15; 34C25.

1. Постановка задачи

Исследуем задачу о построении решений [1]

$$Z(t) \in \mathbb{C}^1_{\alpha\times\beta}[a,b] := \mathbb{C}^1[a,b] \otimes \mathbb{R}^{\alpha\times\beta}$$

задачи Коши для матричного дифференциально-алгебраического уравнения

$$\mathcal{A}Z'(t) = \mathcal{B}Z(t) + \mathcal{F}(t), \quad Z(a) = \mathfrak{A}, \quad \mathfrak{A} \in \mathbb{R}^{\alpha\times\beta}. \tag{1}$$

Здесь [2, 3]

$$\mathcal{A}Z'(t) := \sum_{i=1}^{p} S_i(t)Z'(t)R_i(t), \quad \mathcal{B}Z(t) := \sum_{j=1}^{q} \Phi_j(t)Z(t)\Psi_j(t)$$

— линейные матричные операторы,

$$S_i(t), \Phi_i(t) \in \mathbb{C}_{\gamma\times\alpha}[a,b], \quad R_i(t), \Psi_j(t) \in \mathbb{C}_{\beta\times\delta}[a,b], \quad F(t) \in \mathbb{C}_{\gamma\times\delta}[a,b]$$

— непрерывные матрицы; кроме того $\alpha, \beta, \gamma, \delta \in \mathbb{N}$ — произвольные натуральные числа. Матричное дифференциально-алгебраическое уравнение (1) обобщает традиционные постановки, как для матричных дифференциальных уравнений [4, 5], так и для дифференциально-алгебраических уравнений [6, 7, 8]. Изучение краевых задач, как матричных, так и для дифференциально-алгебраических уравнений основано на исследовании алгебраических матричных уравнений, в частности, результаты, полученные для матричного дифференциального уравнения Риккати [4], опираются на исследования матричного алгебраического уравнения типа Ляпунова [9]; результаты статей [2, 3, 5] опираются на исследования матричных уравнений типа Сильвестра и, в частности, уравнения типа Ляпунова [9, 10, 11, 12].

Особенностью задачи Коши (1) для дифференциально-алгебраических уравнений [6, 7, 8] является некорректность ее постановки в классе $\mathbb{C}^1_{\alpha\times\beta}[a,b]$ при произвольных $\alpha, \beta, \gamma, \delta \in \mathbb{N}$ и $F(t) \in \mathbb{C}_{\gamma\times\delta}[a,b]$ [13, 14]. Поставим следующую задачу: для каких классов задача Коши (1) имеет единственное решение при произвольных $\alpha, \beta, \gamma, \delta \in \mathbb{N}$ и $F(t) \in \mathbb{C}_{\gamma\times\delta}[a,b]$. Обозначим

$$\left\{\Theta_j\right\}_{j=1}^{\beta\cdot\gamma} \in \mathbb{R}^{\beta\times\gamma}$$

естественный базис [15] пространства $\mathbb{R}^{\beta\times\gamma}$. Задача о нахождении решений матричного дифференциально-алгебраического уравнения (1) приводит к задаче о нахождении вектора $z(t) \in \mathbb{C}^1_{\alpha\cdot\beta}[a;b]$, компоненты которого $z_j(t)$ определяют разложение матрицы

$$Z(t) = \sum_{j=1}^{\alpha\cdot\beta} \Xi^{(j)} z_j(t), \quad z_j(t) \in \mathbb{C}^1[a;b], \quad j = 1, 2, \ \dots\ , \ \alpha\cdot\beta.$$

Определим оператор $\mathcal{M}[A]: \ \mathbb{R}^{m\times n} \to \mathbb{R}^{m\cdot n}$ как оператор, который ставит в соответствие матрице $A \in \mathbb{R}^{m\times n}$ вектор-столбец $\mathcal{B} := \mathcal{M}[A] \in \mathbb{R}^{m\cdot n}$, составленный из n столбцов матрицы A, а также обратный оператор [12]

$$\mathcal{M}^{-1}\Big[\mathcal{B}\Big]: \ \mathbb{R}^{m\cdot n} \to \mathbb{R}^{m\times n},$$

который ставит в соответствие вектору $\mathcal{B} \in \mathbb{R}^{m\cdot n}$ матрицу $A \in \mathbb{R}^{m\times n}$. Линейный дифференциально-алгебраический матричный оператор $\mathcal{A}Z'(t)$ по определению представим в виде

$$\mathcal{A}Z'(t) = \sum_{j=1}^{\alpha\beta} \mathcal{A}\ \Xi^{(j)}(t) z_j'(t),$$

при этом

$$\mathcal{M}\Big[\mathcal{A}Z'(t)\Big] = \Omega(t)\cdot z'(t), \quad \Omega(t) := \Big[\Omega_j(t)\Big]_{j=1}^{\alpha\cdot\beta} \in \mathbb{R}^{\gamma\cdot\delta\times\alpha\cdot\beta},$$

где

$$\Omega_j(t) = \mathcal{M}\left[\mathcal{A}\ \Xi^{(j)}(t)\right], \quad j = 1, 2, \ \dots\ , \ \alpha\cdot\beta.$$

Аналогично

$$\mathcal{M}\Big[\mathcal{B}Z(t)\Big] = \Theta(t)\cdot z(t), \quad \Theta(t) := \Big[\Theta_j(t)\Big]_{j=1}^{\alpha\cdot\beta} \in \mathbb{R}^{\gamma\cdot\delta\times\alpha\cdot\beta}, \quad \Theta_j(t) = \mathcal{M}\left[\mathcal{B}\ \Xi^{(j)}(t)\right].$$

Таким образом, задача о построении решений дифференциально-алгебраического уравнения (1) приведена к задаче о нахождении решений

$$z(t) \in \mathbb{C}^1_{\alpha\cdot\beta\times 1}[a; b]$$

традиционного дифференциально-алгебраического уравнения [6, 7]

$$\Omega(t)\cdot z'(t) = \Theta(t)\cdot z(t) + \mathcal{F}(t), \quad \mathcal{F}(t) := \mathcal{M}\Big[F(t)\Big]. \tag{2}$$

2. Основной результат

При условии [2, 3, 5]

$$P_{\Omega^*(t)}\Theta(t) = 0, \quad P_{\Omega^*(t)}\mathcal{F}(t) = 0 \tag{3}$$

система (2) разрешима относительно производной

$$\frac{dz}{dt} = \Omega^+(t)\Theta(t)z + \mathfrak{F}(t, \varphi(t)), \quad \mathfrak{F}(t, \varphi(t)) := \Omega^+(t)\mathcal{F}(t) + P_{\Omega_r}(t)\varphi(t).$$

Здесь $P_{\Omega_r}(t)$ — $(\alpha\cdot\beta\times r)$-матрица, составленная из r линейно-независимых столбцов $(\alpha\cdot\beta\times\alpha\cdot\beta)$-матрицы-ортопроектора $P_\Omega(t):\mathbb{R}^{\alpha\cdot\beta}\to\mathbb{N}(\Omega(t))$. В случае

$$A:=\Omega^+(t)\Theta(t),\ f:=\Omega^+(t)\mathcal{F}(t),\ P_{\Omega_r}(t)=\mathrm{const} \tag{4}$$

система (2) приводится к виду

$$z'=Az+P_{\Omega_r}c_r+f,\quad c_r\in\mathbb{R}^r. \tag{5}$$

Таким образом, при условиях (3) и (4) система (2) имеет положения равновесия $z=\mathrm{const}$, для нахождения которых приходим к уравнению

$$Q\,c+f=0,\quad Q:=\begin{pmatrix} A & P_{\Omega_r}\end{pmatrix},\quad c:=\mathrm{col}\,(z,\ c_r)\in\mathbb{R}^{\alpha\beta+r}. \tag{6}$$

При условии $P_{Q^*}f=0$ (и только при нем) уравнение (6) разрешимо:

$$z=\begin{pmatrix} I_{\alpha\beta} & O\end{pmatrix}(\,P_{Q_\rho}c_\rho-Q^+f),\ \ c_r=\begin{pmatrix} O & I_r\end{pmatrix}(\,P_{Q_\rho}c_\rho-Q^+f).$$

Здесь P_{Q_ρ} — $((\alpha\beta+r)\times\rho)$-матрица, составленная из ρ линейно-независимых столбцов $((\alpha\beta+r)\times(\alpha\beta+r))$-матрицы-ортопроектора $P_Q:\mathbb{R}^{\alpha\cdot\beta+r}\to\mathbb{N}(Q)$. Таким образом, при условиях (3) и (4) система (2) имеет положения равновесия

$$z=D\,c_\rho+K[f],\ \ D:=\begin{pmatrix} I_{\alpha\beta} & O\end{pmatrix}P_{Q_\rho}\in\mathbb{R}^{(\alpha\cdot\beta+r)\times\rho},\ \ K[f]:=-\begin{pmatrix} I_{\alpha\beta} & O\end{pmatrix}Q^+f,$$

определяющие решение задачи Коши (1) в случае

$$D\,c_\rho+K[f]=\mathcal{M}(\mathfrak{A}).$$

Последнее уравнение разрешимо тогда и только тогда, когда

$$P_{D^*}\Big\{\mathcal{M}(\mathfrak{A})-K[f]\Big\}=0. \tag{7}$$

Здесь P_{D^*} — $(\alpha\beta\times\alpha\beta)$-матрица-ортопроектор $P_{D^*}:\mathbb{R}^{\alpha\beta}\to\mathbb{N}(D)$. Итак, при условиях (3), (4) и (7) задача Коши (1) имеет единственное положение равновесия

$$Z(c_r)=\mathcal{M}^{-1}\Big[DD^+\mathcal{M}(\mathfrak{A})\Big]+\mathcal{M}^{-1}\Big\{P_{D^*}K[f]\Big\}.$$

Таким образом, доказано следующее утверждение.

Теорема 0.1 *При условиях (3), (4) и (7) задача Коши (1) имеет единственное положение равновесия*

$$Z(c_r)=W(\mathfrak{A})+K[\mathcal{F}(t)],$$

представленное суммой решения однородного $F(t) = 0$ уравнения (1)

$$Z(c_r) = W(\mathfrak{A}) := \mathcal{M}^{-1}\Big[DD^{+}\mathcal{M}(\mathfrak{A})\Big]$$

и частного решения неоднородной задачи Коши (1)

$$Z(c_r) = K[\mathcal{F}(t)] := \mathcal{M}^{-1}\Big\{P_{D^*}K[f]\Big\}.$$

Пример 0.1 *Условиям доказанной теоремы 0.1 удовлетворяет матричная задачи Коши*

$$\mathcal{A}Z'(t) = \mathcal{B}Z(t) + \mathcal{F}(t), \quad Z(0) = \mathfrak{A}, \quad \mathfrak{A} \in \mathbb{R}^{\alpha\times\beta}; \tag{8}$$

здесь

$$\mathcal{A}Z'(t) = \sum_{i=1}^{2} \mathcal{S}_i\, Z'(t)\, \mathcal{R}_i, \quad \mathcal{S}_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{S}_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{R}_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{R}_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Psi_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\mathcal{B}Z(t) := \sum_{i=1}^{2} \Phi_i\, Z'(t)\, \Psi_i, \quad \Phi_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Phi_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Psi_2 := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathfrak{A} := \frac{1}{5}\begin{pmatrix} 5 & 5 & -1 \\ 5 & 5 & -2 \end{pmatrix}^*, \quad \mathcal{F}(t) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Естественный базис пространства $\mathbb{R}^{3\times 2}$ составляют матрицы

$$\Xi_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^*, \ \Xi_2 := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}^*, \ \dots, \ \Xi_6 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^*.$$

Ключевые при исследовании уравнения (8) матрицы имеют вид

$$\Omega := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^*$$

и

$$\Theta := \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&2&0&0&0&0&0&0 \end{pmatrix}^* ;$$

при этом условие (3) выполнено: $P_{\Omega^*(t)}\Theta(t) = 0, \ P_{\Omega^*(t)}\mathcal{F}(t) = 0$, кроме того

$$A = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&1&0&0&2 \\ 0&0&0&0&0&0 \end{pmatrix}, \ f(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ P_{\Omega_r}(t) = \begin{pmatrix} 5&0&0&0&0 \\ 0&5&0&0&0 \\ 0&0&2&0&0 \\ 0&0&0&5&0 \\ 0&0&0&0&5 \\ 0&0&-1&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}$$

— константы, следовательно, условие (4) также выполнено. Матрицы

$$D = \begin{pmatrix} 5&0&0&0&0 \\ 0&5&0&0&0 \\ 0&0&2&0&0 \\ 0&0&0&5&0 \\ 0&0&0&0&5 \\ 0&0&-1&0&0 \end{pmatrix}, \quad P_{D^*} = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&1&0&0&2 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&2&0&0&4 \end{pmatrix},$$

а также оператор Грина задачи Коши для системы (5)

$$K[f] = \frac{1}{5} \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & -2 \end{pmatrix}$$

позволяют проверить условие (7). Поскольку все требования доказанной теоремы 0.1 выполнены, задача Коши (8) имеет единственное положение равновесия

$$Z(c_r) = W(\mathfrak{A}) + K[\mathcal{F}(t)],$$

где

$$W(\mathfrak{A}) = \begin{pmatrix} 1&1 \\ 1&1 \\ 0&0 \end{pmatrix}, \quad K[\mathcal{F}(t)] = -\frac{1}{5} \begin{pmatrix} 0&0 \\ 0&0 \\ 1&1 \end{pmatrix}.$$

Доказанная теорема может быть использована при решении дифференциальных уравнений Риккати и Бернулли [4], при решении линейных краевых задач для матричных дифференциальных уравнений [3, 5, 16], а также в теории устойчивости движения [17, 18]. Полученные результаты аналогично [19] могут быть перенесены на обобщенные уравнения, содержащие неизвестные матрицы различных размерностей.

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ЛИТЕРАТУРА

1. Boichuk A.A., Samoilenko A.M. Generalized inverse operators and Fredholm boundary-value problems (2-th edition). – Berlin; Boston: De Gruyter, 2016. – 298 p.

2. Чуйко С.М. Обобщенное матричное дифференциально-алгебраическое уравнение // Український математичний вісник, 2015. – **12**, №1. – С. 11 – 26.

3. Chuiko S.M. The Green's operator of a generalized matrix linear differential-algebraic boundary value problem // Siberian Mathematical Journal, 2015. – **56**, №4. – P. 752–760.

4. Boichuk A.A., Krivosheya S.A. A Critical Periodic Boundary Value Problem for a Matrix Riccati Equations // Differential Equations, 2001. – **37**, №4. – P. 464–471.

5. Chuiko S.M. Generalized Green Operator of Noetherian boundary-value problem for matrix differential equation // Russian Mathematics, 2016. – **60**, №8. – P. 64–73.

6. Campbell S.L. Singular Systems of differential equations. – San Francisco – London – Melbourne: Pitman Advanced Publishing Program, 1980. – 178 p.

7. Чистяков В.Ф. Алгебро-дифференциальные операторы с конечномерным ядром. – Новосибирск; Наука, 1996. – 280 с.

8. Boichuk A.A., Pokutnyi A.A., Chistyakov V.F. Application of perturbation theory to the solvability analysis of differential algebraic equations // Computational Mathematics and Mathematical Physics, 2013. – **53**. – №6. – P. 777 – 788.

9. Boichuk A.A., Krivosheya S.A. Criterion of the solvability of matrix equations of the Lyapunov type // Ukrainian Mathematical Journal, 1998. – **50**, №8. – P. 1162 – 1169.

10. Чуйко С.М. О решении матричных уравнений Ляпунова // Вісник Харківського національного університету ім. В.Н.Каразіна. Серія «Математика, прикладна математика і механіка». – № 1120, 2014. – С. 85–94.

11. Беллман Р. Введение в теорию матриц. – М.: Наука, 1969. – 367 с.

12. Чуйко С.М. Элементы теории линейных матричных уравнений. — Славянск: Изд. Б.И. Маторина, 2017. — 164 с.

13. Тихонов А.Н., Арсенин В.Я. Методы решения некорректных задач. – М.: Наука, 1986. – 288 с.

14. Chuiko S.M. On the regularization of a linear Fredholm boundary-value problem by a degenerate pulsed action // Journal of Mathematical Sciences, 2014. – **197**, №1. – P. 138–150.

15. Воеводин В.В., Кузнецов Ю.А. Матрицы и вычисления. – М.: Наука, 1984. – 318 с.

16. Chuiko S. Weakly nonlinear boundary value problem for a matrix differential equation // Miskolc Mathematical Notes, 2016. – **17**, №1. – P. 139–150.

17. Коробов В.И., Бебия М.О. Стабилизация одного класса нелинейных систем, неуправляемых по первому приближению // Доп. НАН України, 2014. – №2. – С. 20–25.

18. Бебия М.О. Стабилизация систем со степенной нелинейностью // Вісник Харківського національного університету ім. В.Н.Каразіна. Серія «Математика, прикладна математика і механіка», 2014. – №1120, Вып. 69. – С. 75–84.

19. Чуйко С.М. О решении билинейного матричного уравнения // Чебышевский сборник, 2016. – **17**, Вып. 2. – С. 196–205.

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Some generalizations of p-loxodromic functions

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The functional equation of the form $f(qz) = p(z)f(z)$, $q \in \mathbb{C}\backslash\{0\}$, $|q| < 1$, $z \in \mathbb{C}\backslash\{0\}$ is considered. For certain fixed elementary functions $p(z)$, meromorphic solutions of this equation are found. These solutions are some generalizations of p-loxodromic functions and can be represented via the Schottky-Klein prime function as well as classic p-loxodromic functions.
Keywords: loxodromic function; p-loxodromic function; the Schottky-Klein prime function.

Христіянин А.Я., Луківська Дз.В. **Деякі узагальнення p-локсодромних функцій.** Розглянуто функціональне рівняння $f(qz) = p(z)f(z)$, $z \in \mathbb{C}\backslash\{0\}$, $q \in \mathbb{C}\backslash\{0\}$, $|q| < 1$. При певних фіксованих елементарних функціях $p(z)$ знайдено його мероморфні розв'язки. Ці розв'язки є деякими узагальненнями p-локсодромних функцій і можуть зображатися за допомогою первинної функції Шотткі-Кляйна, як і класичні p-локсодромні функції.
Ключові слова: локсодромна функція; p-локсодромна функція; первинна функція Шотткі-Кляйна.

Хрыстиянын А.Я., Лукивська Дз.В. **Некоторые обобщения p-локсодромических функций.** Рассмотрено функциональное уравнение $f(qz) = p(z)f(z)$, $z \in \mathbb{C}\backslash\{0\}$, $q \in \mathbb{C}\backslash\{0\}$, $|q| < 1$. При определенных фиксированных элементарных функциях $p(z)$ найдены его мероморфные решения. Эти решения являются некоторыми обобщениями p-локсодромических функций и могут изображаться с помощью первичной функции Шоттки-Кляйна, как и классические p-локсодромические функции.
Ключевые слова: локсодромическая функция; p-локсодромическая функция; первичная функция Шоттки-Кляйна.
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1. Introduction

Denote $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. For $z \in \mathbb{C}^*$ consider the equation of the form

$$f(qz) = p(z)f(z), \tag{1}$$

where $p(z)$ is some function, $q \in \mathbb{C}^*$, $|q| < 1$. If $p(z) \equiv const$, then meromorphic solution of this equation is called p-loxodromic function [5]. In particular, if $p(z) \equiv 1$, we have classic loxodromic function. The class of loxodromic functions is denoted by $\mathcal{L}_q$. It was studied in the works of O. Rausenberger [12], G. Valiron [14] and Y. Hellegouarch [3]. In recent years, A. Kondratyuk and his colleagues also investigated these functions and their various generalizations in other domains (see, for example [4], [6]-[8]).

Loxodromic functions have been used to construct explicit solutions to the rotating Hele-Shaw problem, the viscous sintering problem, the problem of finding vortical equilibria of the Euler equation and the problem of free surface Euler flows of the surface tension [2]. These functions also have a fairly wide range of practical applications, for example see [10], [11].

So, it will be quite interesting to generalize the class of p-loxodromic functions for the case of more general functions $p(z)$ other then the constant ones. The purpose of this article is to obtain meromorphic solutions of the equation (1), where $p(z)$ are some elementary functions. These solutions will be some generalizations of p-loxodromic functions. This task can be viewed as the first step towards more general case where $p(z)$ is an arbitrary rational function, which in turn may lead to further generalizations.

