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Control of wheeled platforms straight motions taking into account jerk restrictions under speeding-up from the state of rest

The generalized mathematical model of wheeled platforms straight motions on the ideal horizontal plane under speeding-up from the state of rest mode is proposed, and the controls satisfying the restrictions of motion jerks are find. The pure mechanical and electromechanical wheeled platforms are considered, as well as the computer simulations of the researched processes are made. The jerks restrictions are reduced to limiting the value of the wheeled platform acceleration time derivative. The proposed approaches are based on the holonomic systems mechanics and on the electromechanical analogies allowing to consider the different kinds of the wheeled platforms taking into account the electric on-board systems like the drive electric motors and the control systems by using the Lagrange equations of second kind. The examples of the proposed approaches using to define the controls satisfying the jerks restrictions under speeding-up from the state of rest are considered for the pure mechanical and electromechanical wheeled platforms. It is obtained the inequality allowing to chose the instantly supplied driving mechanical couple which will provide the admissible jerks of the motion of the wheeled platform under speeding-up from the state of rest. It is shown that the rolling friction and the viscous damping are the principal causes of the wheeled platforms jerks under speeding-up from the state of rest. It is obtained the inequality defining the voltage instantly supplied on the drive electric motors which will provide the admissible jerks of the motion of the electromechanical wheeled platform during speeding-up from the state of rest, and it is shown that the proposed general approaches are suitable for considering the different kinds of wheeled platforms. The computer simulations of the processes of speeding-up from the state of rest for the electromechanical wheeled platform are considered to show results correctness and to illustrate satisfying the restrictions of the motion jerks. The obtained results of the computer simulations are in the full agreement with the well-known fundamental property inherent for the wheeled platforms. The results for the jerks show that the maximum value of the jerk is really at the initial time as was suggested before, and it is shown that the jerks values at the initial time obtained by using the computer simulations are in full agreement with the theoretically defined correspondent exact values. The big jerks of the considered electromechanical wheeled platform are due to the voltage instantly supplying on the drive electric motors at the initial time, and it is understandable that limiting of such instantly supplied voltage value cannot provide any wished small jerks. The smooth time depending for the voltages supplying on the drive electric motors are required to provide any wished small jerks of the electromechanical wheeled platforms.

Keywords: **control; motion; jerk; wheeled platform; mathematical modelling.**

2010 Mathematics Subject Classification: 49K15; 70E60; 70E55.

1. Introduction

Different kinds of wheeled platforms are widely used for human operated transportation systems, but last times it is existed the trend in using them also as the carriers of the different autonomous mobile transportation and technological systems for industrial, military, police, agriculture and house holding purposes. The motions jerks can limit the implementing possibilities of the autonomous wheeled platforms and other robotic systems for automated executing of some kinds of operations. Du to this circumstance, restricting the motions jerks is in current interest problem necessary to increase the operational quality and possibilities of implementing of wheeled platforms [1] and of different kinds of robotic systems. The theme of the proposed research deals with the particular problems about control of wheeled platforms straight motions taking into account jerk restrictions under speeding-up from the state of rest, and this theme is in current interest, because of it is in agreement with the existed general trends in developing the robotic systems directed to extensions of their implementing.

First principal reason for motions jerks limiting is due to the requirements of motion smooth necessary for normal operating of different kinds of robotic systems [2], [3]. The motions smoothness and excluding the jerks can be required for example for delicate or dangerous cargoes transportation [4] as well as for providing the most accurate relative positioning of technological systems parts [5].

It is necessary to note that excluding the motions jerks requires implementing the mechanisms special designs [6], as well as implementing the special control algorithms [4], [5]. So, excluding the wheeled platforms motions jerks is the multidisciplinary problem, and it requires the corresponded developing both the mechanical design both the control systems which must be corresponded with the existed imperfections of the mechanical joints due to the friction and the clearances.

Second principal reason for limiting the motions jerks is due to the motion smooth requirements necessary to provide the normal operation conditions for the sensitive components of on-board measuring systems [7], [8], including the sensors and the complementary electronic devices like analog-to-digital converters and computers for real time processing of the measured signals. Really, motions jerks have influence on on-board sensors like accelerometers or tachometers, and this influence is equivalent to noises disturbing measured signals used for positioning and defining current state parameters like velocities and accelerations [8]. Due to these circumstances, the motions jerks can lead to failures in positioning, in velocities and accelerations defining and in control of planed paths. As the result of all these, normal operation can be broken, and, furthermore, a lot of different dangerous can be created especially in using the fully autonomous wheeled platforms. So, defining the admissible motions jerks providing the normal operation of the wheeled platforms taking into account influencing on on-board measuring systems is the complicated problem required multidisciplinary approaches providing opportunities to consider the interactions between the mechanical, electromechanical and electronic parts [4], [8]. It is naturally that the motions jerks are associated with the accelerations and their changes like was discussed in the research [4] for example, so the quantitative measures of the motions jerks are based on using accelerations and their first and higher derivatives [9]. At the same time, the mechanical motions are represented by the differential equations of second orders, so researching the accelerations derivatives is the special separate problem [10].