2. The case $p(z) = \frac{1}{z}$

Let us consider functional equation

$$f(qz) = \frac{1}{z}f(z),\ z \in \mathbb{C}^*. \tag{2}$$

Our task is to find its meromorphic in $\mathbb{C}^*$ solutions. At first consider the Schottky-Klein prime function [5]

$$P(z) = (1-z)\prod_{n=1}^{\infty}(1-q^n z)\left(1-\frac{q^n}{z}\right). \tag{3}$$

It was introduced by Schottky [13] and Klein [9] for the study of conformal mappings of double-connected domains, see also [1]. This function is holomorphic in $\mathbb{C}^*$ and has zero sequence $\{q^n\}$, $n \in \mathbb{Z}$. The following property of $P(z)$ is well known [3, p. 94]

$$P(qz) = -z^{-1}P(z). \tag{4}$$

Theorem 1 *Let $g \in \mathcal{L}_q$. The meromorphic in $\mathbb{C}^*$ function $f(z) = P(-z)g(z)$ satisfies (2).*

Proof. The proof is by direct calculation. Since g is loxodromic, we have

$$f(qz) = P(-qz)g(qz) = \frac{1}{z}P(-z)g(z) = \frac{1}{z}f(z).$$

We also use here equality (4).

Theorem 2 *Every meromorphic in $\mathbb{C}^*$ solution of (2) can be represented in the form $f(z) = P(-z)g(z)$, where $g \in \mathcal{L}_q$.*

Proof. Let $f(z)$ be a solution of (2). Consider the function $g(z) = \dfrac{f(z)}{P(-z)}$. Since $f(z)$ is meromorphic and $P(-z)$ is holomorphic, it follows that g is meromorphic. Applying equalities (2) and (4), we get

$$g(qz) = \frac{f(qz)}{P(-qz)} = \frac{\frac{1}{z}f(z)}{\frac{1}{z}P(-z)} = g(z).$$

Therefore, for all $z \neq -q^n$, $n \in \mathbb{Z}$ we have $g(qz) = g(z)$. It means that g is loxodromic, which concludes the proof.

We also can reformulate Theorems 1 and 2 in the following forms.

Theorem 3 *The meromorphic in $\mathbb{C}^*$ function*

$$f(z) = C\frac{P\left(\frac{z}{a_1}\right)P\left(\frac{z}{a_2}\right)\dots P\left(\frac{z}{a_m}\right)P\left(\frac{z}{a_{m+1}}\right)}{P\left(\frac{z}{b_1}\right)P\left(\frac{z}{b_2}\right)\dots P\left(\frac{z}{b_m}\right)},$$

where C is a constant, $a_1, a_2, \dots, a_{m+1}$ and $b_1, b_2, \dots, b_m$ are complex numbers, not necessarily distinct, such that $\prod\limits_{j=1}^{m+1} a_j = -\prod\limits_{j=1}^{m} b_j$, satisfies equation (2).

Proof. Indeed, taking into account equality (4),

$$f(qz) = C\frac{P\left(\frac{qz}{a_1}\right)P\left(\frac{qz}{a_2}\right)\dots P\left(\frac{qz}{a_m}\right)P\left(\frac{qz}{a_{m+1}}\right)}{P\left(\frac{qz}{b_1}\right)P\left(\frac{qz}{b_2}\right)\dots P\left(\frac{qz}{b_m}\right)}$$

$$=C\frac{-\frac{a_1}{z}P\left(\frac{z}{a_1}\right)\left(-\frac{a_2}{z}\right)P\left(\frac{z}{a_2}\right)\dots\left(-\frac{a_m}{z}\right)P\left(\frac{z}{a_m}\right)\left(-\frac{a_{m+1}}{z}\right)P\left(\frac{z}{a_{m+1}}\right)}{-\frac{b_1}{z}P\left(\frac{z}{b_1}\right)\left(-\frac{b_2}{z}\right)P\left(\frac{z}{b_2}\right)\dots\left(-\frac{b_m}{z}\right)P\left(\frac{z}{b_m}\right)}$$

$$=C\frac{(-1)^{m+1}a_1a_2\dots a_{m+1}}{(-1)^m b_1b_2\dots b_m}\,\frac{P\left(\frac{z}{a_1}\right)P\left(\frac{z}{a_2}\right)\dots P\left(\frac{z}{a_m}\right)P\left(\frac{z}{a_{m+1}}\right)\left(\frac{1}{z}\right)}{P\left(\frac{z}{b_1}\right)P\left(\frac{z}{b_2}\right)\dots P\left(\frac{z}{b_m}\right)}=\frac{1}{z}f(z)$$

Theorem 4 *Every meromorphic in $\mathbb{C}^*$ solution of equation (2) can be written in the form*

$$f(z) = C\frac{P\left(\frac{z}{a_1}\right)P\left(\frac{z}{a_2}\right)\dots P\left(\frac{z}{a_m}\right)P\left(\frac{z}{a_{m+1}}\right)}{P\left(\frac{z}{b_1}\right)P\left(\frac{z}{b_2}\right)\dots P\left(\frac{z}{b_m}\right)},$$

where C is a constant, $a_1, a_2, \dots, a_{m+1}$ and $b_1, b_2, \dots, b_m$ are complex numbers, not necessarily distinct, such that $\prod\limits_{j=1}^{m+1} a_j = -\prod\limits_{j=1}^{m} b_j$.

Proof. By Theorem 2 we know that

$$f(z) = P(-z)g(z), \tag{5}$$

where $g \in \mathcal{L}_q$. We use the loxodromic function representation via Schottky-Klein prime functions (see [3], [14] for more details). Namely, let $c_1, c_2, ..., c_m$ and $b_1, b_2 ..., b_m$ be the zeros and the poles of function g in the annulus $A_q(R) = \{z \in \mathbb{C} : |q|R < |z| \le R\}$, $R > 0$, respectively, $\partial A_q(R)$ contains neither zeros nor poles of $g \in \mathcal{L}_q$. Note that each loxodromic function g has equal numbers of zeros and poles (counted according to their multiplicities) in every such annulus $A_q(R)$ [3, p. 93]. Then [14, p. 478]

$$g(z) = Kz^p\frac{P\left(\frac{z}{c_1}\right)P\left(\frac{z}{c_2}\right)\cdot ... \cdot P\left(\frac{z}{c_m}\right)}{P\left(\frac{z}{b_1}\right)P\left(\frac{z}{b_2}\right)\cdot ... \cdot P\left(\frac{z}{b_m}\right)}, \tag{6}$$

where

$$\frac{c_1c_2 \dots c_m}{b_1b_2 \dots b_m} = q^{-p}, \quad p \in \mathbb{Z}, \tag{7}$$

and K is a constant. Applying equality (4) to (6), we have

$$g(z) = C\frac{P\left(\frac{z}{q^pc_1}\right)P\left(\frac{z}{c_2}\right)\cdot ... \cdot P\left(\frac{z}{c_m}\right)}{P\left(\frac{z}{b_1}\right)P\left(\frac{z}{b_2}\right)\cdot ... \cdot P\left(\frac{z}{b_m}\right)}. \tag{8}$$

where $C = (-a_1)^p q^{\frac{p(p+1)}{2}} K$. Combining (5) and (8), we obtain

$$f(z) = C\frac{P\left(\frac{z}{q^pc_1}\right)P\left(\frac{z}{c_2}\right)\dots P\left(\frac{z}{c_m}\right)P\left(\frac{z}{-1}\right)}{P\left(\frac{z}{b_1}\right)P\left(\frac{z}{b_2}\right)\dots P\left(\frac{z}{b_m}\right)},$$

Let us denote $a_1 = q^p c_1$, $a_2 = c_2, \dots, a_m = c_m$, $a_{m+1} = -1$. Now we can rewrite f as follows

$$f(z) = C\frac{P\left(\frac{z}{a_1}\right)P\left(\frac{z}{a_2}\right)\dots P\left(\frac{z}{a_m}\right)P\left(\frac{z}{a_{m+1}}\right)}{P\left(\frac{z}{b_1}\right)P\left(\frac{z}{b_2}\right)\dots P\left(\frac{z}{b_m}\right)},$$

where $\prod_{j=1}^{m+1} a_j = -\prod_{j=1}^{m} b_j$, which is clear in view of (7). The theorem is proved.

3. The case $p(z) = \frac{1}{1-z}$

Now, consider functional equation of the form

$$f(qz) = \frac{1}{1-z} f(z),\ z \in \mathbb{C}^*. \tag{9}$$

We also are interested in finding meromorphic in $\mathbb{C}^*$ solutions of (9).

Define the entire function with the zero sequence $\{q^{-n}\}$, $n \in \mathbb{N} \cup \{0\}$, $0 < |q| < 1$,

$$H(z) = \prod_{n=0}^{\infty}(1 - q^n z).$$

Theorem 5 *Let $g \in \mathcal{L}_q$. The meromorphic in $\mathbb{C}^*$ function $f(z) = H(z)g(z)$ satisfies (9).*

Proof. The proof is straightforward. Since g is loxodromic, we have

$$(1-z)f(qz) = (1-z)g(qz)H(qz) = (1-z)g(z)\prod_{n=0}^{\infty}(1 - q^{n+1}z)$$

$$= (1-z)g(z)\prod_{k=1}^{\infty}(1 - q^k z) = g(z)\prod_{n=0}^{\infty}(1 - q^n z) = f(z).$$

Theorem 6 *Every meromorphic in $\mathbb{C}^*$ solution of (9) can be represented in the form $f(z) = H(z)g(z)$, where $g \in \mathcal{L}_q$.*

Proof. The proof is analogous to the proof of Theorem 2. Let f be a solution of equation (9). Consider the function $g = \frac{f}{H}$. Since f is meromorphic and H is holomorphic, it follows that g is meromorphic. Taking into account equality (9), we get

$$g(qz) = \frac{f(qz)}{H(qz)} = \frac{\frac{1}{1-z}f(z)}{\frac{1}{1-z}H(z)} = g(z).$$

Therefore, for all $z \neq q^{-n}$, $n \in \mathbb{N} \cup \{0\}$ we can conclude that $g(qz) = g(z)$. We obtain that g is loxodromic. The proof is completed.

Using the loxodromic function representation via Schottky-Klein prime functions, namely formulas (6) and (7), we also can rewrite Theorems 5 and 6 in the following forms.

Theorem 7 *The meromorphic in $\mathbb{C}^*$ function*

$$f(z) = Cz^p H(z) \frac{P\left(\frac{z}{c_1}\right) P\left(\frac{z}{c_2}\right) \cdot \ldots \cdot P\left(\frac{z}{c_m}\right)}{P\left(\frac{z}{b_1}\right) P\left(\frac{z}{b_2}\right) \cdot \ldots \cdot P\left(\frac{z}{b_m}\right)},$$

where $c_1, c_2 \ldots, c_m$ and $b_1, b_2 \ldots, b_m$ are complex numbers, not necessarily distinct, such that $q^p \prod_{j=1}^{m} c_j = \prod_{j=1}^{m} b_j$, $p \in \mathbb{Z}$ and C is a constant, satisfies (9).

Theorem 8 *Every meromorphic in $\mathbb{C}^*$ solution of (9) can be written in the form*

$$f(z) = Cz^p H(z) \frac{P\left(\frac{z}{c_1}\right) P\left(\frac{z}{c_2}\right) \cdot \ldots \cdot P\left(\frac{z}{c_m}\right)}{P\left(\frac{z}{b_1}\right) P\left(\frac{z}{b_2}\right) \cdot \ldots \cdot P\left(\frac{z}{b_m}\right)},$$

where $c_1, c_2 \ldots, c_m$ and $b_1, b_2 \ldots, b_m$ are complex numbers, not necessarily distinct, such that $q^p \prod_{j=1}^{m} c_j = \prod_{j=1}^{m} b_j$, $p \in \mathbb{Z}$ and C is a constant.

Applying the Schottky-Klein prime function's property (4) to the representation of function f in Theorems 7 and 8 we can reformulate these theorems in the next forms.

Theorem 9 *The meromorphic in $\mathbb{C}^*$ function*

$$f(z) = CH(z) \frac{P\left(\frac{z}{q^p c_1}\right) P\left(\frac{z}{c_2}\right) \cdot \ldots \cdot P\left(\frac{z}{c_m}\right)}{P\left(\frac{z}{b_1}\right) P\left(\frac{z}{b_2}\right) \cdot \ldots \cdot P\left(\frac{z}{b_m}\right)},$$

where $c_1, c_2 \ldots, c_m$ and $b_1, b_2 \ldots, b_m$ are complex numbers, not necessarily distinct, such that $q^p \prod_{j=1}^{m} c_j = \prod_{j=1}^{m} b_j$, $p \in \mathbb{Z}$ and C is a constant, satisfies (9).

Theorem 10 *Every meromorphic in $\mathbb{C}^*$ solution of (9) can be represented in the form*

$$f(z) = CH(z) \frac{P\left(\frac{z}{q^p c_1}\right) P\left(\frac{z}{c_2}\right) \cdot \ldots \cdot P\left(\frac{z}{c_m}\right)}{P\left(\frac{z}{b_1}\right) P\left(\frac{z}{b_2}\right) \cdot \ldots \cdot P\left(\frac{z}{b_m}\right)},$$

where $c_1, c_2 \ldots, c_m$ *and* $b_1, b_2 \ldots, b_m$ *are complex numbers, not necessarily distinct, such that* $q^p \prod_{j=1}^{m} c_j = \prod_{j=1}^{m} b_j$, $p \in \mathbb{Z}$ *and* C *is a constant.*

REFERENCES

1. Crowdy D.G., Geometric function theory: a modern view of a classical subject // IOP Publishing Ltd and London Mathematical Society, Nonlinearity, 2008. – **21** (10). – T205-T219. DOI: 10.1088/0951-7715/21/10/T04

2. P. Ebenfelt, B. Gustafsson, D. Khavinson, M. Putinar. Quadrature Domains and Their Applications: The Harold S. Shapiro Anniversary Volume. – Germany, 2006.

3. Hellegouarch Y. Invitation to the Mathematics of Fermat-Wiles. – Academic Press, 2002.

4. Hushchak O., Kondratyuk A. The Julia exceptionality of loxodromic meromorphic functions // Visnyk of the Lviv Univ., Series Mech. Math., 2013. – **78**. – P. 35-41.

5. Khoroshchak V.S., Khrystiyanyn A.Ya., Lukivska D.V. A class of Julia exceptional functions // Carpathian Math. Publ., 2016. – **8** (1). – C. 172–180. DOI: 10.15330/cmp.8.1.172-180

6. Khoroshchak V.S., Kondratyuk A. A. The Riesz measures and a representation of multiplicatively periodic δ-subharmonic functions in a punctured euclidean space // Mat. Stud., 2015. – **43** (1).– P. 61-65.

7. Khoroshchak V.S., Sokulska N. B. Multiplicatively periodic meromorphic functions in the upper halfplane // Mat. Stud., 2014. – **42** (2). – P. 143-148.

8. Khrystiyanyn A.Ya., Kondratyuk A. A. Modulo-loxodromic meromorphic function in $C\backslash\{0\}$ // Ufa. Math. J., 2016. – **8** (4). – P. 156-162.

9. Klein F. Zur Theorie der Abel'schen Functionen, 1890. – Math. Ann. – **36**. – 1-83. DOI: 10.1007/BF01199432

10. Serdjo Kos, Tibor K. Pogány. On the Mathematics of Navigational Calculations for Meridian Sailing // Electronic Journal of Geography and Mathematics, 2012.

11. James Marcotte, Matthew Salomone. Loxodromic Spirals in M. C. Escher's Sphere Surface // Journal of Humanistic Mathematics, 2014. – **4** (2). – P. 25-46. DOI. 10.5642/jhummath.201402.04

12. Rausenberger O. Lehrbuch der Theorie der Periodischen Functionen Einer variabeln. – Leipzig: Druck und Ferlag von B.G.Teubner, 1884.

13. Schottky F. Über eine specielle Function welche bei einer bestimmten linearen Transformation ihres Arguments unverändert bleibt // J. Reine Angew. Math., 1887. – **101** . – P. 227-272.

14. Valiron G. Cours d'Analyse Mathematique, Theorie des fonctions. 2nd Edition. – Paris: Masson et.Cie., 1947.

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Lower bound on the number of meet-irreducible elements in extremal lattices

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Extremal lattices are lattices maximal in size with respect to the number n of their join-irreducible elements with bounded Vapnik-Chervonekis dimension k. It is natural, however, to estimate the size of a lattice also with respect to the number of its meet-irreducible elements. Although this number may differ for nonequivalent $(n, k+1)$-extremal lattices, we show that each $(n, k+1)$-extremal lattice has k disjoint chains of meet-irreducible elements, each of length $n-k+2$.

Keywords: Extremal lattices, Vapnik-Chervonekis dimension, meet-irreducible elements.

Чорномаз Б. О. **Нижня границя на кількість конерозкладних елементів в екстремальних решітках.** Екстремальними називаються решітки, максимальні за розміром відносно кількості n своїх нерозкладних елементів, при обмеженні на розмірність Вапніка-Червонекіса k. Цілком природньо, з іншого боку, оцінювати розмір решітки також відносно кількості її конерозкладних елементів. Ми покажемо, що в кожній $(n, k+1)$-екстремальній решітці існує k неперетинаючихся ланцюгів конерозкладних елементів, кожний довжини $n-k+2$.

Ключові слова: Екстремальні решітки, розмірність Вапника-Червонекіса, конерозкладні елементи.

Черномаз Б. А. **Нижняя граница на количество конеразложимых элементов в экстремальных решётках.** Экстремальными называются решётки, имеющие максимальный размер относительно количества n своих неразложимых элементов, при ограничении на размерность Вапника-Червонекиса k. Естественно, с друой стороны, оценивать размер решетки также относительно количества её конеразложимых элементов. Мы покажем, что в каждой $(n, k+1)$-экстремальной решетке есть k непересекающихся цепей конеразложимых элементов, каждый длины $n-k+2$.

Ключевые слова: Экстремальные решетки, размерность Вапника-Червонекиса, конеразложимые элементы.

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1. Introduction

This paper deals with extremal problems for lattices with bounded *Vapnik-Chervonekis (VC) dimension.* For a finite lattice L, the VC dimension of L, denoted $vc(L)$, is the maximal k such that that L admit an order embedding of boolean lattice on k generators $\mathcal{B}(k)$. A lattice L is called (n,k)*-free* if it has at most n join-irreducible elements and its VC-dimension is at most $k-1$, that is, if it does not admit order embedding of $\mathcal{B}(k)$. Let us define $f(n,k)$ as

$$f(n,k) = \sum_{i=0}^{k-1} \binom{n}{i}.$$

It is known [2] that $f(n,k)$ is an upper bound on the size of (n,k)-free lattices, which is sharp for all n and k. Thus, we define (n,k)*-extremal lattices* to be (n,k)-free lattices that reach this bound. As a matter of convenience, we will mostly work with $(n,k+1)$-extremal lattices, as they can be considered maximal lattices of VC dimension k.

The idea of constraining VC dimension arises from the fact that, as it was shown in [2], the growth of VC dimension is the only reason for exponential growth of the lattice with respect to $|J(L)|$ or to $|M(L)|$, where $J(L)$ and $M(L)$ are the sets of join-irreducible and of meet-irreducible elements of L correspondingly. The bound $f(n,k)$, restricting the size of L, however, obviously depends only on $|J(L)|$, while it is rather natural to consider either $|J(L)|+|M(L)|$ or $|J(L)||M(L)|$, or other bounds symmetric in $|J(L)|$ and $|M(L)|$, as some natural measure of "complexity" of the lattice. For example, $|J(L)||M(L)|$ is the size of the *reduced formal context* describing L, see [6] for examples.

The first step towards building such symmetric bounds could be an estimation of the size of $M(L)$ for $(n,k+1)$-extremal lattices. As extremal lattices are not unique for $k \geq 2$, this number may vary. Here we are interested in producing a simple lower bound for this case, as we generally seek to maximize $|L|$ and minimize $|J(L)|$ and $|M(L)|$. Namely, we will prove that in L there are at least $k(n-k+2)$ meet-irreducible elements arranged in k disjoint chains, descending from the top of the lattice almost to its bottom. We will also show that this bound is sharp for $k=2$, that is, for $(n,3)$-extremal lattices. It seems that for larger k, however, this is not so, and even the rate of growth of $|M(L)|$ is not clear. As for the upper bound on the size of $|M(L)|$, it can be rather big. We refer to [1] for the example of the family of $(n,k+1)$-extremal lattices, $k \leq n/2$, for which every k-th element is meet-irreducible.