To research the wheeled platforms motions jerks it is necessary to have some general methodology which will allow considering different causes leading to the jerks. There are a lot of causes leading to the wheeled platforms motions jerks [1], and it is necessary to research all of them, but it is the complicated problem not for one research. It is well-known [1] that the jerks are inherent especially for transient modes of wheeled platforms motions. Thus, the purpose of this research is in considering the particular problem about control of wheeled platforms straight motions on the ideal horizontal plane taking into account jerk restrictions under speeding-up from the state of rest. It is understood that the speeding-up is the particular case of transient modes of wheeled platforms, and jerks will be necessarily presented on this mode. Choosing the state of rest as the initial state is to simplify formulating the initial conditions, and such simplification is suitable for obtaining the primary results for planning the further researches in the field of the motion control under jerks restrictions. To realize the purpose of the research the follows tasks will be considered:

• the generalized approaches to define the controls satisfying the straight motions jerks restrictions of wheeled platforms will be developed for the speeding-up from the state of rest modes;

• the examples of the proposed approaches using to define the permissible controls satisfying the jerks restrictions under speeding-up from the state of rest will be considered for the pure mechanical and electromechanical wheeled platforms;

• computer simulations of the processes of speeding-up from the state of rest will be executed for the electromechanical wheeled platforms to show the results correctness and to illustrate satisfying the restrictions of the motions jerks.

Developing all noted above tasks will allow giving the clear imaginations about the proposed generalized approaches and their using in the important particular cases, as well as it will allow illustrating the influence of the researched control processes on the motions jerks for the wheeled platforms under speeding-up from the state of rest.

2. Generalized approaches

Developing the generalized approaches is more suitable than developing the particular approaches for each particular task. The generalized approaches to define the controls satisfying the wheeled platforms jerks restrictions under speeding-up modes from the state of rest are reduced to mathematical modelling of the researched modes and to resolving the formulated restrictions. The mathematical modelling of the wheeled platforms speeding-up modes will be considered under the most generalized assumptions that the researched wheeled platforms can be reduced to the holonomic systems. It is really the serious simplification because of the nonholonomic constraints are inherent for the wheeled platforms in general, but we have the hope that considering the particular case of the straight motions under speeding-up modes from the state of rest allow reducing to the holonomic systems.

It is well-known [11], [12] that the state of the holonomic systems can be defined by using the generalized coordinates:

$$q_k = q_k(t), k = 1, 2, .., N, \tag{1}$$

where $q_k, k = 1, 2, .., N$ are the generalized coordinates; N is the number of the freedom degrees of the holonomic system; $t \geq 0$ is the time.

It is necessary to note that not all generalized coordinates (1) will have the mechanical sense like linear displacements or angles, and some of these coordinates (1) can have the electrical sense like the electrical charges in the case of the electromechanical wheeled platforms. The translational straight motions of the wheeled platform can be imagined as the motions of its mass center, and it can be represented in the natural coordinates, so that we will have for the holonomic system the follows relation:

$$s = s(q_1, q_2, ..., q_N), \tag{2}$$

where s is the length of the arc of the trajectory of the mass center of the considered wheeled platform.

It is not unexpectedly to define the jerk as the time derivative of the acceleration and as the time third derivative of the coordinate:

$$j = \frac{d^3 s}{dt^3}, \tag{3}$$

where j is the estimation of the jerk of the motion of the considered wheeled platform.

Taking into account the used estimation of the motion jerk (3) and the relations (2), (1), we will have the follows:

$$j(t) = \sum_{k=1}^{N} \left(\sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^3 s}{\partial q_k \partial q_i \partial q_j} \frac{dq_k}{dt} \frac{dq_i}{dt} \frac{dq_j}{dt} + 3 \sum_{i=1}^{N} \frac{\partial^2 s}{\partial q_k \partial q_i} \frac{dq_k}{dt} \frac{d^2 q_i}{dt^2} + \frac{\partial s}{\partial q_k} \frac{d^3 q_k}{dt^3} \right). \tag{4}$$

Relation (4) shows that the jerks of the translational motions of the wheeled platforms are depended on the generalized velocities, generalized accelerations and the generalized accelerations time derivatives as well as on the building of the wheeled platform.

The Lagrange equations of second kind give us one of the most general form of the differential equations of dynamics of holonomic systems representing the different kinds of wheeled platform under the different operational modes:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = -\frac{\partial \mathcal{R}}{\partial \dot{q}_k} + Q_k, \ \ k = 1, 2, .., N, \tag{5}$$

where $\mathcal{L}$ is the Lagrange function defined as difference between the kinetic and potential energies of the considered wheeled platform; $\dot{q}_k \equiv dq_k/dt$; $\mathcal{R}$ is the generalized Raleigh function defining all the dissipation for the considered wheeled platform; Q_k are the generalized forces corresponding with the relevant generalized coordinates and defining all the driving forces and couples of the considered wheeled platform.

The equations (5) are the differential equations of second order, so the assumption about the initial state of rest for the considered wheeled platform allows formulating the initial conditions:

$$q_k(0) = 0, \ \ \dot{q}_k = 0, \ \ k = 1, 2, .., N. \tag{6}$$

Thus, the differential equations (5) with the initial conditions (6) generally represent the mathematical model of motion from the state of rest of the wheeled platform considered under the restrictions leading to the correspondent holonomic system with the generalized coordinates (1).

Taking into account the purpose of the research, we will consider further the transient modes from the initial state (6) to some state of uniform motion with the relative small velocity allowing the linearization of the differential equations (5) of the dynamic of the wheeled platform which is considered as the holonomic system.

Such linearization will allow represent the Lagrange $\mathcal{L}$ and Raleigh functions $\mathcal{R}$ in the follows form:

$$\mathcal{L} = \frac{1}{2}\sum_{k=1}^{N}\sum_{i=1}^{N} m_{ki}\dot{q}_k\dot{q}_i - \frac{1}{2}\sum_{k=1}^{N}\sum_{i=1}^{N} c_{ki}q_k q_i, \tag{7}$$

$$\mathcal{R} = \sum_{k=1}^{N} f_k\dot{q}_k + \frac{1}{2}\sum_{k=1}^{N}\sum_{i=1}^{N} \beta_{ki}\dot{q}_k\dot{q}_i, \tag{8}$$

where m_{ki} and c_{ki} are the generalized inertia and stiffness constant parameters of the considered wheeled platform parts; f_k are the parameters defining the non-viscous frictions not depending on the velocities; β_{ki} are the generalized damping parameters satisfying the conditions: $\beta_k \geq 0$, $\beta_{ki} = \beta_{ik}$ and $\sum\limits_{k=1}^{N}\sum\limits_{i=1}^{N} \beta_{ki}\dot{q}_k\dot{q}_i \geq 0$ and defining the linearized viscous damping.

It is naturally to imagine that motion control of the wheeled platforms is realized thru the driving generalized forces. We will assume that the control of the wheeled platform can be reduced to one time depended function:

$$u = u(t), \tag{9}$$

where u is the parameter defining the control influence on the considered wheeled platform.

The assumption (9) limits the possible class of the considered wheeled platforms, but this theoretically limited class can represent the most of actually existed and widely used wheeled platforms. Really, each wheeled platform has the energy source, the transmission as well as the drive and supporting wheels, so that the state of the energy source naturally defines the state of the wheeled platform. Although, the physical essentials of the power produced by the energy source is significantly depended on the type and on the design of the energy source, but it is more principally for us to define the state of the energy source by the power supplied to the transmission to move the drive wheels of the wheeled platform. Due to the noted here circumstances, the assumption (9) seems as the natural because of we have only one principal parameter defining the state of the considered wheeled platform and this parameter is the power supplied from the energy source to the transmission. Of course, the supplied power can be defined by other parameters like the torque, the position of the fuel valve or the voltage supplied to drive electric motors. Exactly, the noted case is the typical for the most of existed and used wheeled platforms. Considering the transient modes from the initial state (6) to some state of the motion with the relative small velocity is in agreement with the purposes of this research, and it allows linearization of the differential equations (5) of the dynamics of the wheeled platform which is considered as some holonomic system. Thus, the driving generalized forces can be represented taking into account the assumption (9) in the follows linearized view:

$$Q_k = \sum_{i=1}^{N} \alpha_{ki}\dot{q}_i + b_k u(t), k = 1, 2, .., N, \tag{10}$$

where α_{ki} are the parameters defining the linearized velocity depending of the driving generalized forces, but b_k are the constant parameters characterizing the sensitivity of the control of the considered wheeled platform.

Taking into account the relations (7), (8) and (10) in the Lagrange equations of second kind (5), we will have the follows linearized differential equations representing the dynamics of the considered wheeled platform:

$$\sum_{i=1}^{N} m_{ki}\ddot{q}_i = -\sum_{i=1}^{N} c_{ki}q_i - \sum_{i=1}^{N} d_{ki}\dot{q}_i - f_k + b_k u(t), k = 1, 2, .., N, \tag{11}$$

where $d_{ki} = \beta_{ki} - \alpha_{ki}$.

Further, it will be suitable to have the vector-matrix representation of the differential equations (11), and to have this representation, we will introduce the follows vectors and matrices:

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{pmatrix}, \bar{\mathbf{f}} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{pmatrix}, \bar{\mathbf{b}} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix},$$
$$\mathbf{M} = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1N} \\ m_{21} & m_{22} & \cdots & m_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ m_{N1} & m_{N2} & \cdots & m_{NN} \end{pmatrix}, \mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1N} \\ c_{21} & c_{22} & \cdots & c_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ c_{N1} & c_{N2} & \cdots & c_{NN} \end{pmatrix}, \tag{12}$$
$$\mathbf{D} = \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1N} \\ d_{21} & d_{22} & \cdots & d_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ d_{N1} & d_{N2} & \cdots & d_{NN} \end{pmatrix}.$$

The introduced above vectors and matrices (12) allow representing the differential equations (11) and the initial conditions (6) in the suitable vector-matrix form:

$$\mathbf{M}\ddot{\mathbf{q}} = -\mathbf{C}\mathbf{q} - \mathbf{D}\dot{\mathbf{q}} - \bar{\mathbf{f}} + \bar{\mathbf{b}}u(t),\ \mathbf{q}(0) = \mathbf{0},\ \dot{\mathbf{q}}(0) = \mathbf{0}, \tag{13}$$

where $\mathbf{0}$ is the zero vector having the correspondent dimension.

Solving the initial-value problem (13) will give the opportunities to find the jerks (4) corresponded to the given control (9), so in the form (13) we have the mathematical model of the considered wheeled platform representing its dynamical properties which must be taken into account to design the controls satisfying the motions jerks restrictions. We will consider further one of the principal kinds of the control (9) defined by the constant:

$$u(t) = u_c, \tag{14}$$

where $u_c > 0$ is the given constant corresponded to some quasi-stationary mode of the motions of the considered wheeled platform characterized by the constant velocity of its mass center.