The structure of the paper is as follows. In Section 2, for the sake of self-sufficiency, we recall some basic facts about extremal lattices, as well as about lattices and partial orders in general. In Section 3 we explore how extremal lattices can be decomposed, and how these decompositions can be stacked. Then, in Section 4, we introduce *extremal decompositions* of lattices, and prove, in Theorem 2, that there is a one-to-one correspondence between extremal lattices and extremal decompositions via *root decompositions*. In Section 5 we use this technique in order to provide, in Theorem 3, the desired lower bound on the number of meet-irreducible elements. Finally, in Section 6 we outline the possible directions for application or extension of obtained results.

2. Preliminary definitions

In this section we recall basic definitions from lattice theory, as well as some facts about extremal lattices. We refer to [6] for further details and for general background.

For a function f and a subset X of the domain of f, we write $f[X]$ to denote the image of X under f. From time to time we deal with unions of disjoint sets, in this case, in order to stress their disjointness, we write $A \bigsqcup B$ instead of $A \bigcup B$. We write $\mathbf{k}$ to denote the standard set of k elements $\{1, \dots, k\}$.

All lattices and other objects considered in this paper are finite. Throughout the text we will be dealing with three types of embeddings of lattices, which we will explicitly differentiate: proper lattice embeddings, that is, $(\vee, \wedge)$-embeddings; $(1, \wedge)$-embeddings, which will be the most common case; and order embeddings, that is, embeddings of lattices as posets. Sometimes we will refer to $(1, \wedge)$-embeddings as simply embeddings, two other cases will always be indicated explicitly.

For a lattice L, an element $x \in L$ is called *join-irreducible* if it does not have proper join-decomposition, that is, if $x = u \vee v$ implies $x = u$ or $x = v$. Meet-irreducible elements are defined dually; the sets of the join-irreducible elements and of the meet-irreducible elements of L are denoted $J(L)$ and $M(L)$ correspondingly. It is a well-known fact that for a finite lattice each element can be represented via join-irreducibles, namely

$$x = \bigvee \{j \in J(L) \mid j \leq x\}.$$

We denote semi-intervals in L as

$$(x] := \{y \mid y \leq x\},$$
$$[x) := \{y \mid y \geq x\}.$$

For each $x \in L$ we introduce notation $J(x) = (x] \cap J(L)$ and $M(x) = [x) \cap M(L)$.

We say that a set $X \subseteq J(L)$ is a *representation of an element* x if $\bigvee X = x$; X is a *minimal representation* if no proper subset of X joins to x. Notice that, in general, minimal representation is not unique, the simplest counterexample provided by the lattice M_3, also called *diamond*. *Atoms* are elements of L that cover 0, the set of all atoms is denoted $A(L)$. Every atom is, obviously, join-irreducible, thus $A(L) \subseteq J(L)$. Lattice is called *atomistic* if each element in it can be represented as a join of atoms. For atomistic lattices it holds that $A(L) = J(L)$. The notion of *coatoms* is defined dually, and the set of coatoms is denoted $Co(L)$.

The notion of *maximal chain of a poset* is quite common in order theory. Here we find useful to introduce a slightly different notion of a *covering chain in a* *poset*. We say that C is a covering chain in P if C, with induced order, is a chain, and if from $x \prec_C y$ it follows that $x \prec_P y$. It is easy to see that for a finite P any covering chain is a subinterval of some maximal chain.

A lattice is called *graded*, if for each element x the lengths of all maximal chains in $(x]$ are equal, in which case this length is called *rank* of x, denoted $r(x)$. In graded lattice, the zero is the only element with rank 0. Note that graded lattice is a particular case of *graded poset*, which, however, would require a little more elaborated definition.

We refer to [2] (or to its extended version [3]) for detailed discussion on extremal lattices and lattices with bounded VC dimension. In particular, we refer to these papers for the proofs of all statements in this section. Note also that there authors would call an (n, k)-free lattice by a more correct, but more cumbersome name a $\mathcal{B}(k)$-free lattice on n join-irreducible elements.

A convenient characterization of $\mathcal{B}(k)$-freeness can be given in terms of minimal generators. The following Proposition is an easy consequence of [3, Lemma 6].

Proposition 1 *Lattice L is $\mathcal{B}(k+1)$-free if the size of each minimal representation is at most k.*

The general bound which connects the lattice size with its VC dimension is as follows:

Theorem 1 (Vapnik-Chervonekis bound) *For a finite lattice L with $vc(L) \leq k$ and $|J(L)| \leq n$ it holds*

$$|L| \leq f(n, k+1). \tag{1}$$

This bound is sharp for all $n, k \geq 1$.

As was mentioned before, lattices reaching the bound (1) are called $(n, k+1)$-extremal. The following proposition states basic properties of extremal lattices, and describes their construction for several simple cases.

Proposition 2

1. *An $(n, 1)$-extremal lattice is a one-element lattice, for all $n \geq 1$;*
2. *an $(n, 2)$-extremal lattice is a chain of length n;*
3. *for $n \leq k$, an $(n, k+1)$-extremal lattice is $B(n)$;*
4. *for $n, k \geq 1$, an $(n, k+1)$-extremal lattice is a graded lattice of height n with $r(x) = |J(x)|$;*
5. *for $n \geq 1$ and $k \geq 2$, every $(n, k+1)$-extremal lattice is atomistic.*

Note. In Theorem 1, which establishes the upper bound on $|L|$, we demand that $vc(L) \leq k$ and $|J(L)| \leq n$, not $vc(L) = k$ and $|J(L)| = n$. This formulation is rather a technicality, as for extremal lattices these inequalities will always turn out to be equalities, for one exception: as stated, an $(n,1)$-extremal lattice is a one-element lattice, its VC dimension is 0, but it has no join-irreducible elements, that is, $|J(L)| = 0 < n$.

In contrast to the general case, in an extremal lattice each element x has a unique minimal representation, which we denote by $G(x)$. Note that the uniqueness of minimal representations can be considered as an alternative definition of the *meet-distributivity* of a lattice. Thus, stating the uniqueness of minimal representations in extremal lattices is equivalent to stating that all extremal lattices are meet-distributive.

As it turns out, all extremal lattices can be iteratively constructed through procedure called *doubling*. For a poset L and its subposet K, doubling of K in L, denoted $L[K]$, is a poset with elements $L \cup \overset{\bullet}{K}$, and order

$$\leq' = \leq \cup \left\{ (x, \overset{\bullet}{y}) \in L \times \overset{\bullet}{K} \mid x \leq y \right\} \cup \left\{ (\overset{\bullet}{x}, \overset{\bullet}{y}) \in \overset{\bullet}{K} \times \overset{\bullet}{K} \mid x \leq y \right\},$$

where $\overset{\bullet}{K}$ is a disjoint copy of K. Although doublings are defined for arbitrary posets, mostly we will be interested in doublings of lattices.

Proposition 3 *If L and K are lattices and K $(1, \wedge)$-embeds into L then $L[K]$ is a lattice.*

The procedure for construction of arbitrary extremal lattices by doublings is provided by the following lemma:

Lemma 1 *For an $(n, k+1)$-extremal lattice L, $n \geq 1$, $k \geq 2$, and an (n, k)-extremal lattice K, order-embedded into L, $L[K]$ is an $(n+1, k+1)$-extremal lattice.*

In the following two sections, which constitute the core of the paper, we will widely generalize the doubling procedure from Lemma 1 above, arriving at *root decompositions* of extremal lattices.

3. Decompositions of extremal lattices

In this section we show that doubling can be used to deconstruct extremal lattices, as well as to construct them. This and the following sections follow in general Section 3 and Section 4 in [4]. The methods that we develop here are, however, far more refined and close connections can only be made at the beginning.

Lemma 2 *Let L be an $(n, k+1)$-extremal lattice, $n \geq 1$, $k \geq 2$. Then there is a one-to-one correspondence between the coatoms of L and the elements of $G(1_L)$, established by:*

$$c \in Co(L) \not\geq a \in G(1_L). \tag{2}$$

Moreover, given such c and a, $K' = [a)$ is an $(n-1, k)$-extremal lattice, $L' = (c] = L - [a)$ is $(n-1, k+1)$-extremal, and the mapping $\delta \colon K' \to L'$,

$$\delta(x) = \bigvee \big(J(x) - \{a\}\big), \tag{3}$$

defines a $(1, \wedge)$*-embedding of* K' *into* L'*. Thus,* $L \cong L'[K']$.

Note. Sometimes, when referring to δ, we write δ_a in order to explicitly identify the element a used in (3).

Proof. We will only establish the correspondence between coatoms and elements of $G(1_L)$, the rest follows verbally [2, Theorem 4].

Let us take $a \in G(1_L)$, $X = \big(J(L) - \{a\}\big)$, and $c = \bigvee X$. As $G(1_L)$ is the only minimal decomposition of 1_L, and as $X \not\supseteq G(1_L)$ (because $a \notin X$), we get $x < 1_G$. It is also clear that x is covered by 1_L, because there is exactly one element (a) in $J(1_L) - J(x)$. Thus, c is a coatom, and it is, trivially, the only coatom not above a. Thus, (2) establishes an injection of $G(1_L)$ into $Co(L)$.

For the bijection, let us note that for any coatom d there is $a \in G(1_L) - J(d)$. Now, if we construct $c = c(a)$ using the procedure above, then c is a coatom, and $d \leq c$, implying $d = c$. ■

We write c_a, or, in a functional form, $c(a)$, to denote the unique coatom of L satisfying (2), for $a \in G(1_L)$. An easy corollary from [4, Proposition 2.3] is the following representation for c_a.

Proposition 4 *In the notation of Lemma 2,* $c_a = \bigvee\big(J(L) - \{a\}\big)$.

For an $(n, k+1)$-extremal lattice L and an element $a \in G(1_L)$, let us denote by L_a a lattice $L - [a) = (c_a]$, and by L^a a $(1, \wedge)$-embedding of semi-interval $[a)$ into L_a by δ from (3). We also use notation $\dot{L}_a$ and $\dot{L}^a$ to denote semi-intervals $(c_a]$ and $[a)$ correspondingly, the notation paralleled with that in the doubling construction. Notice that $L \cong L_a[L^a]$ and $L = L_a \sqcup \dot{L}^a = \dot{L}_a \sqcup \dot{L}^a$. As $L_a = \dot{L}_a$, introducing the latter may seem excessive. We, however, will find it useful further on, when we will be constructing families L_A^B and $\dot{L}_A^B$, for which the mentioned lattices would serve as building blocks. In general, different elements of $G(1_L)$ yield nonequivalent decompositions of L in a sense that $L_a \ncong L_b$ for different $a, b \in G(1_L)$.

Now we are going to prepare a method for stacking decompositions. First of all, we examine how join-irreducible elements behave under decomposition.

Proposition 5 *For an* $(n, k+1)$*-extremal lattice* L *and an element* $a \in G(1_L)$*, holds:*

1. *if* $k \geq 3$ *then* $J(L_a) = J(L^a) = J(L) - \{a\}$*;*

2. *if* $k = 2$ *then* $J(L_a) = J(L) - \{a\}$ *and* $J(L^a)$ *is an* n*-element chain,* $J(L^a) = L^a - \{0_{L^a}\}$*, and there is a natural correspondence between* $J(L_a)$ *and* $J(L^a)$*, established as follows:*

   - *for* $x \in J(L_a)$ *we define* $x' \in J(L^a)$ *as* $\delta(x \vee a)$*,*
   - *for* $y \in J(L^a)$ *we define* $y' \in J(L_a)$ *as a unique* y' *for which* $\delta(y' \vee a) = y$*;*

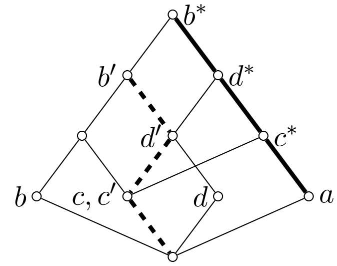

Fig. 1: Correspondence between join-irreducible elements of an $(n, 2)$-extremal lattice embedded into an $(n, 3)$-extremal lattice.

3. *if $k = 1$ then L and L_a are $n+1$ and n-element chains correspondingly, $a = 1_L$, $J(L_a) = J(L) - \{a\}$, and L^a is a one-element lattice, $L^a = 1_{L_a}$, $J(L^a) = \emptyset$.*

Proof. All statements, except for the explicit correspondence in (2), follow from the fact that L_a and L^a are $(n-1, k)$ and $(n-1, k-1)$-extremal correspondingly, and from structural properties of extremal lattices given in Proposition 2. In particular, atomicity implies that L_a and L^a not only have the same number of join-irreducible elements, but that these elements are exactly atoms, and thus coincide.

So we only need to prove the explicit correspondence between join-irreducible elements in an $(n-1, 3)$-extremal lattice L_a and an $(n-1, 2)$-extremal lattice L^a. First of all, as L^a is a chain, $J(L^a) = L^a - \{0_{L^a}\}$. Now, as L is $(n, 3)$-extremal, each subset of $J(L)$ of size at most 2 is a unique minimal representation for some $x \in L$. Thus, the set $A = \{x \vee a \mid x \in J(L)\}$ contains exactly n elements, all lying above a. But, as $[a) = \overset{\bullet}{L}{}^a$, we get $|[a)| = |A| = n$. Thus, the mapping $x \mapsto x^* = x \vee a$ establishes a one-to-one correspondence between $J(L)$ and $\overset{\bullet}{L}{}^a$. Moreover, as $a^* = a \vee a = a = 0_{\overset{\bullet}{L}{}^a}$, this is also a correspondence between $J(L_a) = J(L) - \{a\}$ and $J(\overset{\bullet}{L}{}^a)$. The application of δ to the right-hand side establishes the desired correspondence.

See Figure 1 for the illustration of the argument. ■

From Proposition 5 also easily follows the correspondence between $J(L^a)$ and $J(\overset{\bullet}{L}{}^a)$:

Corollary 1 *In terms of Proposition 5,*

$$J(\overset{\bullet}{L}_a) = J(L_a);$$
$$J(\overset{\bullet}{L}{}^a) = a \vee J(L^a) = a \vee J(L_a),$$

where $a \vee J = \{a \vee j \mid j \in J\}$.

Proposition 6 *For an $(n, k+1)$-extremal lattice L and an element $a \in G(1_L)$, holds:*

1. $G(1_{L^a}) = \begin{cases} G(1_L) - \{a\}, & k \geq 3, \\ \big(G(1_L) - \{a\}\big)', & k = 2, \\ \emptyset, & k = 1; \end{cases}$

2. $G(1_{L_a}) = \begin{cases} G(1_L) - \{a\}, & n \leq k, \\ G(1_L) - \{a\} + \{b\} \text{ for some } b \in J(L) - G(1_L), & n > k. \end{cases}$

Proof.

Item (2) is exactly [4, Lemma 3.4], and cases $k = 1, 2$ in (1) are trivial, so we need only to prove (1) for $k \geq 3$. Let us observe that the lattice L^a is an $(n-1, k)$-extremal with $J(L) = J(L^a)$ and $|G(1_{L^a})| = k - 1 = |G(1_L) - \{a\}|$. On the other hand, notice that

$$\bigvee_L G(1_L) = a \vee \bigvee_L \big(G(1_L) - \{a\}\big) = \bigvee_L \big\{x \vee a \mid x \in G(1_L) - \{a\}\big\}$$
$$= \bigvee_{L^a} \big(G(1_L) - \{a\}\big).$$

Thus, $G(1_L) - \{a\}$ is a minimal representation of 1_{L^a} in L^a. ■

With Proposition 6 we now can take two (or more) elements $a, b \in G(1_L)$ and construct lattices $L_{ab} = \big(L_a\big)_b$ and $L_{ba} = \big(L_b\big)_a$. Fortunately, these lattices are equal, as we will soon prove. First of all, however, we need to introduce some intermediary terminology, which is a technical, but necessary step.

Note. The terminology developed below, until Lemma 3, will only be used in formulation and the proof of the lemma, which then would enable us to drop it and introduce a more concise formulations.

For an $(n, k+1)$-extremal lattice L, let A and B be disjoint, and possibly empty, subsets of $G(1_L)$, $|A| + |B| = p \leq k$. Let $X = x_1, \ldots, x_p$ be an enumeration of $A \bigsqcup B$. We denote by $L_X^{A,B}$ the lattice, embedded into L and obtained as the result of the following process: $L_0 = L$, $L_{i+1} = \big(L_i\big)_{x_i}$ if $x_i \in A$ and $L_{i+1} = \big(L_i\big)^{x_i}$ if $x_i \in B$, $L_X^{A,B} = L_n$. We write simply L_X, if A and B are clear from the context. When X is an enumeration of a set $\{a, b\}$ with only two elements, we use instead a simplified notation L_{ab}, ${L_a}^b$, ${L^b}_a$ and L^{ab} to denote four possible ways of decomposition.

Similarly, we define $\overset{\bullet}{L}_X$ by putting $L_{i+1} = (\overset{\bullet}{L}_i)^{x_i}$ instead of $\big(L_i\big)^{x_i}$, and $L_{i+1} = (\overset{\bullet}{L}_i)_{x_i}$ instead of $\big(L_i\big)_{x_i}$, in the iterative definition above. Note that $\overset{\bullet}{L}{}_X^{A,B}$ is embedded into L itself, for all A, B and X.

Proposition 7 *For an $(n, k+1)$-extremal lattice L and $a, b \in G(1_L)$, holds:*

$$c_{L_a}(b) = c_{L^a}(b) = c_a \wedge c_b,$$
$$c_{\overset{\bullet}{L}_a}(b) = c_a \wedge c_b,$$
$$c_{\overset{\bullet}{L}{}^a}(b^*) = c_b,$$

where $b^* = b \vee a \in J(\dot{L}^a)$.

Proof. By Proposition 4, $c_a = \bigvee\big(J(L) - \{a\}\big)$, thus

$$\begin{aligned} c_{L_a}(b) = c_{(c_a]}(b) &= \bigvee\big(J(c_a) - \{b\}\big) = \bigvee\big(J(L) - \{a, b\}\big) \\ &= \bigvee\big(J(c_a) \cap J(c_b)\big) = c_a \wedge c_b = c_{L_b}(a). \end{aligned}$$

As $L_a = \dot{L}_a$, the second equation is obvious, for the third one we use the representation $J(\dot{L}^a) = a \vee J(L_a)$ from Corollary 1, to get:

$$\begin{aligned} c_{\dot{L}_a}(b^*) &= \bigvee\big(J(\dot{L}_a) - \{b^*\}\big) = \bigvee\big(a \vee J(L_a) - \{a \vee b\}\big) \\ &= a \vee \bigvee\big(J(L_a) - \{b\}\big) = a \vee \bigvee\big(J(L) - \{a, b\}\big) \\ &= \bigvee\big(J(L) - \{b\}\big) = c_b. \end{aligned}$$

■

Lemma 3 $L_X^{A,B}$ *is independent of enumeration* X*, for an* $(n, k+1)$*-extremal lattice* L *and disjoint* $A, B \subseteq G(1_L)$*. That is,* $L_X = L_Y$*, for all enumerations* X, Y *of* $A \sqcup B$*.*

Thus, L_X *depends only on* A *and* B*, and we denote it by* L_A^B*. Moreover,* L_A^B *is an* $(n - |A| - |B|, k + 1 - |B|)$*-extremal lattice.*

Proof. Trivially, all we have to do is to prove three cases, namely that $L_{ab} = L_{ba}$, ${L_a}^b = {L^b}_a$ and $L^{ab} = L^{ba}$, for an $(n, k+1)$-extremal L with $n \geq 2$ and $k \geq 1$, 2 and 3 correspondingly. Figure 2 below depicts this equivalence.

The proof itself, however, is a straightforward application of Proposition 7:

1. $L_{ab} = (c_{L_a} b] = (c_a \wedge c_b] = L_{ba}$;

2.
$$\begin{aligned} {L_a}^b = \delta_b[b)_{L_a} &= \delta_b[b, c_a] = \{\delta_b(x) \mid b \leq x, x \leq c_a\} \\ &= \{\delta_b(x) \mid b \leq x, \delta_b(x) \leq \delta_b(c_a)\} \\ &= \delta_b[b) \cap (\delta_b(c_a)] = \delta_b[b) \cap (c_a \wedge c_b] = {L^b}_a; \end{aligned}$$

3.
$$\begin{aligned} L^{ab} &= \delta_b[b)_{L^a} = \delta_b\big([b) \cap \delta_a[a)\big) \\ &= \delta_b \circ \delta_a\big([a \vee b)\big) = \delta_a \circ \delta_b\big([a \vee b)\big) = L^{ba}. \end{aligned}$$

■

Lemma 3 then easily extends to $\dot{L}_A^B$.