Considering the particular case (14) of the control (9) is really very important from the point of view on designing the control of wheeled platforms speeding-up from the state of rest taking account the motion jerks restriction. Really, the motions defined by the differential equations and the initial conditions (13) for the control (14) represent the transient characteristics of the considered wheeled platform, and exactly these transient characteristics define the transient processes including the jerks during speeding-up of the wheeled platform from the state of rest. It is naturally to assume that the maximum jerks of the wheeled platform are at the beginning of the motions, because of exactly in this moment the motion is created from the state of rest, and further we will have only increasing of the velocity of the already existed motion, until this velocity will achieve the steady value, corresponded to the control (14). Taking into account the initial conditions (6), the relation (4) allows defining the wheeled platform jerk at the initial time of the speeding-up process:

$$j(0) = j_0, \; j_0 = \sum_{k=1}^{N} j_k \tfrac{d^3 q_k}{dt^3}(0), \tag{15}$$

where j_0 is the jerk at the initial time; $j_k = \frac{\partial s}{\partial q_k}\Big|_{\substack{q_i=0\\ i=1,2,\dots,N}}$.

To restrict the jerks of the considered wheeled platform it is naturally to limit the initial jerk (15):

$$|j_0| \le [j]\,, \tag{16}$$

where $[j] \ge 0$ is the admissible jerk of the considered wheeled platform.

Considering the transient process (13) during the wheeled platform speeding-up for the control (14) will allow defining the control satisfying the jerk restriction (16), but to do this it is principally more suitable to represent the mathematical model (13) representing the considered wheeled platform in the form of the system of first ordered differential equations. To represent the second ordered differential equations (13) as the system of the first ordered differential equations we will introduce the follows phase state space:

$$x_1 = q_1, \; x_2 = q_2, \dots, \; x_N = q_n, \; x_{N+1} = \dot{q}_1, \; x_{N+2} = \dot{q}_2, \dots, \; x_{2N} = \dot{q}_n, \tag{17}$$

where $x_k, k = 1, 2, ...2N$ are the phase coordinates.

It is suitable to represent the phase coordinate (17) as the vector:

$$\mathbf{x} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^T, \tag{18}$$

where $n = 2N$ is the dimension of the state phase space and T is the transpose operation symbol.

The introduced vector (18) and the assumption (14) about the control allow representing the differential equations and the initial conditions (13) in the follows suitable form:

$$\tfrac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} - \mathbf{f} + \mathbf{b}u_c, \; \mathbf{x}(0) = \mathbf{0}, \tag{19}$$

where $\mathbf{A}$ is some matrix, $\mathbf{f}$ and $\mathbf{b}$ are some vectors; $\mathbf{0}$ is the zero vector.

Comparing the equation (19) and the equation (13) allow us to write the matrix $\mathbf{A}$ and the vectors $\mathbf{f}$ and $\mathbf{b}$ included in the equation (19):

$$\mathbf{A} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{C} & -\mathbf{M}^{-1}\mathbf{D} \end{pmatrix}, \mathbf{f} = \begin{pmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\bar{\mathbf{f}} \end{pmatrix}, \mathbf{b} = \begin{pmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\bar{\mathbf{b}} \end{pmatrix}, \tag{20}$$

where $\mathbf{O}$ and $\mathbf{I}$ are the zero and unit matrix, but $\mathbf{0}$ is the zero vector of the correspondent dimensions.

Taking into account the introduced above vector (17), (18), the initial jerk (15) of the wheeled platform can be represented in the follows view:

$$j_0 = \mathbf{j}\frac{d^2\mathbf{x}}{dt^2}(0),\ \mathbf{j} = \begin{pmatrix} \underbrace{0 \quad 0 \quad \cdots \quad 0}_{N} & j_1 & j_2 & \cdots & j_N \end{pmatrix}. \tag{21}$$

Solution of the initial-value problem (19), (20) and the relation (21) allow finding the initial jerk j_0 required for the jerk restriction (16) of the considered wheeled platform. Really, the solution of the problem (19) can be represented in the follows form:

$$\mathbf{x}(t) = \left(e^{\mathbf{A}t} - \mathbf{I}\right)\left(\mathbf{A}^{-1}\left(\mathbf{b}u_c - \mathbf{f}\right)\right). \tag{22}$$

The solution (22) and the relation (21) allow finding the initial jerk of the motion for the considered wheeled platform:

$$j_0 = \mathbf{jA}\left(\mathbf{b}u_c - \mathbf{f}\right). \tag{23}$$

Relation (23) and the the restriction (16) will allow defining the control (14) and representing this control thru the primary linearized differential equations (13). To do this, it is necessary to take into account the relations (20) and (21), so the result of all these will lead to the restriction of the control (14) in the follow view:

$$\left|\bar{\mathbf{j}}\mathbf{M}^{-1}\mathbf{D}\mathbf{M}^{-1}\bar{\mathbf{f}} - \left(\bar{\mathbf{j}}\mathbf{M}^{-1}\mathbf{D}\mathbf{M}^{-1}\bar{\mathbf{b}}\right)u_c\right| \le [j], \tag{24}$$

where $\bar{\mathbf{j}} = \begin{pmatrix} j_1 & j_2 & \cdots & j_N \end{pmatrix}$

The relation (24) is actually gave the restriction of the considered wheeled platform control (14) providing speeding-up from the state of rest under the limited motion jerks. We can see from the relation (24) that the jerks can be only due to existing the linear dissipative and gyroscopic generalized forces, because the zero matrix $\mathbf{D}$ allows satisfying the jerk restriction (24) for any control (14). These dissipative forces are usually the result of the aerodynamic and hydrodynamic frictions; the Coriolis forces are the example of the gyroscopic forces.

The constant generalized forces of the wheeled platform are represented by the vector $\bar{\mathbf{f}}$ and are had the significant influencing on the motions jerks. These constant generalized forces are usually for example the gravity forces acting on the wheeled platforms moved on the inclining road or the rolling friction couples of the wheels interacting with the soil.

3. Examples

The developed approaches reduced to the inequality (24) for control of the straight motion under speeding-up from the state of rest mode taking into account the jerks restrictions can be used for different kinds of the wheeled platforms. Further, we will illustrate the mechanics foundations of the developed approaches as well as we will consider the particular application of these developed approaches deals with the control of autonomous electromechanical wheeled platform.

Example 1. The simple schematization (fig. 1a) of the four-wheeled platform will be considered firstly to illustrate the mechanical foundations of the proposed approaches reduced to the inequality (24). This schematization (fig. 1a) is based on the assumption (1) about the generalized coordinates, and in this particular case it will be assumed that the straight motion of the considered four-wheeled platform can be defined by one generalized coordinate q_1 representing the rotation angle of its wheels, so the straight motion can be defined as follows (fig. 1a):

$$s = q_1 r, \tag{25}$$

where s is the linear coordinate defining the straight motion; q_1 is the rotation angle and r is the radius of the wheels of the considered platform.

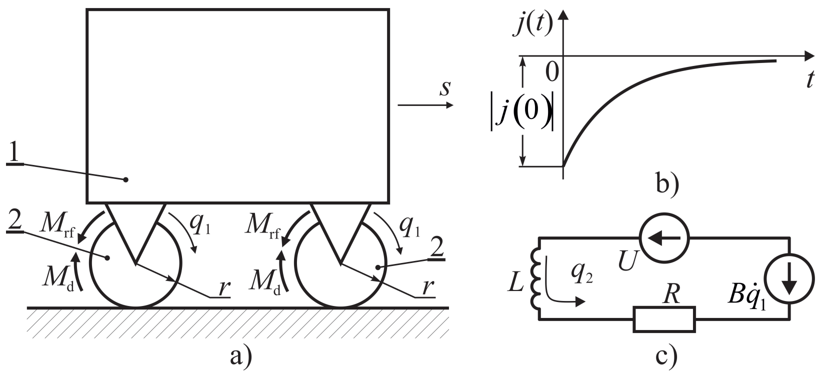

Fig. 1. Schematizing of the four-wheeled platform (a) with the housing-1 and the wheels-2, as well as the result for the jerk (b) of this platform and the equivalent scheme of the drive electric motors (c)

The relation (25) actually is the particular case of the generalized form relation (2), so the relation (4) defining the jerk (3) will have the more simple view:

$$j(t) = r\tfrac{d^3 q_1}{dt^3}. \tag{26}$$

For the assumed schematization (fig. 1a) of the considered four-wheeled platform we will have the follows Lagrange function $\mathcal{L}$, the Raleigh function $\mathcal{R}$ and the driving generalized force Q_1:

$$\mathcal{L} = \tfrac{1}{2} J \dot{q}_1^2, J = m_p r^2 + 4J_w, \tag{27}$$

$$\mathcal{R} = 4M_{rf}\dot{q}_1 + \tfrac{1}{2}\beta\dot{q}_1^2, \tag{28}$$

$$Q_1 = 4M_d,\ u_c = M_d, \tag{29}$$

where m_p is the total mass and J_w is the inertia moment of the wheel of the considered platform (fig. 1a); $M_{rf} = const$ is the rolling friction couple; β is the parameter defining the viscous linear damping; M_d is the driving couple acting to each of the wheels.

The relations (27)-(29) and the Lagrange equations (5) with the assumed initial conditions (6) in the considered case of the system with one freedom degree ($N = 1$) allow writing the follows differential equation and the initial conditions:

$$J\ddot{q}_1 + \beta\dot{q}_1 = 4\left(M_d - M_{rf}\right),\ q_1(0) = 0,\ \dot{q}_1(0) = 0. \tag{30}$$

Solution of the Cauchy linear problem (30) can be represented in the follows view:

$$q_1(t) = \tfrac{4}{\beta}\left(M_d - M_{rf}\right)\left(t - \tfrac{J}{\beta}\left(1 - e^{-\frac{\beta}{J}t}\right)\right). \tag{31}$$

The solution (32) allows finding the jerk of the considered wheeled platform using the relation (26):

$$j(t) = -\tfrac{4\beta r}{J^2}\left(M_d - M_{rf}\right)e^{-\frac{\beta}{J}t}. \tag{32}$$

Solution (32) shows (fig. 1b) that the maximal jerk of the motion is in the initial time moment corresponding to the beginning of speeding-up of the considered wheeled platform from the state of rest, and this circumstance in the full agreement with the previously used limitation of the jerks which was represented by the inequality (16). Thus, the maximal jerk of the considered wheeled platform (fig. 1b) can be defined by the relation (32) at the initial time moment $t = 0$:

$$j(0) = -\tfrac{4\beta r}{J^2}\left(M_d - M_{rf}\right). \tag{33}$$

Due to the relation (33), it is possible to have the particular representation of the generalized inequality (16):

$$\tfrac{4\beta r}{J^2}\left|M_d - M_{rf}\right| \leq [j]. \tag{34}$$

To provide the motion of the considered wheeled platform it is necessary to satisfy the follows relation:

$$M_d \geq M_{rf}. \tag{35}$$

Due to the inequalities (34) and (35), it is possible to have the condition on the driving couple:

$$M_d \leq M_{rf} + \tfrac{J^2}{4\beta r}[j]. \tag{36}$$

The inequality (36) allows choosing the driving couple which will provide the admissible jerks of the motion of the wheeled platform speeding up from the state of rest. The inequality (36) shows that the rolling friction and the viscous damping are the principal causes of the jerks of the wheeled platforms under speeding up from the state of rest. Besides, the obtained result (34) is the illustration of the generalized approaches reduced to the inequality (24). Really, the result (34) can be obtained by using the generalized inequality (24), if are will be assumed the follows:

$$\bar{\mathbf{j}} = (r),\ \mathbf{M} = (J),\ \mathbf{D} = (\beta),\ \bar{\mathbf{f}} = (4M_{rf}),\ \bar{\mathbf{b}} = (4). \tag{37}$$