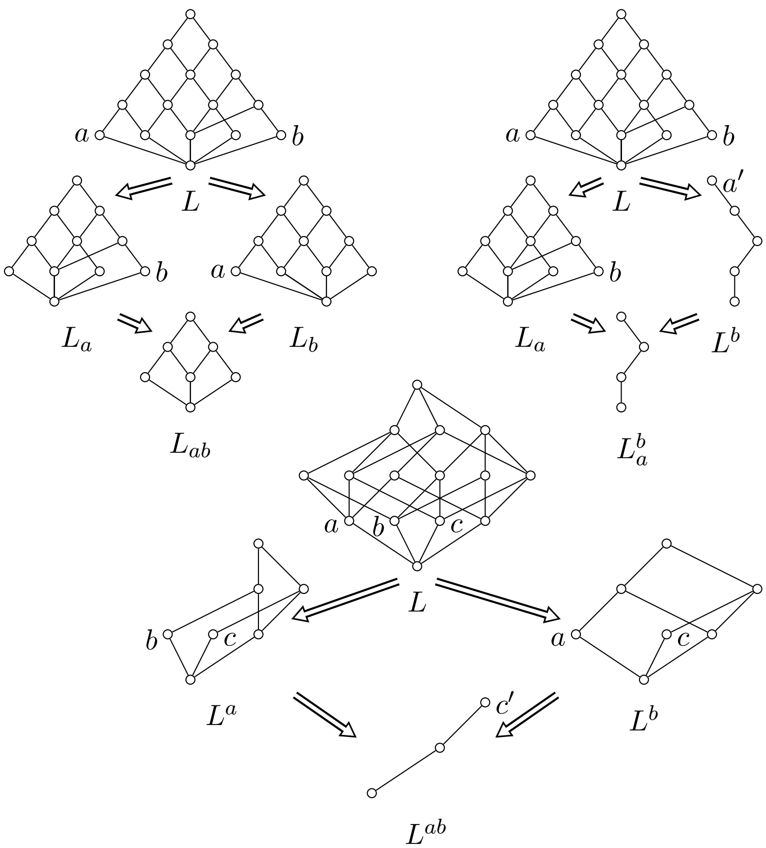

Fig. 2: Equivalence of enumerations.

Proposition 8 *In terms of Lemma 3,* $\overset{\bullet}{L}{}_X^{A,B}$ *is independent of enumeration, thus justifying the notation* $\overset{\bullet}{L}{}_A^B$*. Moreover,*

$$\overset{\bullet}{L}{}_A^B = \left[\bigvee B, \bigwedge c[A]\right].$$

Mapping $\delta_B\colon \left[\bigvee B\right) \to \left(\bigwedge c[B]\right]$*, defined as*

$$\delta_B(x) = \bigvee\bigl(J(x) - B\bigr),$$

establishes an isomorphism between $\overset{\bullet}{L}{}_A^B$ *and* L_A^B*, and*

$$\delta_B^{-1}(x) = \bigvee\bigl(J(x) + B\bigr).$$

Proof. The independence proof follows the one from Lemma 3, and is only simplified by the fact we do not use δ. Namely, we get $\overset{\bullet}{L}_{ab} = (c_a \wedge c_b] = \overset{\bullet}{L}_{ba}$, $\overset{\bullet}{L}_a{}^b = [b, c_a] = \overset{\bullet}{L}{}^b{}_a$ and $\overset{\bullet}{L}_{ab} = [a \vee b) = \overset{\bullet}{L}_{ba}$. Same argument also yields two other statements of the proposition. ∎

Corollary 2 *In terms of Lemma 3,* L_A^B $(1,\wedge)$*-embeds into* $L_{A\sqcup B}$*. Moreover, for any disjoint* A' *and* B'*, such that* $A' \sqcup B' = A \sqcup B$*, a lattice* L_A^B $(1,\wedge)$*-embeds into* $L_{A'}^{B'}$*, whenever* $A \subseteq A'$*.*

Note. At this point, as mentioned above, we no longer need any notation involving enumerations, like $L_X^{A,B}$. Further on we will write simply L_A^B.

4. Root decompositions

Notions of *root* and *root decomposition* were introduced in [4] in order to count isomorphism classes of $(n,3)$-extremal lattices. There it was shown that for $k \leq 3$, but not for larger ones, isomorphism of decompositions is equivalent to isomorphism of lattices. Here we generalize root decompositions in order to obtain similar equivalence for larger k, thus, our definition of root decomposition will be different. The definition of root, however, stays the same.

Definition 1 (Extremal decomposition) *An* $(n,k{+}1)$-extremal decomposition *is a family* $\{L_X^*\}$ *of extremal lattices, parametrized by a set* $X \subseteq \mathbf{k}$*, together with a family* $\{\phi_{X,Y}\colon L_X^* \to L_Y^*\}_{Y\subseteq X\subseteq \mathbf{k}}$ *of embeddings, such that:*

1. L_X^* *is* $(n, k - |X| + 1)$*-extremal, for all* X*;*
2. $\phi_{X,Y}$ *is a* $(1,\wedge)$*-embedding of* L_X^* *into* L_Y^**, for all* $Y \subseteq X$*;*
3. $\phi_{X,X} = id$ *and* $\phi_{Y,Z} \circ \phi_{X,Y} = \phi_{X,Z}$*, for all* $Z \subseteq Y \subseteq Z \subseteq \mathbf{k}$*; that is, all embeddings are compatible.*

We typically denote an $(n, k+1)$-extremal decomposition by $\mathcal{L} = (L^*_X, \phi_{X,Y})$ and often omit the word extremal, whenever the context is clear. The lattice $L^*_\emptyset$, or simply L^*, is called a *root* of the decomposition, all other lattices L^*_X are embedded into it. With abuse of notation we denote embedding of L^*_X into L^* as ϕ_X. We will write simply ϕ if the image and the domain are clear from the context. At times, instead of $\mathbf{k}$ we use custom fixed set of size k, which we call a *base set* of $\mathcal{L}$.

Note. For most embeddings in an $(n, k+1)$-decomposition, the unit preservation is automatically satisfied, as, by Proposition 2, for $k \geq 1$ all $(n, k+1)$-extremal lattices are graded of height n. The only place where it matters is when we considering embeddings of $L^*_{\mathbf{k}}$, as it is a one-element $(n, 1)$-extremal lattice, and the condition ensures that this element is always mapped to the unit element.

Our goal is to show that every $(n+k, k+1)$-extremal lattice can be in a unique way put to correspondence with an $(n, k+1)$-decomposition. The road for such correspondence is already paved by Lemma 3.

Definition 2 (Root decomposition) *For an $(n+k, k+1)$-extremal lattice L we define its* root decomposition $\mathcal{L}(L)$ *as an $(n, k+1)$-extremal decomposition $\mathcal{L}(L) = (L^*_X, \phi_{X,Y})$, where $G = G(1_L)$, $L^*_X = L^X_{G-X}$, and $\phi_{X,Y} \colon L^*_Y \to L^*_X$ is a natural embedding of $L^*_Y = L^Y_{G-Y}$ into $L^*_X = L^X_{G-X}$, for $X \subseteq Y$.*

It follows from Corollary 2 that all ϕ are $(1, \wedge)$-embeddings and the compatibility is straightforward. We also define a *root element* of an extremal lattice L as $x^* = \bigwedge G(1_L)$ and a *root* L^* of L as $L^* = [0, x^*]_L$. Note, that L^* will also be the root of $\mathcal{L}(L)$, and further we will not distinguish these definitions.

For further justification of putting $\mathcal{L}(L)$ to correspondence with L, we make the following digression.

Proposition 9 *For an extremal lattice L it holds*

$$L = \bigsqcup_{X \subseteq G} \dot{L}^X_{G-X},$$

where $G = G(1)$.

Proof. We recall from Proposition 8 that $\dot{L}^X_{G-X} = \left[\bigvee X, \bigwedge c[G-X]\right]$. For $x \in L$, let us introduce $H(x) = G \cap J(x)$ (we recall that the suggestive notation $G(x)$ is already used to denote the minimal representation of x). Then $H(x) \subseteq G$ and $x \geq \bigvee H(x)$. Moreover, Lemma 2 implies that $j \leq x \Leftrightarrow x \not\leq c_j$, for $h \in J(L)$. Thus, $x \leq c_h$ for all $h \in G - H(x)$, and consequently $x \leq \bigwedge c[G - H(x)]$, that is, $x \in \dot{L}^{H(x)}_{G-H(x)}$.

On the other hand, if $x \in \dot{L}^H_{G-H}$, for some $H \subseteq G$, then $x \geq h$ for all $h \in H$, and $x \leq c_j \Leftrightarrow j \not\leq x$ for all $j \in G - H$, from which it follows that $H = H(x)$.

All in all, the family $\dot{L}^X_{G-X}$ is nonintersecting and covers entire L, so the statement of the proposition follows. ■

While root decomposition gives us a transition from extremal lattices to extremal decompositions, the following definition enables us to pass from decompositions back to lattices.

Definition 3 (Canonical lattice) Canonical lattice $L(\mathcal{L})$ *of an* $(n,k+1)$*-decomposition* $\mathcal{L}$ *is a poset*

$$\{(X,x) \mid X \subseteq \mathbf{k}, x \in L_X\},$$

with an order defined by

$$(X,x) \leq (Y,y) \Leftrightarrow X \subseteq Y \text{ and } x \leq \phi y.$$

The fact that $L(\mathcal{L})$ is a lattice, let alone an extremal one, is not that trivial and we pose it as a separate lemma.

Lemma 4 *The canonical lattice of an* $(n,k+1)$*-extremal decomposition is* $(n+k,k+1)$*-extremal.*

Proof. For $k=1$ the statement is trivial, so we consider $k>1$, fix an $(n,k+1)$-extremal decomposition $\mathcal{L}=(L^*_X,\phi_{X,Y})$, and denote its canonical lattice by L. The lattice $L^*_{\mathbf{k}}$ is $(n,1)$-extremal and thus, by Proposition 2, is a one-element lattice. We denote this element by u and note that $\phi_{\mathbf{k},X}$ always maps u to $1_{L^*_X}$. Thus, $(\mathbf{k},u)$ is the largest element, that is, a unit, of L.

Now, for $x \in L^*_X$ and $y \in L^*_Y$, let us consider the element $z=(X\cap Y, \phi(x)\wedge\phi(y))$, which is trivially a lower bound of (X,x) and (Y,y). On the other hand, if some (W,w) is another lower bound, then $W \subseteq X,Y$ and thus $W \subseteq X\cap Y$. Finally, $w \leq \phi_{X,W}(x)$ and $w \leq \phi_{Y,W}(y)$. Thus

$$\begin{aligned} w \leq \phi_{X,W}x \wedge \phi_{Y,W}(y) &= \phi_{X\cap Y,W}\circ\phi_{X,X\cap Y}(x)\wedge\phi_{X\cap Y,W}\circ\phi_{Y,X\cap Y}(y) \\ &= \phi_{X\cap Y,W}\circ\left(\phi_{X,X\cap Y}(x)\wedge\phi_{Y,X\cap Y}(y)\right), \end{aligned}$$

and $(W,w)\leq z$, that is, z is a meet of (X,x) and (Y,y). All in all, L is a $(1,\wedge)$-semilattice and, consequently, a lattice. Still, let us describe join in L explicitly. We claim that

$$\bigvee_i (A_i,a_i) = (A,a),$$

where $A=\bigcup_i A_i$, $a=\bigvee_i \psi_{A_i,A}(a_i)$ and $\psi_{X,Y}(x)=\bigwedge\{y\in Y \mid x\leq\phi_{Y,X}(y)\}$ for $X\subseteq Y$. As it is with ϕ, we write simply ψ when the image and the domain are clear. Trivially, $\psi_{X,Y}\circ\phi_{Y,X}(x)=x$, for all $X\subseteq Y$.

The proof of explicit construction of joins is almost immediate. Indeed, $(A_i,a_i)\leq(A,a)$ for all i. Now, if we take (W,w) such that $(A_i,a_i)\leq(W,w)$ for all i, then $A=\bigcup_i A_i\subseteq W$ and $a_i\leq\phi_{W,A_i}(w)$, for all i. But then

$$\begin{aligned} \psi_{A_i,A}(a_i) &\leq \psi_{A_i,A}\circ\phi_{W,A_i}(w) = \psi_{A_i,A}\circ\phi_{A,A_i}\circ\phi_{W,A}(w) \\ &= \phi_{W,A}(w), \end{aligned}$$

and $a = \bigvee_i \psi a_i \leq \phi w$, implying $(A, a) \leq (W, w)$.

Now we consider join-irreducible elements of L. We claim that there are exactly two kinds of them: n elements $(\emptyset, j)$ for $j \in J(L^*)$, and k elements $(\{i\}, 0)$ for $i \in \mathbf{k}$. Indeed, all these $n + k$ elements cover $(\emptyset, 0)$, that is, the zero of L, and so they are trivially join irreducible. Let us show that no other join irreducible element exists.

First of all, for $X \subseteq \mathbf{k}$, if $|X| \geq 2$ then $(X, x) = (X - a, \phi x) \vee (X - b, \phi x)$, where a and b are any two distinct elements of X, and thus (X, x) is not join irreducible. If $|X| = 1$ and $x > 0$ then $(X, x) = (X, 0) \vee (\emptyset, \phi x)$, and (X, x) is again not join irreducible. Finally, if $x \in L^*$ and $x = y \vee z$ is a proper join decomposition of x then $(\emptyset, x) = (\emptyset, y) \vee (\emptyset, z)$ is a proper join decomposition of $(\emptyset, x)$, which finishes our claim about the structure of $J(L)$.

Simple manipulation with binomial coefficients show that L has

$$\sum_{X \subseteq \mathbf{k}, i \leq k - |X|} \binom{n}{i} = \sum_{l \leq k} \binom{k-l}{l} \sum_{i \leq k-l} \binom{n}{i} = \sum_{l \leq k} \binom{k}{l} \sum_{i \leq l} \binom{n}{i}$$
$$= \sum_{l \leq k} \sum_{i \leq l} \binom{k}{l} \binom{n}{i} = \sum_{j \leq n+k} \binom{n+k}{j}$$

elements, so in order to finish the proof we only need to show that L is $\mathcal{B}(k+1)$-free.

To show this, we employ Proposition 1 and argue that each minimal join representation has at most k elements. Let us fix an element (X, x) of L, and let $H \subseteq J(L)$ be a minimal representation of (X, x). Again, we may take $X \subsetneq \mathbf{k}$, for otherwise the statement holds trivially, and denote $l = |X|$. Recalling the structure of $J(L)$, we split H into $H' = H \cap \{(\emptyset, j) \mid J \in J(L^*)\}$ and $H'' = H \cap \{(\{i\}, 0) \mid i \in \mathbf{k}\}$. Then

$$\begin{aligned}
(X, x) &= \bigvee H' \vee \bigvee H'' \\
&= \bigvee \{(\emptyset, j) \mid (\emptyset, j) \ \in H'\} \vee \bigvee \{(\{i\}, 0) \mid (\{i\}, 0) \in H''\} \\
&= \Big(\emptyset, \bigvee \{j, \mid (\emptyset, j) \ \in H'\}\Big) \vee \Big(\{i \mid (\{i\}, 0) \in H''\}, 0\Big) \\
&= (Y, y)\,,
\end{aligned}$$

where $Y = \{i \mid (\{i\}, 0) \in H''\}$ and $y = \bigvee \{\psi_{\emptyset, Y}(j) \mid (\emptyset, j) \ \in H'\}$. We may conclude that $X = Y$ and $x = y$. From the first equation we get $X = \{i \mid (\{i\}, 0) \in H''\}$ and thus $H'' = \{(\{i\}, 0) \mid i \in X\}$. In particular, $|H''| = |X| = l$. Now, let us notice that for $j \in J(L^*)$, $\psi_{\emptyset, X}(j)$ lies in $J(L^*_X)$: if $|X| \leq k - 2$ then $J(L^*_X) = \psi\,[J(L_*)]$, and if $|X| = k - 1$ then L^*_X is a chain and all its nonzero elements are join irreducible. As H is minimal, then the representation $y = \bigvee \{\psi_{\emptyset, X}(j) \mid (\emptyset, j) \ \in H'\}$ is also minimal, for otherwise we could exclude some elements from H'' without changing the join of H. However, this representation is in L^*_X, which is $(n, k - l + 1)$-extremal. Thus, $|H''| \leq k - l$, and $|H| = |H'| + |H''| \leq l + k - l = k$. ■

Corollary 3 *For an $(n, k+1)$-extremal decomposition $\mathcal{L}$ there are $n+k$ elements in $J = J(L(\mathcal{L}))$, which have form $J = J' \sqcup J''$, where*

$$J' = \{(\emptyset, j) \mid J \in J(L^*)\},$$
$$J'' = \{(\{i\}, 0) \mid i \in \mathbf{k}\},$$

$|J'| = n$ and $|J''| = k$.

Corollary 4 *For an $(n, k+1)$-extremal decomposition $\mathcal{L}$, joins and meets in $L(\mathcal{L})$ are defined as*

$$\bigvee_i (A_i, a_i) = (A, a)\,,$$

where $A = \bigcup_i A_i$, $a = \bigvee_i \psi_{A_i,A}(a_i)$, $\psi_{X,Y}(x) = \bigwedge\{y \in Y \mid x \leq \phi_{Y,X}(y)\}$, for $X \subseteq Y$. And

$$\bigwedge_i (B_i, b_i) = (B, b)\,,$$

where $B = \bigcap_i B_i$ and $b = \bigwedge_i \phi_{B_i,B}(b_i)$.

To establish a correspondence between extremal lattices and decompositions we now clarify which decompositions we consider isomorphic.

Definition 4 Isomorphism of $(n, k+1)$ decompositions $\mathcal{L} = (L^*_X, \phi_{X,Y})$ and $\mathcal{K} = (K^*_X, \varphi_{X,Y})$ *is a pair (σ, ε) where σ is a permutation of $\mathbf{k}$, and $\varepsilon\colon K^* \to L^*$ is an isomorphism from K to L, such that $\phi^{-1}_{\sigma(X)} \circ \varepsilon \circ \varphi_X$ is an isomorphism of K^*_X into L^*_X, for all $X \subseteq \mathbf{k}$. Decomposition $\mathcal{K}$ is* isomorphic *to $\mathcal{L}$ if there is an isomorphism between them.*

It is trivial to check that, thus defined, isomorphism is an equivalence relation, and that canonical lattices and root decompositions are preserved under isomorphisms.

Proposition 10 *For $(n+k, k+1)$-extremal lattices L and L', and $(n, k+1)$-extremal decompositions $\mathcal{L}$ and $\mathcal{L}'$ holds:*

- *$\mathcal{L}(L) \cong \mathcal{L}(L')$, whenever $L \cong L'$;*
- *$L(\mathcal{L}) \cong L(\mathcal{L}')$, whenever $\mathcal{L} \cong \mathcal{L}'$.*

Finally, the following Lemma shows that the operations of constructing canonical lattice and root decomposition are inverse up to isomorphism, which establishes the correspondence between extremal lattices and extremal decompositions.

Lemma 5 *For an $(n+k, k+1)$-extremal lattice L and an $(n, k+1)$-extremal decomposition $\mathcal{L}$ holds:*

$$L(\mathcal{L}(L)) \cong L,$$
$$\mathcal{L}(L(\mathcal{L})) \cong \mathcal{L}.$$

Proof. $L(\mathcal{L}(L)) \cong L$. Let us denote $G = G(1_L)$ and

$$L' = L(\mathcal{L}(L)) = \left\{(X, x) \mid X \subseteq G, x \in L^X_{G-X}\right\},$$

and let us recall that by Proposition 9

$$L = \bigsqcup_{X \subseteq G} \dot{L}^X_{G-X},$$

and that by Proposition 8, the mapping $\delta_X \colon \dot{L}^X_{G-X} \to L^X_{G-X}$,

$$\delta_X(x) = \bigvee\nolimits_{L^*}\big(J_L(x) - X\big),$$
$$\delta_X^{-1}(x) = \bigvee\nolimits_{L}\big(J_{L^*}(x) \sqcup X\big),$$

is an isomorphism between $\dot{L}^X_{G-X}$ and L^X_{G-X}.