Example 2. In the previously considered example, the control was reduced to the drive couple (29) immediately acting on the wheel. At the same time, the drive couples are often the results of some power source operating, and it is possible only the indirect control of the drive couples due to the controlling of the power source state. This circumstance make more difficult the wheeled platforms control under the motions jerks restrictions because of the power sources have the own inherent properties and can have additional influence on the wheeled platforms. To show this, we will consider the same four-wheeled platform (fig. 1a), but driving by means the direct current electric motors schematized as shown on the fig. 1c. In this case the generalized coordinate q_2 representing the electric charge in the equivalent electric circuits of the electric motors actually defines the state of the drive electric motors, and the voltage $U = U(t)$ supplied to the each of these drive electric motor actually controls the drive couple M_d on the wheels. So, the Lagrange function, the generalized Raleigh function and the generalized forces representing the four-wheeled platform (fig. 1a) with the driving electric couples (fig. 1c) on each of the wheels will have the follows view:

$$\mathcal{L} = \tfrac{1}{2}J\dot{q}_1^2 + \tfrac{1}{2}4L\dot{q}_2^2, \tag{38}$$

$$\mathcal{R} = 4M_{rf}\dot{q}_1 + \tfrac{1}{2}\beta\dot{q}_1^2 + \tfrac{1}{2}4R\dot{q}_2^2, \tag{39}$$

$$Q_1 = 4M_d,\ M_d = B\dot{q}_2,\ Q_2 = 4(U - B\dot{q}_1),\ u_c = U, \tag{40}$$

where L is the inductance, R is the resistance of the equivalent electric circuit and B is the electromechanical parameter of the drive direct current electric motor; U is the supplied voltage on the drive electric motors.

The relations (38)-(40) and the Lagrange equations (5) with the assumed initial conditions (6) in the considered case of the system with two freedoms degree ($N = 2$) allow writing the follows differential equations and the initial conditions:

$$J\ddot{q}_1 = -\beta\dot{q}_1 + 4B\dot{q}_2 - 4M_{rf}, 4L\ddot{q}_2 = -4B\dot{q}_1 - 4R\dot{q}_2 + 4U, \tag{41}$$

$$q_1(0) = 0,\ q_2(0) = 0,\ \dot{q}_1(0) = 0,\ \dot{q}_2(0) = 0. \tag{42}$$

The differential equations (41) with the initial conditions (42) can be represented in the generalized form (13) in which we will have the follows vectors and matrices:

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \bar{\mathbf{f}} = \begin{pmatrix} 4M_{rf} \\ 0 \end{pmatrix}, \bar{\mathbf{b}} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \tag{43}$$

$$\mathbf{M} = \begin{pmatrix} J & 0 \\ 0 & 4L \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} \beta & -4B \\ 4B & 4R \end{pmatrix}. \qquad (44)$$

Substituting the vectors (43) and matrices (44) to the inequality (24) allows obtaining the limitation on the supplied voltage on the drive electric motors providing the required restriction of the jerk of the straight motion under the speeding-up of the four-wheeled electromechanical platform:

$$j_0 = \tfrac{4r}{J}\left(\tfrac{\beta}{J}M_{rf} + \tfrac{B}{L}U\right), \ \tfrac{4r}{J}\left|\tfrac{\beta}{J}M_{rf} + \tfrac{B}{L}U\right| \le [j]. \qquad (45)$$

The results (34) and (45) allow showing that increasing the inertia of the wheeled platform represented by the generalized inertia moment J leads to decreasing the straight motion jerks under speeding-up from the state of rest. So, in the case of importance of limiting the jerks it is necessary to increase the mass of the wheeled platform. The results (34) and (45) also showing that decreasing the radius of the wheels of the platform leads to decreasing the jerks of the straight motion under speeding-up from the state of rest. Both the results (34) and (45) show that the rolling friction will necessarily lead to the jerks. At the same time, the result (35) shows that choosing the drive couple allows provide any wished small jerk, even if the rolling friction is presented, but the result (45) shows that it is impossible to have any wished small jerks of the electromechanical wheeled platform, if the rolling friction is presented, and it is only possible to minimize the jerks. This difference in the results (34) and (45) is due to that the properties of the sources of the drive mechanical torque of the wheels are not considered in the result (34), but this was considered in the result (45). So, properties of the the power source have the significant influence on the control providing the jerks restrictions of the straight motion under speeding-up from the state of rest of the wheeled platform.