Thus, the mapping $\alpha\colon L' \to L$, defined by $\alpha(X, x) = \delta_X^{-1}(x)$, provides a bijection between L' and L with $\alpha^{-1}(x) = (H(x), \delta_{H(x)}(x))$, where $H(x) = J(x) \cap G$. Now we recall, that $\phi_{X,Y}$ for $\mathcal{L}(L)$ is provided by the natural embedding of L^Y_{G-Y} into L^X_{G-X}, which means that $(X, x) \le (Y, y)$ if and only if $X \subseteq Y$ and $x \le_{L^*} y$.

Note that for $x \in \dot{L}^X_{G-X}$ holds $J(\delta_X(x)) = J(x) - X$, and thus for $x \in L^X_{G-X}$ holds $J_L(\delta_X^{-1}(x)) = J_{L^*}(x) \sqcup X$. Thus

$$\begin{aligned}(X, x) \le (Y, y) &\Leftrightarrow X \subseteq Y, x \le_{L^*} y\\ &\Leftrightarrow X \subseteq Y, J_{L^*}(x) \subseteq J_{L^*}(y)\\ &\Leftrightarrow X \sqcup J_{L^*}(x) \subseteq Y \sqcup J_{L^*}(y)\\ &\Leftrightarrow J\big(\delta_X^{-1}(x)\big) \subseteq J\big(\delta_Y^{-1}(y)\big)\\ &\Leftrightarrow \alpha(X, x) \le_L \alpha(Y, y),\end{aligned}$$

and the isomorphism of L and L' follows.

$\mathcal{L}(L(\mathcal{L})) \cong \mathcal{L}$. Let $\mathcal{L} = (L_X, \phi_{X,Y})$, $L = L(\mathcal{L})$ and $\mathcal{L}' = \mathcal{L}(L) = (L'_X, \phi'_{X,Y})$. By Corollary 3, L is $(n+k, k+1)$-extremal and $J(L) = J' \sqcup J''$, where

$$J' = \{(\emptyset, j) \mid J \in J(L^*)\},$$
$$J'' = \{(\{i\}, 0) \mid i \in \mathbf{k}\}.$$

Trivially, $\bigvee J'' = (X, 1) = 1_L$. Thus, $G = G(1_L) = J''$, and

$$\begin{aligned} L'^* = L'_G &= \Big[0_L, \bigvee J(L) - G(L)\Big]_L \\ &= \Big[0_L, \bigvee J'\Big]_L = \Big[(\emptyset, 0_{L^*}), \Big(\emptyset, \bigvee J(L^*)\Big)\Big]_L \\ &= [(\emptyset, 0_{L^*}), (\emptyset, 1_{L^*})]_L \cong L^*. \end{aligned}$$

We define $\sigma\colon \mathbf{k} \to J''$ and $\varepsilon\colon L^* \to L'^*$ as $\sigma(i) = (\{i\}, 0)$ and $\varepsilon(x) = (\emptyset, x)$. We claim that (σ, ε) is an isomorphism of $\mathcal{L}$ and $\mathcal{L}'$. Indeed, let $X' \subseteq J'' = \big\{(\{i\}, 0) \mid i \in X\big\} = \sigma[X]$ for $X \subseteq \mathbf{k}$. Then

$$\begin{aligned} L'^*_{X'} = L'^{X'}_{G-X'} &= \Big[\bigvee X', \bigwedge c[G - X']\Big]_L \\ &= \Big[\bigvee X', \bigvee J - (G - X')\Big]_L = \Big[\bigvee X', \bigvee J' \vee \bigvee X'\Big]_L \\ &= [(X, 0_{L^*}), (\emptyset, 1_{L^*}) \vee (X, 0_{L^*})]_L \\ &= [(X, 0_{L^*}), (X, 1_{L^*})]_L \cong L^*_X, \end{aligned}$$

and the isomorphism between L^*_X and $L'^*_{X'}$ is established by the mapping $\alpha\colon L^*_X \to L'^*_{X'}$, $\alpha(x) = (X', x) = \phi^{-1}_{X'}\big(0, \phi_X(x)\big) = \phi^{-1}_{\sigma[X]} \circ \varepsilon \circ \phi(x)$. ∎

As an easy consequence, we now obtain the most important structural result of the paper, which establishes a correspondence between extremal lattices and decompositions.

Theorem 2 *For an $(n + k, k + 1)$-extremal lattice L and an $(n, k + 1)$-extremal decomposition $\mathcal{L}$, $L \cong L(\mathcal{L})$ if and only if $\mathcal{L} \cong \mathcal{L}(L)$.*

Although technical details in this section were rather involved, the basic fact of the correspondence between extremal lattices and decompositions is quite transparent. Apart from providing structural information about extremal lattices, decompositions can be quite handy in depicting them, as illustrated by Figures 3 and 4.

5. Meet-irreducible elements in extremal decompositions

One possible application of structural insight we gain from extremal decompositions is the estimation of the number of meet-irreducible elements. We start by characterizing meet-irreducible elements of canonical lattices of extremal decompositions. We then apply this characterization to get a simple lower bound on the number of meet-irreducible elements of extremal lattices.

Proposition 11 *For an $(n, k+1)$-extremal decomposition $\mathcal{L}$, its canonical lattice $L = L(\mathcal{L})$, and elements (X, x) and $(Y, y) \in L$, (X, x) is covered by (Y, y) if and only if either $x = y$ and $X \prec Y$, or if $X = Y$ and $x \prec_{L^*_X} y$.*

Proof. It is trivial that under given conditions, $(X, x) \prec (Y, y)$, so we show that these conditions are also necessary. Indeed, if $(X, x) \prec (Y, y)$ then $(X, x) < (Y, y)$,

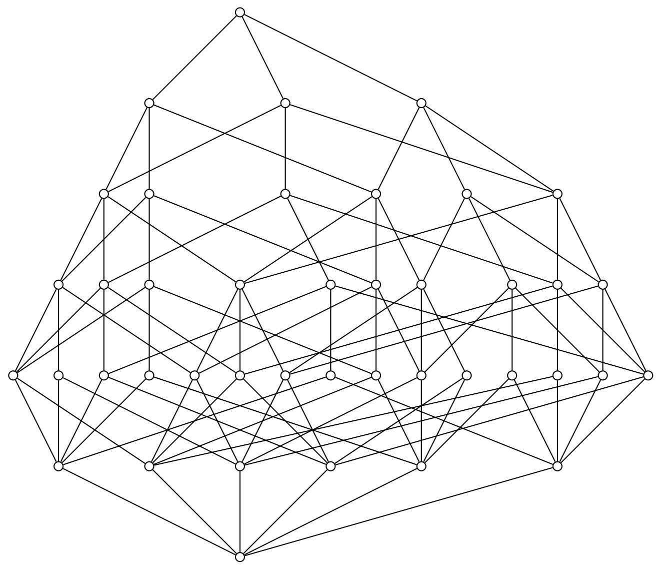

Fig. 3: $(6, 4)$-extremal lattice.

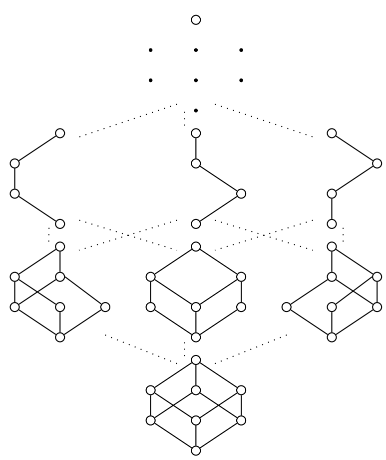

Fig. 4: $(3, 4)$-extremal decomposition of lattice on Figure 3.

meaning that $X \subseteq Y$ and $x \subseteq y$, and that at least one inequality is strict. If $X \not\preccurlyeq Y$ then there is Z such that $X \subsetneq Z \subsetneq Y$ and thus $(X,x) < (Z,x) < (Y,y)$, contradicting the covering. Thus, $X \preccurlyeq Y$, and similarly $x \preccurlyeq y$. Finally, if in both cases there is a proper covering, that is, if $X \prec Y$ and $x \prec y$, then $(X,x) < (X,y) < (Y,y)$, again contradicting the covering. ■

Lemma 6 *For an $(n, k+1)$-extremal decomposition $\mathcal{L}$ and its canonical lattice $L = L(\mathcal{L})$, an element $(X,x) \in L$ is meet-irreducible if and only if one of two mutually exclusive conditions hold:*

- *either x is meet-irreducible in L^*_X and $x \notin \phi_{Y,X}[L^*_Y]$, for all $Y \supsetneq X$;*
- *or x is a unit in L^*_X and $X = \mathbf{k} - a$, for some $a \in \mathbf{k}$.*

Note. By this lemma, all elements of the form $(\mathbf{k} - a, x)$ are meet-irreducible.
Proof. We use the property that an element is meet-irreducible if and only if it is covered by exactly one element. The statement follows from step by step classification of the elements of L:

- $\overline{x} = (\mathbf{k}, 1)$. $\overline{x}$ is a unit in L and thus is not meet-irreducible. Neither it satisfies any of two given conditions;

- $\overline{x} = (\mathbf{k} - a, 1)$*, for some* $a \in \mathbf{k}$. The only element covering $\overline{x}$ is $(\mathbf{k}, 1)$, thus it is meet-irreducible, while it also satisfies the second condition, and does not satisfy the first, because the unit element is not meet-irreducible;

- $\overline{x} = (\mathbf{k} - a, x)$*, for some* $a \in \mathbf{k}$ *and* $x < 1$. As $L^*_{\mathbf{k}-a}$ is a chain, x is meet-irreducible in $L^*_{\mathbf{k}-a}$, and thus $\overline{x}$ satisfies the first condition. The element $\overline{x} = (\mathbf{k} - a, x)$ is covered by $(\mathbf{k} - a, x')$, for a unique $x' \in L^*_{\mathbf{k}-a}$, covering x. In the same time $L^*_{\mathbf{k}}$ contains only unit, and $(\mathbf{k}, x)$ is not a cover of $\overline{x}$. Thus, $\overline{x}$ has a unique cover and thus it is meet-irreducible;

- $\overline{x} = (X, 1)$*, for* $X \subseteq \mathbf{k}$, $|X| \leq k - 2$. $\overline{x}$ does not satisfy neither of two conditions. In the same time for $a, b \in \mathbf{k} - X$, $a \neq b$, elements $(X \sqcup a, 1)$ and $(X \sqcup b, 1)$ cover $\overline{x}$, thus $\overline{x}$ is not meet-irreducible;

- $\overline{x} = (X, x)$*, for* $X \subseteq \mathbf{k}$, $|X| \leq k - 2$, $x < 1$. If x is meet-irreducible in L^*_X and $x \notin \phi_{Y,X}[L^*_Y]$ for all $Y \supsetneq X$, then $\overline{x}$ satisfies the first condition, does not satisfy the second, and (X, x') for a unique cover $x' \in L^*_X$ of x is a unique cover of $\overline{x}$ in L. Otherwise both conditions are not satisfied and there is a proper meet-decomposition of $\overline{x}$. If x is not meet-irreducible in L^*_X then this decomposition is given by $(X,x) = (X,x') \wedge (X,x'')$ for proper meet decomposition $x = x' \wedge x''$. Or, if $x \in L^*_Y$ for some $X \supsetneq Y$ then the decomposition is given by $(X,x) = (X,1) \wedge (Y,x)$.

■

Theorem 3 *Any $(n+k, k+1)$-extremal lattice L has at least $k(n+1)$ meet-irreducible elements, arranged in k disjoint covering chains of length n each. Each of these chains contains exactly one element of rank i, for $i \in k-1, \ldots n+k-1$.*

Any covering chain of meet-irreducible elements in L of length n is one of those chains.

Proof. Trivially, if we denote the initial lattice by L and denote $G = G(1_L)$ and $\mathcal{L} = \mathcal{L}(L)$, then $L \cong L(\mathcal{L})$ and by Lemma 6 set $\{(G-a, x) \mid x \in L^*_{G-a}\} \cong [\bigvee G - a, 1)$ gives such chain.

For the second statement, let us take a covering chain C of meet-irreducible elements in L of length n. Again, from Lemma 6 it easily follows that all these elements should lie in L^*_X for some fixed X. Otherwise there are elements $\overline{x} = (X, x)$ and $\overline{y} = (Y, y)$ in C such that $\overline{x} \prec \overline{y}$ and $X \subsetneq Y$, which is impossible. Now, let us note that height of L^*_X is n, and thus in order to fit in such chain, the unit of L^*_X should also be meet-irreducible, which is only the case for $X = G - a$. ■

We call meet-irreducible elements from Theorem 3 *canonical meet-irreducible elements*, and corresponding chains *canonical chains*. Note also that in an $(n+k, k+1)$-extremal lattice there are at least k meet-irreducible elements of rank $k-1$. However, all elements of rank lower than $k-1$ are situated trivially, as was shown in Lemma 3.2 in [4], which we repeat below as a Proposition 12. Its easy corollary is that there are no meet-irreducible elements of smaller rank.

Proposition 12 *For an element x of an $(n, k+1)$-extremal lattice, $G(x) = J(x)$, whenever $G(x) \leq k-1$.*

Corollary 5 *The smallest rank of a meet-irreducible element in an $(n+k, k+1)$-extremal lattice is $k-1$.*

Proof. Theorem 3 states that there are k meet-irreducible elements of rank $k-1$. On the other hand, let us take x such that $r(x) = |J(x)| \leq k-2$, and let us take two distinct join-irreducible elements $a, b \in J(L) - J(X)$. Then there are two elements x_a and x_b such that $G(x_a) = J(x) + a$ and $G(x_b) = J(x) + b$. By Proposition 12, $J(x_a) = G(x_a) = J(x) + a$ and $J(x_b) = J(x) + b$. Consequently, $J(x_a \wedge x_b) = J(x_a) \cap J(x_b) = J(x)$, and thus $x_a \wedge x_b = x$, which gives a proper meet-decomposition of x. ■

The easiest example of canonical chains in extremal lattice can be given by an *interval lattice on $n+2$ elements*, which is $(n+2, 3)$-extremal. This lattice is the lattice of all intervals of $[1, \ldots, n+2]$, including the empty one, ordered by set inclusion. The meet-irreducible elements are $\{[1, i] \mid i = 1, \ldots n+1\}$ and $\{[i, n+2] \mid i = 2, \ldots n+2\}$ and there are $2(n+1)$ of them. Figure 5 provides an illustration of these lattices.

As it turns out, interval lattice has no meet-irreducible elements, other than those, provided by Theorem 3. Moreover, it is, in essence, the only extremal lattice with that property. All other $(n, 3)$-extremal lattices will have some additional elements, and for $k > 2$, construction of an $(n+k, k+1)$-extremal lattice with $k(n+1)$ elements is impossible for large n.

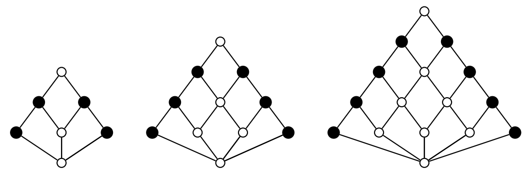

Fig. 5: Interval lattices. Black dots indicate meet-irreducible elements.

Lemma 7 *An interval lattice on $n+2$ atoms is $(n+2,3)$-extremal with $2(n+1)$ meet-irreducible elements, that is, it reaches the lower bound on the number of meet-irreducible elements from Theorem 3.*

Moreover, if an $(n+2,3)$-extremal lattice has $2(n+1)$ meet-irreducible elements, then it is isomorphic to the interval lattice on $n+2$ elements.

Proof. The fact that the interval lattice is extremal an reaches the lower bound is trivial, so we only need to prove the second statement. Let L be an $(n+2,3)$-extremal lattice with $J(L) = \mathbf{n+2}$. By Theorem 3, in L there are two disjoint chains $\{m_1, \ldots, m_{n+1}\}$ and $\{l_1, \ldots, l_{n+1}\}$ of meet-irreducible elements, such that $r(m_i) = r(l_i) = i$, for $i = 1, \ldots, n+1$. By proposition of the lemma, those are the only meet-irreducible elements of L. Let us additionally put $m_{n+2} = l_{n+2} = 1_L$. This way, chains $\{m_i\}$ and $\{l_i\}$ are still covering, but we can now state that each $x \in L$ can be represented, not necessarily in a unique way, as $x = m_i \wedge l_j$, for some i and j.

Without losing generality, we suppose that $J(m_i) = \mathbf{i}$, for all i: this can always be achieved by reordering of $J = J(L)$. Note that for each $x = m_i \wedge l_j$ we have $J(x) = J(m_i) \cap J(l_j)$.

We claim that $J(l_j) = [n+3-j, n+2]$. In this case the elements of L will be x such that $J(x) = \emptyset$, or such that $J(x) = [a,b]$ for $1 \leq a \leq b \leq n+2$. This structure will correspond exactly to the interval lattice and that would finish the proof of the lemma.

Let us suppose the contrary, and fix the largest j such that $J(l_j) \neq [n+3-j, n+2]$, notice that $j < n+2$ as $J(l_{n+2}) = J(1) = [1, n+2]$. As $J(l_{j+1}) = [n+2-j, n+2]$, we get $J(l_j) = J(l_{j+1}) - a = [n+2-j, n+2] - a$, for some $a \in [n+3-j, n+2]$.

Let us recall that $a = m_{i_0} \wedge l_{j_0}$, where i_0 and j_0 are smallest such that $a \leq m_{i_0}$ and $a \leq l_{j_0}$. Thus, $a = m_a \wedge l_{j+1}$ and

$$\begin{aligned} 1 = |J(a)| &= \big|[1,a] \cap [n+2-j, n+2]\big| \\ &= \big|[n+2-j, a]\big| \geq \big|[n+2-j, n+3-j]\big| = 2, \end{aligned}$$

a contradiction. ■

6. Discussion and open problems

As was mentioned in Introduction, the ultimate goal of our exploration is to arrive at bounds on the size of lattices with bounded VC dimension, symmetric with respect to $|J(L)|$ and $M(L)$. However far we may be from this goal, several improvements certainly can be made.

First, it seems that for $k \geq 3$ our lower bound may be significantly improved, although this would require a certain elaboration. As a first step, we are interested in the possibility of constructing extremal lattices with only canonical meet-irreducible elements:

Question 1 *Is there an $(n+k, k+1)$-extremal lattice with only canonical meet-irreducible elements, for $k \geq 3$ and for sufficiently large n*

The tentative answer to Question 1 is: no. What we can realistically expect is some limit on the rate of growth on the number of meet-irreducible elements, which we anticipate to be linear.

Question 2 *For given k, is there a constant $C = C(k)$ such that for all n there exists an $(n+k, k+1)$-extremal lattice L with $|M(L)| \leq C \cdot n$?*

An obvious way of providing an upper bound on the minimal number of meet-irreducible elements is to try and construct a family of lattices obtaining such bound. Obviously, root decompositions can be a handy tool for this. As a step in this direction, one can ask for an algorithm that, starting from given $(n, k+1)$-extremal lattice L, would go through all its $(n+k, k+1)$-*extensions*, possibly with repetitions. Here we use term *extensions* to denote extremal lattices, which share a given root, at least up to isomorphism.

Devising such algorithm may, on the other hand, be nontrivial, as it involves generating (n, l)-extremal lattices, embedded into intersection of several $(n, l+1)$-extremal lattices. This, apart from problems of practical realization, may require further theoretical elaboration.

Problem 1 *Device an effective algorithm for enumerating, possibly with repetitions, all extensions of an $(n, k+1)$-extremal lattice.*

As a curiosity, which, on the other hand, can help in shaping the theory, let us recall that the root decomposition in this paper appear as a generalization of a more simple construction from [4], which was used to count all possible non-isomorphic $(n, 3)$-extremal lattices. Now, we may ask the similar question about the number of non-isomorphic lattices for lager k, and see if our advanced decomposition can help in finding them.

Question 3 *How many non-isomorphic $(n+k, k+1)$-extremal lattices exist?*

REFERENCES

1. Albano A. The implication logic of (n, k)-extremal lattices // ICFCA 2017. Lecture Notes in Computer Science, – V. 10308, – 2017. – P. 39–55.

2. Albano A., Chornomaz B. Why concept lattices are large. Extremal theory for the number of minimal generators and formal concepts // Proceedings of the Twelfth International Conference on Concept Lattices and Their Applications, 2015. – P. 73–86.

3. Albano A., Chornomaz B. Why concept lattices are large: Extremal theory for generators, concepts and VC-dimension // International Journal of General Systems, 2017, – to appear.

4. Chornomaz B. Counting extremal lattices // Journal Algebra and Discrete Mathematics, submitted, 2017.

5. Ganter B., Wille R. Formal Concept Analysis: Mathematical Foundations. 1999. Springer, Berlin-Heidelberg, 284 p.

6. Grätzer G. General Lattice Theory, 2nd ed., 1998. – Birkhäuser-Verlag, Basel, 663 p.

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The Kharitonov theorem and robust stabilization via orthogonal polynomials

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Kharitonov's theorem for interval polynomials is given in terms of orthogonal polynomials on $[0, +\infty)$ and their second kind polynomials. A family of robust stabilizing controls for the canonical system is proposed.
Keywords: Kharitonov theorem; orthogonal polynomials; Hurwitz polynomials; stabilization of control systems.

Абдон Е. Чоке-Ріверо **Теорема Харитонова та робастна стабілізація, засновані на ортогональних поліномах.** Представлена теорема Харитонова для інтервальних поліномів у термінах ортогональних поліномів на $[0, +\infty)$ та їх поліномів другого роду. Запропонований клас керувань, які робастно стабілізують канонічну систему.
Ключові слова: теорема Харитонова; ортогональні поліноми; поліноми Гурвиця; стабілізація керованих систем.

Абдон Э. Чоке-Риверо. **Теорема Харитонова и робастная стабилизация, основанные на ортогональных полиномах.** Представлена теорема Харитонова для интервальных полиномов в терминах ортогональных полиномов на $[0, +\infty)$ и их полиномов второго рода. Предложено семейство управлений, робастно стабилизирующее каноническую систему.
Ключевые слова: теорема Харитонова; ортогональные полиномы; полиномы Гурвица; стабилизация управляемых систем.

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1 Introduction

Throughout this paper, let n and m be positive integers. We will use $\mathbb{C}$ and $\mathbb{R}$ to denote the set of all complex numbers and the set of all real numbers, respectively.

The aim of this work is to rewrite Kharitonov's well-known theorem [26] on the Hurwitness of interval polynomials through orthogonal polynomials $[0, \infty)$ and their second kind polynomials; see Proposition 2 and Theorem 2. We will also construct positional robust controls $u = u_n(x)$ for the Brunovsky system of degree n via two sets of Markov parameter sequences or equivalently by using two families of Hurwitz polynomials; see Definition 8 and Theorem 3.

The motivation for present work comes from two sources. One comes from the interrelations between the Markov parameters, orthogonal polynomials and Hurwitz polynomials and their practical application on control theory. The second comes from the generalization of the indicated results for the matrix case.

The present work is based on the Markov parameter approach which is thoroughly studied in [20, Chapter XV]. We decisively use the explicit interrelation between the coefficients of given polynomials and their Markov parameters; see remark 1 or [10, Lemma 3.1]. This interrelation together with the Hurwitness criteria in terms of the positive definiteness of two Hankel matrices; see lemma 1 or [10, Theorem 3.4]. The explicit representation of a Hurwitz polynomial through orthogonal polynomials, allows us to rewrite the Kharitonov theorem on interval polynomials with the help of orthogonal polynomials; see Proposition 1 or [9, Theorem 7.10].

In this sense, the following notions play a relevant role for the present paper:

- The truncated Stieltjes moment problem,
- Orthogonal polynomials,
- Hurwitz polynomials.

In contrast to Kharitonov's theorem, instead of verifying the Hurwitzness of four polynomials of degree $n = 2m$ (resp. $n = 2m+1$), we propose checking four polynomials of the degree $[\frac{n}{2}]$ (resp. $[\frac{n+1}{2}]$). To this end, the notion of Kharitonov quadruples is introduced. Roughly speaking, this notion highlights the fact that every stable interval polynomial can be constructed by two ordered sequences of Markov parameters. The latter means that the corresponding orthogonal polynomials and their second kind polynomials satisfy a certain order; see Definition 8.

The paper contains three conjectures. The first one states that every stable interval polynomial generates four sequences of ordered Markov parameters. The second conjecture says that the ordering of the quadruple $(h_n^{(\max)}, g_n^{(\max)}, h_n^{(\min)}, g_n^{(\min)})$ can be written in terms of the degree of the corresponding interval polynomial p_n. Finally, the third conjecture states the necessary and sufficient conditions for an interval polynomial to be a stable interval polynomial in terms of the Kharitonov quadruples.

The construction of robust controls of control systems in terms of the coefficient of certain interval polynomials was considered in [1], [25], [19], and references therein. In contrast to these works, we apply the Markov parameter approach. The advantages of using Markov parameters are explained in [22]. These consist mainly of the fact that the stable region in the coefficient space of a given polynomial is not convex, while the stable region in terms of the Markov parameters s_j with positive definite Hankel matrices (2) is convex set [24].

Future work can be devoted to the comparison of the descending degree procedure of the interval polynomial proposed in the present work (as in example 1) with the Routh procedure considered in [3]. Furthermore, future research on the characterization of two Markov sequences to be ordered sequences which generate Kharitonov quadruples is relevant. Such characterization could notably improve Algorithm 3.1.

This work is organized as follows. A brief summary of the truncated Stieltjes moment problem, orthogonal polynomials and the Hurwitz polynomial are given in the Introduction. In section 2, the Kharitonov theorem is represented via orthogonal polynomials on $[0, +\infty)$ and their second kind polynomials. An example of constructing a stable interval polynomial of degree $n = 7$ starting from two sequences of Markov parameters is given. Additionally, in remark 4 an example of a family of interval polynomials is proposed. In section 3, a result on the construction of stable interval polynomials via orthogonal polynomials is given; see Theorem 3. In subsection 3.1, an algorithm for the construction of a robust control is suggested. Following this algorithm, a family of robust controls is written; see examples 2 and 3. Finally, in section 4, the conclusion and three conjectures what develop or complete some results of section 2 are presented.

In the subsequent three subsections, we recall the definitions and relevant results concerning the Stieltjes moment problem, orthogonal polynomials on $[0, +\infty)$ and Hurwitz polynomials.

Note that in [12] the stabilization of the canonical system through orthogonal polynomials on $[0, +\infty)$ is treated.

1.1 The truncated Stieltjes moment problem and extremal solutions

The truncated Stieltjes moment problem is stated as follows: Let n be greater than or equal to 2. Given a sequence $(s_j)_{j=0}^{n-1}$ of real numbers, find the set $\mathcal{M}$ of nondecreasing functions σ of bounded variation on $[0, \infty)$ such that

$$s_j = \int_0^\infty t^j d\sigma(t), \quad 0 \leq j \leq n-1. \tag{1}$$

This problem was considered in [29, Page 176 and Page 192].

In case of an infinite sequence $(s_k)_{k=0}^\infty$ with (1) for $j \geq 0$, the stated problem is called the classical Stieltjes moment problem.

Let

$$\mathbf{H}_{1,j} := \begin{pmatrix} s_0 & s_1 & \dots & s_j \\ s_1 & s_2 & \dots & s_{j+1} \\ \vdots & \vdots & \vdots & \vdots \\ s_j & s_{j+1} & \dots & s_{2j} \end{pmatrix}, \quad \mathbf{H}_{2,j} := \begin{pmatrix} s_1 & s_2 & \dots & s_{j+1} \\ s_2 & s_3 & \dots & s_{j+2} \\ \vdots & \vdots & \vdots & \vdots \\ s_{j+1} & s_{j+2} & \dots & s_{2j-1} \end{pmatrix}. \tag{2}$$

It is known [16], [17] that the truncated Stieltjes moment problem with given moments $(s_j)_{j=0}^{2m+1}$ (resp. $(s_j)_{j=0}^{2m}$) as a solution if and only if $\mathbf{H}_{1,m}$ and $\mathbf{H}_{2,m-1}$

(resp. $\mathbf{H}_{1,m-1}$ and $\mathbf{H}_{2,m-1}$) are positive semidefinite. In [16], [17], the complete set of solutions of the truncated Stieltjes moment problem when $\mathbf{H}_{1,m}$ and $\mathbf{H}_{2,m-1}$ (resp. $\mathbf{H}_{1,m-1}$ and $\mathbf{H}_{2,m-1}$) are positive definite was given.

With the help of the analytic function in $\mathbb{C} \setminus [0, \infty)$

$$s(z) := \int_0^\infty \frac{d\sigma(t)}{t-z},$$

called associated solution with $\sigma \in \mathcal{M}$, the truncated Stieltjes moment problem is reduced to finding a set of associated analytic functions $s \in \mathcal{Z}$ such that

$$s(z) = -\frac{s_0}{z} - \frac{s_1}{z^2} - \dots - \frac{s_{n-1}}{z^n} - \dots .$$

Assume that σ is normalized as $\sigma(t) = \frac{\sigma(t+0)+\sigma(t-0)}{2}$, and $\sigma(0) = 0$. From the Stieltjes inverse formula [2, Page 631], one gets a corresponding measure by

$$\sigma(t) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_0^t \operatorname{Im} s(x + i\epsilon) dx.$$

1.2 Orthogonal polynomials on $[0, +\infty)$

Orthogonal polynomials [6], [39] play an important role in a number mathematical areas. On one hand, orthogonal polynomials have been extensively used in applications for solving practical problems, such as in signal processing [32] and in filter design [38], [30]. On the other hand, the zeros of a certain family of orthogonal polynomials can be interpreted as the electrostatic energy for a system of a finite number of charges; see [43].

In the present subsection, we focus on truncated families of orthogonal polynomials on $[0, +\infty)$.

Definition 1 *The sequence $(s_j)_{j=0}^{2m}$ (resp. $(s_j)_{j=0}^{2m-1}$) is called a Stieltjes positive definite sequence if $\mathbf{H}_{1,m}$ and $\mathbf{H}_{2,m-1}$ (resp. $\mathbf{H}_{1,m-1}$ and $\mathbf{H}_{2,m-1}$) are positive definite matrices.*

In the sequel, we consider only Stieltjes positive definite sequences.

Definition 2 *Let $(s_j)_{j=0}^{2m-1}$ and $(s_j)_{j=0}^{2m}$ be Stieltjes positive definite sequences. For $k = 1, 2$, let*

$$\mathbf{D}_{k,j}(z) := \begin{pmatrix} s_{k-1} & s_k & \dots & s_{j+k-1} \\ s_k & s_{k+1} & \dots & s_{j+k} \\ \dots & \dots & \dots & \dots \\ s_{j+k-2} & s_{j+k-1} & \dots & s_{2j+k-2} \\ 1 & z & \dots & z^j \end{pmatrix},$$

$$\mathbf{E}_{k,j}(z) := \begin{pmatrix} s_{k-1} & s_k & \dots & s_{j+k-1} \\ s_k & s_{k+1} & \dots & s_{j+k} \\ \dots & \dots & \dots & \dots \\ s_{j+k-2} & s_{j+k-1} & \dots & s_{2j+k-2} \\ e_{k,0}(z) & e_{k,1}(z) & \dots & e_{k,j}(z) \end{pmatrix},$$

where $(e_{1,0}(z), e_{1,1}(z), \dots, e_{1,j}(z)) := (0, -s_0, -zs_0 - s_1, \dots, -\sum_{l=0}^{j-1} z^{j-l-1} s_l)$ *and*
$(e_{2,0}(z), e_{2,1}(z), \dots, e_{2,j}(z)) := (-s_0, -zs_0 - s_1, \dots, -\sum_{l=0}^{j} z^{j-l} s_l)$.
Denote by $p_{1,0}(z) := 1$*,* $q_{1,0}(z) := 0$*,* $p_{2,0}(z) := 1$*, and* $q_{2,0}(z) := s_0$*. For* $j \geq 1$ *and* $k = 1, 2$*, let*

$$p_{k,j}(z) := \frac{\det \mathbf{D}_{k,j}(z)}{\det \mathbf{H}_{k,j-1}}, \quad q_{k,j}(z) := \frac{\det \mathbf{E}_{k,j}(z)}{\det \mathbf{H}_{k,j-1}}. \tag{3}$$

The polynomials $q_{k,j}$ *are called second kind polynomials.*

Note that in [9] a matrix version of $p_{k,j}$ and $q_{k,j}$ is considered. In the proof of [8, Remark 2.6], the transformation from the matrix form to the determinant form (3) is performed.

Definition 3 *Let* $n = 2m$ *(resp.* $n = 2m+1$*). Let* $\sigma(t)$ *be a positive distribution on* $[0, \infty)$ *such that all moments* $s_j := \int_0^\infty t^j d\sigma(t)$ *are finite for* $0 \leq j \leq n-1$*. The sequence of monic polynomials* $(p_{1,j})_{j=0}^m$

$$\int_0^\infty p_{1,j}(t) p_{1,k}(t) d\sigma(t) = \begin{cases} 0, & j \neq k, \\ c_j, & j = k, \end{cases} \qquad c_j > 0$$

and respectively

$$\int_0^\infty p_{2,j}(t) p_{2,k}(t) t d\sigma(t) = \begin{cases} 0, & j \neq k, \\ d_j, & j = k, \end{cases} \qquad d_j > 0$$

are called the sequences of monic orthogonal polynomials on $[0, \infty)$ *with respect to* $d\sigma(t)$ *(resp* $t d\sigma(t)$*).*

For completeness, we recall two special, associated solutions of the truncated Stieltjes moment problem for $n = 2m+1$ (resp. $n = 2m$) called extremal solutions:

$$s_M^{(2m)}(z) =: -\frac{q_{1,m}(z)}{p_{1,m}(z)}, \; s_\mu^{(2m)}(z) =: -\frac{q_{2,m}(z)}{z\, p_{2,m}(z)}, \tag{4}$$

$$s_M^{(2m-1)}(z) =: -\frac{q_{1,m}(z)}{p_{1,m}(z)}, \; s_\mu^{(2m-1)}(z) =: -\frac{q_{2,m-1}(z)}{z\, p_{2,m-1}(z)}. \tag{5}$$

These solutions, introduced by Yu. Dyukarev in [18], play a relevant role as proving Proposition 1.

1.3 Hurwitz polynomials and Markov parameters

The real polynomial of degree n

$$f_n(z) := a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n$$

can be written as with the help of two polynomials h_n and g_n such that

$$f_n(z) = h_n(z^2) + z g_n(z^2),$$

where

$$h_n(z) := \begin{cases} a_0 z^m + a_2 z^{m-1} + \ldots + a_{n-2} z + a_n, & n = 2m, \\ a_1 z^m + a_3 z^{m-1} + \ldots + a_{n-2} z + a_n, & n = 2m+1, \end{cases} \tag{6}$$

$$g_n(z) := \begin{cases} a_1 z^{m-1} + a_3 z^{m-2} + \ldots + a_{n-3} z + a_{n-1}, & n = 2m, \\ a_0 z^m + a_2 z^{m-1} + \ldots + a_{n-3} z + a_{n-1}, & n = 2m+1. \end{cases} \tag{7}$$

A polynomial f_n is called a Hurwitz polynomial if all its roots have negative real parts.

Definition 4 *The numbers $(s_j)_{j=0}^{2m-1}$ (resp. $(s_j)_{j=0}^{2m}$) appearing in the asymptotic expansions*

$$\frac{g_{2m}(-z)}{h_{2m}(-z)} = -\frac{s_0}{z} - \frac{s_1}{z^2} - \frac{s_2}{z^3} - \ldots - \frac{s_{2m-2}}{z^{2m-1}} - \frac{s_{2m-1}}{z^{2m}} - \ldots, \tag{8}$$

$$\frac{h_{2m+1}(-z)}{(-z)\, g_{2m+1}(-z)} = -\frac{s_0}{z} - \frac{s_1}{z^2} - \frac{s_2}{z^3} - \ldots - \frac{s_{2m-1}}{z^{2m}} - \frac{s_{2m}}{z^{2m+1}} + \ldots. \tag{9}$$

are called Markov parameters of the polynomials f_n

Note that the expansion (8) appears in [20, Chapter XV], meanwhile expansion (9) was first introduced in [9] in the matrix case.

Here we highlight two of the Hurwitzness criteria.

- The algebraic Routh-Hurwitz criterion [23], [34], [4], which is given in terms of the coefficients a_k, of the polynomial f_n. More precisely, one should verify whether the so-called Hurwitz matrix, constructed by the coefficients a_k has positive principal minors; see [23], [34], [4].

- The Markov parameter criterion [20, Chapter XV] given in terms of the Markov parameters s_k. This criteria consists of finding out whether two Hankel matrices of the form (2) are positive definite; see lemma 1.

Lemma 1 *[10, Theorem 3.4] Let n be greater than or equal to 2. The polynomial f_{2m+1} (resp. f_{2m}) is a Hurwitz polynomial if and only if the associated Hankel matrices $\mathbf{H}_{1,m}$ and $\mathbf{H}_{2,m-1}$ (resp. $\mathbf{H}_{1,m-1}$ and $\mathbf{H}_{2,m-1}$) associated with f_n are positive definite matrices.*

The following remark proved in [10] allows the calculation of the Markov parameters s_k from the coefficients a_j of the polynomial f_n.

Remark 1 *[10, Lemma 3.1] Let f_n be a real polynomial of degree n, and let h_n, g_n be as in (6) and (7). The Markov parameter sequence $(s_j)_{j=0}^{2m}$ (resp. $(s_j)_{j=0}^{2m-1}$) from the relations (8) and (9) is determined by the following equalities:*

$$(s_0, s_1, \ldots, s_{2m-1})^\mathsf{T} = \mathcal{A}_{2m}^{-1}(a_1, a_3, \ldots, a_{2m-1}, 0, \ldots, 0)^\mathsf{T}, \quad n = 2m, \tag{10}$$

$$(s_0, s_1, \ldots, s_{2m})^\mathsf{T} = \mathcal{A}_{2m+1}^{-1}(a_1, a_3, \ldots, a_{2m+1}, 0, \ldots, 0)^\mathsf{T}, \quad n = 2m+1, \tag{11}$$

where

$$\mathcal{A}_n := \begin{pmatrix} a_0 & 0 & \dots & 0 & 0 \\ a_2 & -a_0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ a_{2(n-1)} & -a_{2(n-2)} & \dots & (-1)^n a_2 & (-1)^{n+1} a_0 \end{pmatrix},$$

for $n \geq 2$ *is the* $n \times n$ *matrix with* $a_k = 0$ *for* $k > n$.

In [9, Theorem 6.1], it was proven that every Hurwitz polynomial can be written in terms of orthogonal polynomials $p_{k,j}$, $k = 1, 2$, on $[0, \infty)$ and their second kind polynomials $q_{k,j}$; see [7, Equality E.2]. We reformulate the latter as a proposition.

Proposition 1 *Every real Hurwitz polynomial* f_n *with* $a_0 = 1$ *admits the following representation*

$$f_n(z) = \begin{cases} (-1)^m (p_{1,m}(-z^2) - z\, q_{1,m}(-z^2)), & n = 2m, \\ (-1)^m (q_{2,m}(-z^2) + z\, p_{2,m}(-z^2)), & n = 2m+1. \end{cases} \tag{12}$$

Here $p_{k,j}$, $k = 1, 2$ *are orthogonal polynomials on* $[0, \infty)$, *and* $q_{k,j}$ *are their second kind polynomials defined as in Definition 2.*

To prove Proposition 1, the subsequent, explicit relation between polynomials h_n, g_n as in (6), (7) and orthogonal polynomials (3) was introduced in [9, Pages 78 and 79]:

$$h_{2m}(z) = (-1)^m p_{1,m}(-z), \quad g_{2m}(z) = (-1)^{m+1} q_{1,m}(-z), \tag{13}$$

$$g_{2m+1}(z) = (-1)^m p_{2,m}(-z), \quad h_{2m+1}(z) = (-1)^m q_{2,m}(-z). \tag{14}$$

2 Kharitonov's theorem via orthogonal polynomials

In this section, we propose a new form of the Kharitonov theorem which first appeared in [26] in 1978. This representation consists of writing the $h_n^{(r)}$ (resp. $g_n^{(r)}$) part of each of the four Kharitonov polynomials via a member of a family of orthogonal polynomials on $[0, \infty)$ and their second kind polynomials. Such a procedure is based on the Markov parameters generated by the Kharitonov polynomials $K_n^{(r)}$.