4. Computer simulations

Further, we will consider the computer simulation of the wheeled electromechanical platform defined by the mathematical model (41), (42). This computer simulation will be reduced to the numerical solving of the initial value problem (41), (42), which will be represented as the system of the first ordered differential equations with the initial conditions (19). To have the required representation (19) of the initial value problem (41), (42) we will use new variables (17) with the $N = 2$ generalized coordinates and the control $u_c = U$, as it was defined in the last relation (40). Thus, taking into account the relations (20), (43) and (44), we will have the vector $\mathbf{x}$, the matrix $\mathbf{A}$ as well as the vectors $\mathbf{f}$ and $\mathbf{b}$ defining the linear differential equations (19) in the follows view:

$$\mathbf{x} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^T, \qquad (46)$$

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\beta/J & 4B/J \\ 0 & 0 & -B/L & -R/J \end{pmatrix}, \ \mathbf{f} = \begin{pmatrix} 0 \\ 0 \\ 4M_{rf}/J \\ 0 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/L \end{pmatrix}. \qquad (47)$$

The involved in the differential equations (41) numerical parameters representing the characteristics of the wheeled electromechanical platform will be considered as follows:

$$J = 80\ \mathrm{kg} \cdot \mathrm{m}^2,\ r = 0,15\ \mathrm{m},\ \beta = 2,5\ \tfrac{\mathrm{kg} \cdot \mathrm{m}^2}{\mathrm{s}},\ M_{rf} = 515\ \mathrm{N} \cdot \mathrm{m}, \tag{48}$$

$$L = 2,6\mathrm{mH},\ R = 1,18\ \Omega,\ B = 4\ \tfrac{\mathrm{N} \cdot \mathrm{m}}{\mathrm{A}}. \tag{49}$$

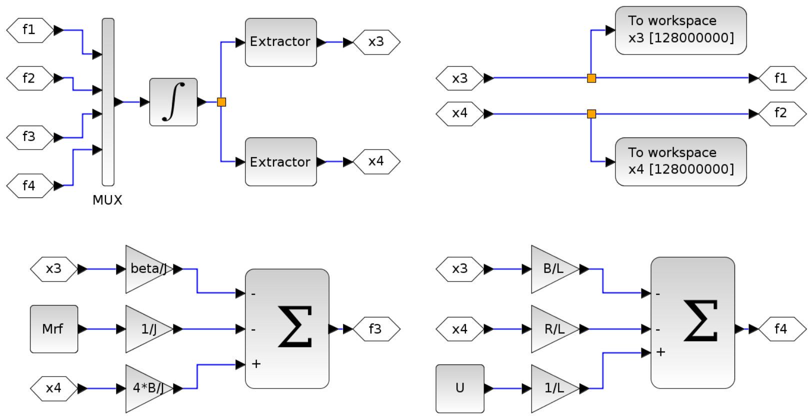

Fig. 2. Graphical representation of the model of the electromechanical wheeled platform in the Scilab free open source software

To solve the initial value problem (41), (42), (46)-(49) we will use the Scilab free open source scientific software in which we will use the especially designed graphical representation of the model of the considered electromechanical wheeled platform as shown on the fig. 2. This computer model (fig. 2) allows having different results, but further, we will consider only the follows:

$$v\left(t\right) = r x_3\left(t\right), \tag{50}$$

$$j\left(t\right) = r \tfrac{d^2 x_3}{dt^2}\left(t\right), \tag{51}$$

where v is the velocity and j is the jerk of the motion of the considered wheeled electromechanical platform.

Numerical solving of the initial value problem (41), (42), (46)-(49) allows having only the approximate solution for the $x_3\left(t\right)$, but this approximate solution will be close to the exact solution of this problem, so we can have the correct results for the velocity (50) of the considered wheeled platform. At the same time, it is well known that differentiation of the approximate solution $x_3\left(t\right)$ is incorrect in the Hadamard sense, and due to this we cannot have the correct results for the jerk of the considered electromechanical wheeled platform, if the formula (51) will be used directly. To exclude the Hadamard incorrectness to have the correct results for the jerk (51) it is necessary to represent the derivative $d^2 x_3/dt^2$ thru the $\mathbf{x}$

vector. It is not difficult in the considered example; really, taking into account the relation $x_3 = \dot{q}_1$ and first differential equation (41), we will have the follows relation:

$$\frac{d^2x_3}{dt^2} = -\frac{\beta}{J}\ddot{q}_1 + \frac{4B}{J}\ddot{q}_2. \tag{52}$$

Further, it is necessary to exclude the second derivatives of the generalized coordinates from the obtained relation (52) using the differential equations (41). All these and the definitions (17) will allow having the follows:

$$\frac{d^2x_3}{dt^2} = \left(\frac{\beta^2}{J^2} - \frac{4B^2}{JL}\right)x_3 - \left(\frac{4\beta B}{J^2} + \frac{4BR}{JL}\right)x_4 + \frac{4\beta M_{rf}}{J^2} + \frac{4B}{JL}U. \tag{53}$$

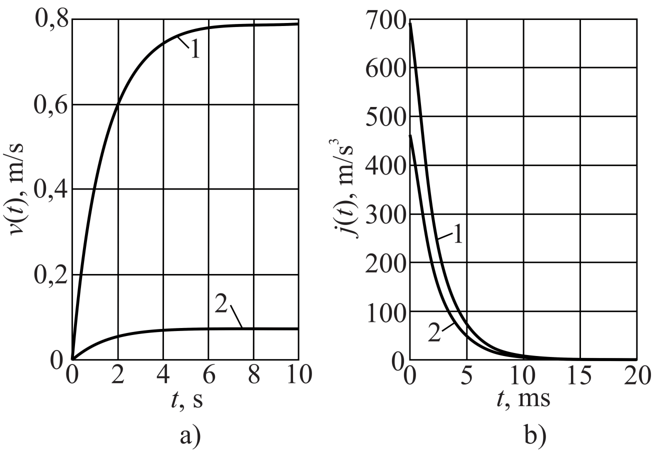

Fig. 3. Velocity (a) and jerk (b) of the electromechanical wheeled platform corresponded to the voltages $U = 60$ V (curve 1) and $U = 40$ V (curve 2) supplied on the drive electric motors