Let $\delta \in \mathcal{R}^{n+1}$, and let $\mathcal{P}_n$ be a family of monic interval polynomials:

$$p_n(z, \delta) := \sum_{j=0}^{n} \delta_{n-j} z^j, \tag{15}$$

with

$$x_j \leq \delta_{n-j} \leq y_j, \quad j = \{0, 1, \dots, n\}. \tag{16}$$

Denote

$$h_n^{(1)}(z) := x_0 + y_2 z + x_4 z^2 + \ldots, \tag{17}$$
$$g_n^{(1)}(z) := x_1 + y_3 z + x_5 z^2 + \ldots, \tag{18}$$
$$h_n^{(2)}(z) := y_0 + x_2 z + y_4 z^2 + \ldots, \tag{19}$$

and

$$g_n^{(2)}(z) := y_1 + x_3 z + y_5 z^2 + \ldots. \tag{20}$$

Definition 5 *Let p_n be an interval polynomial as in (15), and let $h_n^{(k)}$, $g_n^{(k)}$ be polynomials as in (17)-(20). The following four polynomials*

$$K_n^{(1)}(z) = h_n^{(1)}(z^2) + z g_n^{(1)}(z^2), \tag{21}$$
$$K_n^{(2)}(z) = h_n^{(1)}(z^2) + z g_n^{(2)}(z^2), \tag{22}$$
$$K_n^{(3)}(z) = h_n^{(2)}(z^2) + z g_n^{(1)}(z^2), \tag{23}$$

and

$$K_n^{(4)}(z) = h_n^{(2)}(z^2) + z g_n^{(2)}(z^2) \tag{24}$$

are called Kharitonov polynomials of the interval polynomial p_n.

Note that the Kharitonov polynomials are usually defined in the following form:

$$K_n^{(1)}(z) = x_0 + x_1 z + y_2 z^2 + y_3 z^3 + x_4 z^4 + x_5 z^5 + \ldots, \tag{25}$$
$$K_n^{(2)}(z) = x_0 + y_1 z + y_2 z^2 + x_3 z^3 + x_4 z^4 + y_5 z^5 + \ldots, \tag{26}$$
$$K_n^{(3)}(z) = y_0 + x_1 z + x_2 z^2 + y_3 z^3 + y_4 z^4 + x_5 z^5 + \ldots, \tag{27}$$
$$K_n^{(4)}(z) = y_0 + y_1 z + x_2 z^2 + x_3 z^3 + y_4 z^4 + y_5 z^5 + \ldots, \tag{28}$$

The equivalence between (21)-(23) and (25)-(28) is obvious.

Definition 6 *Let $\alpha := (\alpha_0, \alpha_1, \ldots, \alpha_n)$ where α_j are real numbers. An interval polynomial $p_n(z, \delta)$ as in (15) is said to be a stable interval polynomial if for each $\alpha_j \in [x_j, y_j]$ all the zeros of $p_n(z, \alpha)$ are strictly in the left-hand complex plane.*

Let us recall the celebrated Kharitonov theorem [26].

Theorem 1 *Let p_n be an interval polynomial as in (15). Furthermore, let $K_n^{(r)}$ for $r = 1, 2, 3, 4$ be Kharitonov polynomials as in Definition 5. The interval polynomial p_n (15) is stable if and only if the four Kharitonov polynomials $K_n^{(r)}$ for $r = 1, 2, 3, 4$ are stable.*

In the present work, we restrict ourselves to the case where the leading interval coefficient δ_0 is equal to $[1,1]$.

Definition 7 *Let the polynomials $h_n^{(k)}$, $g_n^{(k)}$ for $k = 1,2$ be defined as in (17)-(20). For $n = 2m$, define*

$$s^{(1)}(z) := \frac{g_n^{(1)}(-z)}{h_n^{(1)}(-z)}, \quad s^{(2)}(z) := \frac{g_n^{(2)}(-z)}{h_n^{(1)}(-z)}, \tag{29}$$

$$s^{(3)}(z) := \frac{g_n^{(1)}(-z)}{h_n^{(2)}(-z)}, \quad s^{(4)}(z) := \frac{g_n^{(2)}(-z)}{h_n^{(2)}(-z)}. \tag{30}$$

Similarly for $n = 2m+1$, define

$$s^{(1)}(z) := \frac{h_n^{(1)}(-z)}{(-z)g_n^{(1)}(-z)}, \quad s^{(2)}(z) := \frac{h_n^{(2)}(-z)}{(-z)g_n^{(1)}(-z)}, \tag{31}$$

$$s^{(3)}(z) := \frac{h_n^{(1)}(-z)}{(-z)g_n^{(2)}(-z)}, \quad s^{(4)}(z) := \frac{h_n^{(2)}(-z)}{(-z)g_n^{(2)}(-z)}. \tag{32}$$

Each of these rational functions $s^{(r)}$ can be expanded as in (8) and (9), respectively. Every sequence $(s_j^{(r)})_{j=0}^{n-1}$ corresponding to such expansions is called the Markov parameter sequence, which is associated with the polynomial $K_n^{(r)}$.

Under the assumption that $K_n^{(r)}(z)$ are monic Hurwitz polynomials, we will prove that the functions $s^{(r)}(z)$ are in fact extremal solutions of truncated Stieltjes moment problems.

Lemma 2 *Let the polynomials $K_n^{(r)}(z)$ for $r = 1,2,3,4$ be monic Hurwitz polynomial, then the following is valid.*
a) The Markov parameter sequence $(s_j^{(r)})_{j=0}^{n-1}$ associated with the polynomial $K_n^{(r)}$ is a truncated Stieltjes positive definite sequence for $r = 1,2,3,4$.
b) The functions $s^{(r)}(z)$ defined by (29)-(32) are extremal solutions of the truncated Stieltjes moment problem with $(s_j^{(r)})_{j=0}^{n-1}$ for $r = 1,2,3,4$.

Proof 1 Part a) is a direct consequence of lemma 1. Part b) is verified by employing (4), (5) and equalities in lines 12, 22 *on [9, Page 80].*

The following Proposition can be readily verified by applying Proposition 1 for every $r = 1,2,3,4$.

Proposition 2 *The interval polynomial (15) with $\delta_0 = [1,1]$ is stable if and only if the four Kharitonov polynomials $K_n^{(r)}$ for $r = 1,2,3,4$ as in (21)-(24) admit the following representation*

$$K_n^{(r)}(z) = \begin{cases} (-1)^m(p_{1,m}^{(r)}(-z^2) - zq_{1,m}^{(r)}(-z^2)), & n = 2m, \\ (-1)^m(q_{2,m}^{(r)}(-z^2) + zp_{2,m}^{(r)}(-z^2)), & n = 2m+1 \end{cases} \quad r = 1,2,3,4, \tag{33}$$

where $p_{1,m}^{(r)}$ *and* $q_{1,m}^{(r)}$ *(resp.* $p_{2,m}^{(r)}$ *and* $q_{2,m}^{(r)}$*) are orthogonal polynomials on* $[0,+\infty)$ *and second kind polynomials.*

To write Kharitonov's theorem of two sequences of Markov moments, we introduce the following notion.

Definition 8 *Let* $n=2m$ *(resp.* $n=2m+1$*). Let* $((s_j^{(\min)})_{j=0}^{n-1},\ (s_j^{(\max)})_{j=0}^{n-1})$ *be Stieltjes positive definite sequences such that* $s_j^{(\min)} \leq s_j^{(\max)}$, $0\leq j\leq n-1$ *with at least one strict inequality. Furthermore, let* $(p_{k,m}^{(\max)},\ q_{k,m}^{(\min)})$, *for* $k=1,2$, *the polynomials as in Definition 2. The quadruple*

$$\mathcal{P}_{2m} := (p_{1,m}^{(\min)}, q_{1,m}^{(\min)}, p_{1,m}^{(\max)}, q_{1,m}^{(\max)}) \tag{34}$$

and

$$\mathcal{P}_{2m+1} := (p_{2,m}^{(\min)}, q_{2,m}^{(\min)}, p_{2,m}^{(\max)}, q_{2,m}^{(\max)}) \tag{35}$$

are called Kharitonov quadruple if the Markov parameter sequences

$$((s_j^{(i_1)})_{j=0}^{2m-1},\quad (s_j^{(i_2)})_{j=0}^{2m-1})\quad (resp.\quad ((s_j^{(i_1)})_{j=0}^{2m}),\quad (s_j^{(i_2)})_{j=0}^{2m})) \tag{36}$$

generated by

$$\left(-\frac{p_{1,m}^{(\min)}(z)}{q_{1,m}^{(\max)}(z)}, -\frac{p_{1,m}^{(\max)}(z)}{q_{1,m}^{(1\min)}(z)}\right)\quad (resp.\quad \left(-\frac{p_{2,m}^{(\min)}(z)}{zq_{2,m}^{(\max)}(z)}, -\frac{p_{2,m}^{(\max)}(z)}{zq_{2,m}^{(\min)}(z)}\right)) \tag{37}$$

are Stieltjes positive definite sequences.

Remark 2 *The Markov parameters (36) can be calculated by Laurent series expansion of the rational functions appearing in (37), respectively.*

Alternatively, to determine the Markov parameters (36) one can use remark 1 with

$$(h_n(z), g_n(z)) = ((-1)^m p_{1,m}^{(\min)}(-z), (-1)^{m+1} q_{1,m}^{(\min)}(-z)),\quad n=2m \tag{38}$$

and

$$(h_n(z), g_n(z)) = ((-1)^m q_{2,m}^{(\min)}(-z), (-1)^m p_{2,m}^{(\min)}(-z)),\quad n=2m+1. \tag{39}$$

Definition 9 *Let* $n=2m$ *(resp.* $n=2m+1$*), and let* $K_n^{(r)}$ *for* $r=1,2,3,4$ *be the monic Kharitonov polynomials as in Definition 5, which correspond to the interval polynomial (15) with the leading coefficient* $\delta_0=[1,1]$. *Furthermore, let* $h_n^{(k)}$, $g_n^{(k)}$ *for* $k=1,2$ *be as in (17)-(20). We say that the Kharitonov polynomials* $K_n^{(r)}$ *form a Kharitonov quadruple if between the polynomials* $h_n^{(k)}$, $g_n^{(k)}$, $k=1,2$ *there are quadruples*

$$\left((-1)^m h_{2m}^{(i_1)}(-z), (-1)^{m+1} g_{2m}^{(i_2)}(-z), (-1)^m h_{2m}^{(i_3)}(-z), (-1)^{m+1} g_{2m}^{(i_4)}(-z)\right)$$

and

$$\left((-1)^m g_{2m+1}^{(i_2)}(-z), (-1)^m h_{2m+1}^{(i_1)}(-z), (-1)^m g_{2m+1}^{(i_4)}(-z), (-1)^m h_{2m+1}^{(i_3)}(-z)\right),$$

that are Kharitonov quadruples. Here (i_j) *is one of the superscripts* (1) *or* (2).

Now we state the main result of the present work.

Theorem 2 *Let* $n = 2m$ *(resp.* $n = 2m+1$*) and* $K_n^{(r)}$ *for* $r = 1, 2, 3, 4$ *be monic Kharitonov polynomials as in Definition 5. If the polynomials* $K_n^{(r)}$ *form a Kharitonov quadruple, then the corresponding interval polynomial* p_n *is a stable interval polynomial.*

P r o o f 2 The proof follows by using Propositon 2 and Equalities (13)-(14).

Note that the converse statement to Theorem 2 appears in Conjecture 3.

The following remark verifies, for $2 \le j \le 7$, some ordering of the pairs $(h_j^{(i_k)}, g_j^{(i_k)})$ appearing in (17)-(20). This ordering allows the identification of the pairs $(h_j^{(\max)}, g_j^{(\max)})$ and $(h_j^{(\min)}, g_j^{(\min)})$. For $j \ge 7$, the corresponding equalities are stated in Conjecture 2.

Remark 3 *Let* $h_n^{(1)}$, $g_n^{(1)}$, $h_n^{(2)}$, $g_n^{(2)}$ *be as in (17)-(20). Furthermore, let the pairs* $(h_n^{(i_1)}, g_n^{(i_2)})$ *for* $i_k = 1$ *or* $i_k = 2$ *with* $k = 1, 2$*. be Kharitonov quadruples as in definition 9. Thus, the following equalities hold.*

$$(h_2^{(\max)}, g_2^{(\max)}) = (h_2^{(2)}, g_2^{(2)}), \quad (h_2^{(\min)}, g_2^{(\min)}) = (h_2^{(1)}, g_2^{(1)}), \tag{40}$$
$$(h_3^{(\max)}, g_3^{(\max)}) = (h_3^{(1)}, g_3^{(2)}), \quad (h_3^{(\min)}, g_3^{(\min)}) = (h_3^{(2)}, g_3^{(1)}), \tag{41}$$
$$(h_4^{(\max)}, g_4^{(\max)}) = (h_4^{(1)}, g_4^{(1)}), \quad (h_4^{(\min)}, g_4^{(\min)}) = (h_4^{(2)}, g_4^{(2)}), \tag{42}$$
$$(h_5^{(\max)}, g_5^{(\max)}) = (h_5^{(2)}, g_5^{(1)}), \quad (h_5^{(\min)}, g_5^{(\min)}) = (h_5^{(1)}, g_5^{(2)}), \tag{43}$$
$$(h_6^{(\max)}, g_6^{(\max)}) = (h_6^{(2)}, g_6^{(2)}), \quad (h_6^{(\min)}, g_6^{(\min)}) = (h_6^{(1)}, g_6^{(1)}), \tag{44}$$
$$(h_7^{(\max)}, g_7^{(\max)}) = (h_7^{(1)}, g_7^{(2)}), \quad (h_7^{(\min)}, g_7^{(\min)}) = (h_7^{(2)}, g_7^{(1)}). \tag{45}$$

P r o o f 3 Equalities (40)-(45) can be verified by using lemma 1: see also [10, Remark 3.1].

Example 1 *Let the following Stieltjes positive definite sequences be given*

$$\{s_k^{(\max)}\}_{k=0}^{6} = \{\frac{19}{2}, \frac{913}{4}, \frac{49959}{8}, \frac{2753481}{16}, \frac{151846263}{32}, \frac{8374343913}{64}, \frac{461849056119}{128}\}$$
$$\{s_k^{(\min)}\}_{k=0}^{6} = \{9, \frac{415}{2}, 5538, \frac{596853}{4}, \frac{16095575}{4}, \frac{868194535}{8}, 2926929877\}.$$

Clearly, $s_k^{(\min)} < s_k^{(\max)}$ *for* $0 \le k \le 6$*. The corresponding orthogonal polynomials and second kind polynomials (see Definition 2) are given by*

$$p_{2,3}^{(\max)}(z) = z^3 - \frac{63z^2}{2} + 111z - \frac{153}{2}, \quad q_{2,3}^{(\max)}(z) = \frac{19z^3}{2} - 71z^2 + \frac{219z}{2} - 12,$$
$$p_{2,3}^{(\min)}(z) = z^3 - 31z^2 + \frac{223z}{2} - 76, \quad q_{2,3}^{(\min)}(z) = 9z^3 - \frac{143z^2}{2} + 109z - \frac{25}{2}.$$

Let

$$K_7^{(\min)}(z) := -\,(q_{2,3}^{(\min)}(-z^2) + z\,p_{2,3}^{(\min)}(-z^2)), \tag{46}$$
$$K_7^{(\max)}(z) := -\,(q_{2,3}^{(\max)}(-z^2) + z\,p_{2,3}^{(\max)}(-z^2)), \tag{47}$$
$$K_7^{(3)}(z) := -\,(q_{2,3}^{(\max)}(-z^2) + z\,p_{2,3}^{(\min)}(-z^2)), \tag{48}$$

and

$$K_7^{(4)}(z) := -\,(q_{2,3}^{(\min)}(-z^2) + z\,p_{2,3}^{(\max)}(-z^2)). \tag{49}$$

By applying remark 2, we calculate the Markov parameters

$$\{s_k^{(3)}\}_{k=0}^6 = \{\frac{19}{2}, \frac{447}{2}, \frac{23910}{4}, 161131, \frac{34763331}{8}, \frac{937569489}{8}, \frac{50573020801}{16}\} \tag{50}$$
$$\{s_k^{(4)}\}_{k=0}^6 = \{9, 212, 5788, 159466, 4396929, \frac{242490639}{2}, \frac{13373470377}{4}\}. \tag{51}$$

Next, we verify that (50) and (51) are Stieltjes positive definite sequences; see Definition 1. Furthermore, we construct the corresponding orthogonal polynomials $p_{2,3}^{(3)}$, $p_{2,3}^{(4)}$ *and their second kind polynomials* $q_{2,3}^{(3)}$, $q_{2,3}^{(4)}$*. These are the following:*

$$p_{2,3}^{(1)}(z) = z^3 - 31z^2 + \frac{223z}{2} - 76, \quad q_{2,3}^{(1)}(z) = \frac{19z^3}{2} - 71z^2 + \frac{219z}{2} - 12,$$
$$p_{2,3}^{(4)}(z) = z^3 - \frac{63z^2}{2} + 111z - \frac{153}{2}, \quad q_{2,3}^{(4)}(z) = 9z^3 - \frac{143z^2}{2} + 109z - \frac{25}{2}.$$

By Proposition 1, the corresponding Kharitonov polynomials $K_7^{(\min)}$, $K_7^{(\max)}$, $K_7^{(3)}$ *and* $K_7^{(4)}$ *are Hurwitz polynomials. Finally, by Theorem 2 the interval polynomial*

$$f_7(z,\delta) := \delta_0 z^7 + \delta_1 z^6 + \delta_2 z^5 + \delta_3 z^4 + \delta_4 z^3 + \delta_5 z^2 + \delta_6 z + \delta_7 \tag{52}$$

is a stable interval polynomial. Here $\delta_0 \in [1,1]$, $\delta_1 \in [9,9.5]$, $\delta_2 \in [31,31.5]$, $\delta_3 \in [71,71.5]$, $\delta_4 \in [111,111.5]$, $\delta_5 \in [109,109.5]$, $\delta_6 \in [76,76.5]$ *and* $\delta_7 \in [12,12.5]$.

The interval coefficients δ_j are attained from the coefficients of $K_7^{(\min)}$, $K_7^{(\max)}$, $K_7^{(3)}$ and $K_7^{(4)}$, which in fact are the Kharitonov polynomials of f_7.

Note that the interval polynomial (52) was considered in [4, Example 5.4, Chapter 5].

Remark 4 *By using the moments of example 1 and Definition 2, we construct the polynomials* $((p^{(r)}_{1,m})^3_{m=0}, (q^{(r)}_{1,m})^3_{m=0}, (p^{(r)}_{2,m})^3_{m=0}, (q^{(r)}_{2,m})^3_{m=0})$*. With the help of these polynomials and (12), we establish four finite sequences of Hurwitz polynomials:*

$$f^{(r)}_k(z) := z^k + a^{(r)}_{k,1} z^{k-1} + \ldots + a^{(r)}_{k,k}, \quad r = 1, \ldots, 4,$$

and $k \in \mathbb{Z}^6_1$*. Here* $\mathbb{Z}^p_1 := \{1, 2, \ldots, p\}$*. For every* k*, each interval coefficient of the interval polynomial is defined by*

$$[\min_{r\in\mathbb{Z}^4_1} a^{(r)}_{k,j}, \max_{r\in\mathbb{Z}^4_1} a^{(r)}_{k,j}].$$

The family of stable interval polynomials in descending order with an initial interval polynomial (52) is then given by

$$\begin{aligned} f_6(z) =& z^6 + [9, 9.5]z^5 + [30.57, 30.05]z^4 + [66.61, 67.74]z^3 + [97.24, 99.18]z^2 \\ &+ [83.30, 85.84]z + [47.27, 48.74], \\ f_5(z) =& z^5 + [9, 9.5]z^4 + [29.33, 29.94]z^3 + [54.87, 57.75]z^2 \\ &+ [61.43, 67.49]z + [26.00, 29.92], \\ f_4(z) =& z^4 + [9, 9.5]z^3 + [28.41, 29.11]z^2 \\ &+ [46.39, 50.06]z + [38.96, 43.07], \\ f_3(z) =& z^3 + [9, 9.5]z^2 + [26.69, 27.35]z + [30.63, 33.71], \\ f_2(z) =& z^2 + [9, 9.5]z + [23.05, 24.02], \quad f_1(z) = z + [9, 9.5]. \end{aligned}$$

3 Robust stabilization of the canonical system

Let $x := \text{column}(x_1, x_2, \ldots, x_n)$. Consider the linear system

$$\dot{x} = \mathbf{A}_n x, \tag{53}$$

where

$$\mathbf{A}_n := \begin{pmatrix} 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 1 \\ -\alpha_n & -\alpha_{n-1} & \ldots & -\alpha_2 & -\alpha_1 \end{pmatrix}$$

with $\alpha_j \in [\underline{\alpha_j}, \overline{\alpha_j}]$ for $1 \le j \le n$. System (53) represents a linear system subject to some uncertainties, which may be caused by unknown perturbations with entries within a given interval; see [25].

Definition 10 *Let* $\mathbf{A}_n$ *be a matrix as in (53).*
a) The interval polynomial

$$p_{\mathbf{A}_n}(t) := (-1)^n (t^n + \alpha_1 t^{n-1} + \alpha_1 t^{n-2} + \ldots + \alpha_{n-1} t + \alpha_n) \tag{54}$$

is called the characteristic interval polynomial of the matrix $\mathbf{A}_n$*.*
b) System (53) is called stable if $(-1)^n p_{\mathbf{A}_n}$ *is a stable interval polynomial.*

Now consider the linear control system

$$\dot{x} = \mathbf{A}_n x + b_n u_n, \tag{55}$$

with

$$b_n := \text{column}\,(0, \dots, 0, 1). \tag{56}$$

Definition 11 *The system (55) is robustly stabilizable if there exists a* $1 \times n$ *interval matrix* $\gamma := -(\gamma_1, \gamma_2, \dots, \gamma_n)$ *where* $\gamma_j \in [\underline{\gamma_j}, \overline{\gamma_j}]$ *for* $1 \le j \le n$ *such that the linear system* $\dot{x} = (\mathbf{A}_n + b\gamma)x$ *is stable. Here*

$$\mathbf{A}_n + b\gamma = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 \\ -\delta_n & -\delta_{n-1} & \dots & -\delta_2 & -\delta_1 \end{pmatrix},$$

with $\delta_j = [x_j, y_j]$ *and*

$$[x_j, y_j] = [\underline{\alpha_j} + \underline{\gamma_j}, \overline{\alpha_j} + \overline{\gamma_j}]. \tag{57}$$

The linear interval function

$$u_n(x, \gamma) := -\gamma_n x_1 - \gamma_{n-1} x_2 - \dots - \gamma_1 x_n \tag{58}$$

is called the robust stabilizing control of the system (55).

In (57), we used interval arithmetic. For completeness, let us recall endpoint formulas for the arithmetic operations of intervals; see [31].

Remark 5 *Let* $[a, b]$ *and* $[c, d]$ *be closed intervals. The addition, subtraction, multiplication and division of intervals are defined respectively as follows:*

$$\begin{aligned} [a, b] + [c, d] :=& [a + c, b + d], \\ [a, b] - [c, d] :=& [a - d, b - c], \\ [a, b] \cdot [c, d] :=& [\min\{ac, ad, bc, bd\}, \max\{ac, ad, bc, bd\}], \\ \frac{[a, b]}{[c, d]} :=& \left[\min\{\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\}, \max\{\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\} \right], \quad 0 \notin [c, d]. \end{aligned}$$

Remark 6 *System (55) with* $\mathbf{A}_n = \mathbf{A}_n(\alpha)$ *and* $u_n = u_n(x, \gamma)$ *a is parametric differential equation*

$$\dot{x} = \mathbf{A}_n(\alpha)x + b_n u_n(x, \gamma). \tag{59}$$

In turn, differential equation (59) is a special case of the differential equation

$$\dot{x} = f(x, \alpha, \gamma),$$

where α *and* γ *are parameters taking certain given values within certain closed intervals. See for example [33, Equality (1)], [21] and [35].*

Now we turn to the problem of the robust stabilization of the Brunovsky system. Let

$$\mathbf{A}_n^{(0)} := \begin{pmatrix} 0_{n-1\times 1} & \mathbf{I}_{n-1} \\ 0 & 0_{1\times n-1} \end{pmatrix},$$

where $\mathbf{I}_n$ and $0_{p\times q}$ denotes the identity matrix and $p\times q$ zero matrix. The system

$$\dot{x} = \mathbf{A}_n^{(0)}x + b_n u_n \tag{60}$$

is called the Brunovsky system or canonical system. System (60) is a widely used control system for the study of the controllability and feedback stabilizability of linear and nonlinear systems, with the latter after a certain transformation; see [37], [36]. The Brunovsky system as the basic control model is used for testing results or approximating more general systems for controllability, time optimal control and stability problems; see [5], [37], [40], [41], [42], [44], [15], [13], and [11]. In particular, we emphasize the relevance of the controllability function method created by B.I. Korobov in 1979 [27]. This method allows stabilization at a finite time of the Brunovksy system and more general control systems under bounded controls [28]. See also [14].

The following result allows the construction of a robust control that stabilizes system (60) by employing the Kharitonov quadruples as in Definition 8.

Theorem 3 *Let $n = 2m$ (resp. $n = 2m+1$). Let p_n be the interval polynomial of the form (15) with interval coefficients δ_j constructed via the Kharitonov quadruples $(p_{1,m}^{(\min)}, q_{1,m}^{(\min)}, p_{1,m}^{(\max)}, q_{1,m}^{(\max)})$, respectively $(p_{2,m}^{(\min)}, q_{2,m}^{(\min)}, p_{2,m}^{(\max)}, q_{2,m}^{\max})$. Thus, the linear interval function*

$$u_n(x) = -\delta_n x_1 - \delta_{n-1}x_2 - \ldots - \delta_1 x_n \tag{61}$$

is a robustly stabilizing control for system (60).

Proof 4 Let $\delta^{(n)} := -(\delta_1, \delta_2, \ldots, \delta_n)$. Write the positional control u_n (61) as $u_n(x) = \delta^{(n)}x$. Substitute u_n for $u_n(x) = \delta^{(n)}x$ in (60). The right-hand side of (60) can be written in the form $\dot{x} = \widetilde{\mathbf{A}}_n x$, where

$$\widetilde{\mathbf{A}}_n := \mathbf{A}_n^{(0)} + b_n\delta^{(n)}.$$

The characteristic polynomial of $\widetilde{\mathbf{A}}_n$ has the form

$$p_{\widetilde{\mathbf{A}}_n}(t) := \det(tI - \widetilde{\mathbf{A}}_n) = (-1)^n(t^n + \delta_1 t^{n-1} + \delta_1 t^{n-2} + \ldots + \delta_{n-1}t + \delta_n)$$

Clearly $(-1)^n p_{\widetilde{\mathbf{A}}_n}$ coincides with the stable interval polynomial p_n of the form (15) with coefficients $(1, \delta_1, \delta_2, \ldots, \delta_n)$. Consequently, the control (61) robustly stabilizes system (60).

3.1 An algorithm for constructing a robust control

Let $n = 2m$ (resp. $n = 2m+1$).

1) Find two Stieltjes positive sequences $(s_j^{(\min)})_{j=0}^{n-1}$, $(s_j^{(\max)})_{j=0}^{n-1}$ such that $s_j^{(\min)} \leq s_j^{(\max)}$ with at least one strict inequality.

2) Construct polynomials $(p_{1,m}^{(\min)}, q_{1,m}^{(\min)}, p_{1,m}^{(\max)}, q_{1,m}^{(\max)})$, and $(p_{2,m}^{(\min)}$, $q_{2,m}^{(\min)}$, $p_{2,m}^{(\max)}$, $q_{2,m}^{(\max)})$ as in Definition 2.

3) In the case that the polynomials constructed in 2) form a Kharitonov quadruple, using lemma 1 and remark 2 calculate the interval coefficients. In the opposite case, return to Step 1).

4) With the help of (61), write the stabilizing robust control u_n.

Example 2 *Consider the system (60) with $n = 7$. We use example 1, which in fact follows the suggested algorithm. Thus, we attain the positional control*

$$\begin{aligned} u_7(x) = &-[12.12.5]x_1 - [76, 76.5]x_2 - [109, 109.5]x_3 - [111, 111.5]x_4 - [71, 71.5]x_5 \\ &- [31, 31.5]x_6 - [9, 9.5]x_7, \end{aligned}$$

which robustly stabilizes system (60).

Example 3 *As in a similar manner for $2 \leq n \leq 6$, system (60) can be robustly stabilized by*

$$\begin{aligned} u_6(x) &= -[47.27, 48.47]x_1 - [83.3, 85.84]x_2 - [97.24, 99.18]x_3 \\ &\quad - [66.61, 67.74]x_4 - [30.57, 30.05]x_5 - [9, 9.5]x_6, \\ u_5(x) &= -[26, 29.92]x_1 - [61.43, 67.49]x_2 - [54.87, 57.75]x_3 \\ &\quad - [29.33, 29.94]x_4 - [9, 9.5]x_5, \\ u_4(x) &= -[38.96, 43.07]x_1 - [46.39, 50.06]x_2 - [28.41, 29.11]x_3 - [9, 9.5]x_4, \\ u_3(x) &= -[30.63, 33.71]x_1 - [26.69, 27.35]x_2 - [9, 9.5]x_3 \end{aligned}$$

and

$$u_2(x) = -[23.05, 24.02]x_1 - [9, 9.5]x_2.$$

4 Conclusion and conjectures

In the present work, a reformulation of the Kharitonov theorem via quadruple polynomials is given. A family of decreasing degrees stable interval polynomials is proposed. With the help of constructed stable interval polynomials, a family of robust controls is formulated.

Next we present three conjectures concerning the results of section 1.

Conjecture 1 *Let $n = 2m$ (resp. $n = 2m + 1$) and let p_n be a stable interval polynomial of the form (15). Furthermore, for $r = 1, 2, 3, 4$ let $\left(s_j^{(r)}\right)_{j=0}^{n-1}$ be Markov parameters*

corresponding to Kharitonov polynomials $K_n^{(r)}$ of p_n. Thus, the following order yields

$$s_j^{(\min)} \leq s_j^{(i_2)} \leq s_j^{(i_3)} \leq s_j^{(\max)}, \quad 0 \leq j \leq n-1 \tag{62}$$

where $(\min)$, (i_2), (i_3), and $(\max)$ take one of the values $1, 2, 3$ or 4. Furthermore, at least one of the inequalities in (62) is a strict inequality.

Conjecture 2 *Let $h_n^{(1)}$, $g_n^{(1)}$, $h_n^{(2)}$, $g_n^{(2)}$ be as in (17)-(20). The following equalities hold.*

$$(h_{4\ell-2}^{(\max)}, g_{4\ell-2}^{(\max)}) = (h_{4\ell-2}^{(2)}, g_{4\ell-2}^{(2)}), \quad (h_{4\ell-2}^{(\min)}, g_{4\ell-2}^{(\min)}) = (h_{4\ell-2}^{(1)}, g_{4\ell-2}^{(1)}), \tag{63}$$

$$(h_{4\ell-1}^{(\max)}, g_{4\ell-1}^{(\max)}) = (h_{4\ell-1}^{(1)}, g_{4\ell-1}^{(2)}), \quad (h_{4\ell-1}^{(\min)}, g_{4\ell-1}^{(\min)}) = (h_{4\ell-1}^{(2)}, g_{4\ell-1}^{(1)}), \tag{64}$$

$$(h_{4\ell}^{(\max)}, g_{4\ell}^{(\max)}) = (h_{4\ell}^{(1)}, g_{4\ell}^{(1)}), \quad (h_{4\ell}^{(\min)}, g_{4\ell}^{(\min)}) = (h_{4\ell}^{(2)}, g_{4\ell}^{(2)}), \tag{65}$$

$$(h_{4\ell-3}^{(\max)}, g_{4\ell-3}^{(\max)}) = (h_{4\ell-3}^{(2)}, g_{4\ell-3}^{(1)}), \quad (h_{4\ell-3}^{(\min)}, g_{4\ell-3}^{(\min)}) = (h_{4\ell-3}^{(1)}, g_{4\ell-3}^{(2)}). \tag{66}$$

This conjecture is a generalization of remark 3. It says that the superindex (min) and (max) can be related to the degree of the interval polynomial p_n (15).

Conjecture 3 *Let $n = 2m$ (resp. $n = 2m + 1$). The interval polynomial p_n is a stable if and only if the Kharitonov polynomials $K_n^{(r)}$ for $r = 1, 2, 3, 4$ form Kharitonov quadruples.*

Note that the sufficient condition of Conjecture 3 is proven in Theorem 2.

REFERENCES

1. J. Ackermann, Robust control-system with uncertain physical parameters, 1993. – Springer-Verlag, New York. – 406 p.

2. F.V. Atkinson, Discrete and continuous boundary problems (Russian translation), 1964. – Mir, Moscow. – 750 p.

3. Bandyopadhyay B., Sreeram V., Shingare P., Stable $\gamma - \delta$ Routh approximation of interval systems using Kharitonov polynomials. // International Journal of Information and Systems Sciences, 1996. – **6**(4). – P. 1–12.

4. S.P. Bhattacharyya, H. Chapellat and L.H. Keel, Robust control. The parametric approach, 1995.– Prentice–Hall. – 672 p.

5. Brunovsky P., A classification of lineal controllable systems. // Kybernetika, 1970. – **6**. – P. 176–188.

6. T.S. Chihara, An introduction to orthogonal polynomials (Mathematics and its Applications), 1978. – Dover Publications, INC, New York. – 249 p.

7. Choque Rivero A.E., On Dyukarev's resolvent matrix for a truncated Stieltjes matrix moment problem under the view of orthogonal matrix polynomials. // Linear Algebra Appl., 2015. – **474**. – P. 44–109.

8. Choque Rivero A.E., From the Potapov to the Krein-Nudel'man representation of the resolvent matrix of the truncated Hausdorff matrix moment problem. // Bol. Soc. Mat. Mexicana, 2015. – **21**(2). – P. 233–259.

9. Choque Rivero A.E., On matrix Hurwitz type polynomials and their interrelations to Stieltjes positive definite sequences and orthogonal matrix polynomials. // Linear Algebra Appl., 2015. – **476**. – P. 56–84.

10. Choque Rivero A.E., Orthogonal polynomials and Hurwitz polynomials generated by Routh-Markov parameters. // Submitted to Mediterr. J. Math., 2017. – P. 1–16.

11. Choque Rivero A.E., On the solution set of the admissible control problem via orthogonal polynomials, // IEEE Trans. Autom. Control, 2017. – **62**(10). – P. 5213–5219.

12. Choque-Rivero A.E., González Hernández O.F., Stabilization via orthogonal polynomials, to appear in IEEE Xplore, 2017 IEEE International Autumn Meeting on Power, Electronics and Computing (ROPEC 2017), Ixtapa, México, 2017. – P. 1–4.

13. Choque Rivero A.E., Karlovich Yu., The time optimal control as an interpolation problem, // Commun. Math. Anal., 2011. – **3**. – P. 1–11.

14. Choque Rivero A.E., Korobov V.I., Skoryk V.O., Controllability function as time of motion. I, (in Russian) // Mat. Fiz. Anal. Geom., 2004. – **11**(2). – P. 208–225. English translation in arxiv.org/abs/1509.05127.

15. Choque Rivero A.E., Korobov V.I., Skylar G.M., The admissible control problem from the moment problem point of view, // Appl. Math. Lett., 2010. – **23**(1). – P. 58–63.

16. Dyukarev Yu.M., The Stieltjes matrix moment problem, // Deposited in VINITI (Moscow) at 22.03.81, No. 2628-81, 37 p.

17. Dyukarev Yu.M., A general scheme for solving interpolation problems in the Stieltjes class that is based on consistent representations of pairs of nonnegative operators. I. (Russian) // Mat. Fiz. Anal. Geom., 1999. – **6**. – P. 30–54.

18. Dyukarev Yu.M., Indeterminacy criteria for the Stieltjes matrix moment problem, // Math. Notes, 2004. – **75**(1-2). – P. 66–82.

19. T.A. Ezangina, S.A. Gayvoronskiy, S.V. Efimov, Construction of robustly stable interval polynomial, Mechatronics Engineering and Electrical Engineering, 2015. – Sheng (Ed.) Taylor and Francis Group.

20. F.R. Gantmacher, Matrix Theory, Vol. 2, 1959. AMS Chelsea Publishing, 276 p.

21. Hernández V.M., Sira-Ramírez H., On the robustness of generalized pi control with respect to parametric uncertainties, // European Control Conference, 2003. – P. 1–6.

22. Hollot C.V., Kharitonov-like results in the space of Markov parameters, // IEEE Trans. Autom. Control, 1989. – **34**(5). – P. 536–538.

23. Hurwitz A., Über die Bedingungen unter welchen eine Gleichung nur Wurzeln mit negativen reellen Teilen besitzt, // Math. Ann., 1895. – **46**. – P. 273–284.

24. Karlin S. and Shapley L.S., Geometry of reduced moment spaces, // Proc. Natl. Acad. Sci. USA, 1949. – **35**(12). – P. 673—677.

25. Kawamura T., Shima M., Robust stability analysis of characteristic polynomials whose coefficients are polynomials of interval parameters, // J. Math. Syst. Est. Control, 1996. – **6**(4). – P. 1–12.

26. Kharitonov V.L., Asymptotic stability of an equilibrium position of a family of systems of linear differential equations. // Diff. Eq., 1979. – **14**. – P. 1483–1485.

27. Korobov V.I., A general approach to the solution of the problem of synthesizing bounded controls in a control problem, // Mat. Sb., 1979. – **109**(151). – P. 582–606.

28. V.I. Korobov, Controllability function method, 2007. NITS, Inst. Comp. Research, M-Ighevsk.

29. M.G. Krein and A.A. Nudel'man , The Markov moment problem and extremal problems (Translations of Mathematical Monographs, vol. 50) Providence, 1977. – American Mathematical Society. – 417 p.

30. L. Lindtrön, Signal filtering using orthogonal polynomials and removal of edge effects, 2007. – US7221975B2.

31. R.E. Moore, R.B. Kearfott, M.J. Cloud, Introduction to interval analysis, 2009. – SIAM. Philadelphia. – 223 p.

32. Sandryhaila A., Kovačević J., Püschel M., Algebraic signal precessing theory: 1-D nearest neighbor models, // IEEE Trans. Signal Process, 2012. – **60**(5). – P. 2247–2259.

33. Strebel O., A preprocessing method for parameter estimation in ordinary differential equations, // Chaos, Solitons and Fractals, 2013. – **57**. – P. 93–104.

34. M.M. Postnikov, Stable polynomials (in Russian), 1981. – Nauka, Moscow.

35. Pulch R., Polynomials chaos for linear differential algebraic equations with random parameters, // International Journal for Uncertainty Quantification, 2011. – bf 1(3). – P. 223–240.

36. G. Rigatos, Nonlinear control and filtering using differential flatness theory approaches: Applications to electromechical systems, 2015. – Springer.

37. E.D. Sontag, Mathematical control theory: deterministic finite-dimensional systems, 1998. – Revised 2nd edition. – Springer.

38. Stojanović N., Stamenković N., Živaljević D., Monotonic, critical monotonic, and nearly monotonic low-pass filters designed by using the parity relation for Jacobi polynomials, // Int. J. Circ., 2017. – **12**. – P. 1978–1992. DOI: 10.1002/cta.2375

39. G. Szegő, Orthogonal polynomials, 1975. – Amer. Math. Soc. Colloq. Publ. Series. Vol. 23, Amer. Math. Soc., Providence, Rhode Island, 4^{th} edition.

40. Ovseevich A., A local feedback control bringing a linear system to equilibrium, // J. Optim. Theory Appl., 2015. – **165**(2). – P. 532–544.

41. Patil D.U., Chakraboty D., Computation of time optimal feedback control using Groebner basis, // IEEE Trans. Control, 2014. – **59**(8). – P. 2271–2276.

42. Polyakov A., Efimov D., and Perruquetti W., Finite-time stabilization using implicit Lyapunov function technique, // IFAC Proceedings, 2013. – **46**(23). – P. 140–145.

43. Valent G., Van Assche W., The impact of Stieltjes' work on continued fractions and orthogonal polynomials: additional material, // J. Comput. Appl. Math., 1995. – **65**. – P. 419–447.

44. Walther U., Georgiou T.T., and Tannenbaum A., On the computation of switching surfaces in optimal control: A Gröner basis approach, // IEEE Trans. Control, 2001. – **46**(4). – P. 534–540.

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