It is necessary to note, that instead the particular result (53) it is possible to use the generalized result obtained from the differential equations (19):

$$\frac{d^2x}{dt^2} = (\mathbf{AA})\,\mathbf{x} - \mathbf{Af} + (\mathbf{A})\,\mathbf{b}u_c. \tag{54}$$

The opportunities of representing the jerk thru the vector $\mathbf{x}$ in the general form (54) for the linearized problem (19) are really very important to exclude the differentiation of the $\mathbf{x}$ vector leading to the Hadamard incorrectness in the case of using the numerical methods for finding the $\mathbf{x}$ vector. The most interested quantitative results obtained by using the computer simulations (fig. 2) for the velocity (50) and for the jerk (51), (53) of the considered wheeled electromechanical platform are presented on the fig. 3. We can see (fig. 3a) that the velocity of the wheeled platform is directed to the maximum value corresponding to equilibrium between the viscous damping and the driving couples which are depended on the voltage supplied to the drive electric motors. This is in the full agreement with the well-known fundamental property inherent for the wheeled platforms. The results for the jerk (fig. 3b) show that the maximum value of the jerk is really at the initial time moment as was suggested before in the relations (16) and (24). The jerks values at the initial time moment (fig. 3b) obtained by using the computer simulations are in full agreement with the correspondent exact values defined theoretically by using first relation (45). Aspiration of the jerk's value to zero value during the time is in the agreement with aspiration of the acceleration value to zero. We can see (fig. 3b) the significant values of the jerks of the considered electromechanical wheeled platform due to instant voltage supplying on the drive electric motors at the initial time moment, and it is understandable that limiting of the value of the instantly supplied voltage cannot provide any given small jerks. Thus, to provide any small given jerks of the electromechanical wheeled platforms the smooth time's depending for the voltages supplying on the drive electric motors is required, and it is looked understandable.

Conclusion

The researches of the particular problem about control of wheeled platforms straight motions on the ideal horizontal plane taking into account jerk restrictions under speeding-up from the state of rest allowed obtaining some results, and due to these results it is possible to have the follows conclusions.

First of all, the generalized approaches to define the controls satisfying the straight motions jerks restrictions of wheeled platforms are developed for the modes of speeding-up from the state of rest. The jerks restrictions are reduced to limiting of the time derivative value of the wheeled platform acceleration. These generalized approaches based on the holonomic systems mechanics and on the electromechanical analogies allow considering the different kinds of the wheeled platforms taking into account the electric on-board systems like the drive electric motors and the control systems by using the Lagrange equations of second kind. Although, holonomic systems can represent only some particular motions of the wheeled platforms, but such particular cases are really important for solving the problems about the speeding-up and slowing-up straight motions of wheeled platforms. Considering the nongolonomic systems which can represent all the modes of the motions of wheeled platforms is planned for the future researches.

Secondly, the examples of the proposed approaches using to define the controls satisfying the jerks restrictions under speeding-up from the state of rest are considered for the pure mechanical and electromechanical wheeled platforms. It is obtained the inequality which allows choosing the instantly supplied driving mechanical couple which will provide the admissible motion jerks of the wheeled platform under speeding-up from the state of rest. It is shown, the rolling friction and the viscous damping are the principal causes of the motion jerks of the wheeled platforms under speeding-up from the state of rest. It is obtained the inequality defining the voltage instantly supplied on the drive electric motors which will provide the admissible motion jerks of the electromechanical wheeled platform under speeding-up from the state of rest, and it is shown that the proposed general approaches are suitable also for considering the jerks of different kinds of wheeled platforms.

Thirdly, the computer simulations of the processes of speeding-up from the state of rest for the electromechanical wheeled platform are considered to show the results correctness and to illustrate satisfying the motions jerks restrictions. The obtained results of the computer simulations are in the full agreement with the well-known fundamental property inherent for the wheeled platforms. The results for the jerk show that the maximum value of the jerk is really at the initial time moment as was suggested before, and it is noted that the jerks values at the initial time moment obtained by using the computer simulations are in full agreement with the correspondent exact values defined theoretically. The big values obtained for the jerks of the considered electromechanical wheeled platform are due to instant voltage supplying on the drive electric motors at the initial time moment, and it is understandable that limiting of the value of the instantly supplied voltage cannot provide any wished small jerks. To provide any wished jerks of the electromechanical wheeled platforms it is required to have the smooth time depending for the voltages supplying on the drive electric motors.

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Керування прямолінійним рухом колісних платформ з урахуванням обмежень на ривки при розганянні зі стану спокою

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Запропоновано узагальнену математичну модель процесу розганяння колісних платформ на ідеальній горизонтальній площині зі стану спокою та одержано керування, що задовольняє обмеження на ривки відповідних прямолінійних рухів. Розглянуті чисто механічна та електромеханічна колісні платформи, виконано комп'ютерне моделювання досліджуваних процесів. Узагальнені підходи засновані на механіці голономних систем та електромеханічних аналогіях, що дозволяють за допомогою

рівнянь Лагранжа другого роду розглядати різні типи колісних платформ з урахуванням електричних бортових систем, таких як приводні електродвигуни та системи керування. Хоча голономні системи відображають лише деякі окремі рухи колісних платформ, але такі окремі випадки дійсно важливі для розв'язування задач про прискорення та уповільнення рухів колісних платформ з урахуванням обмежень на ривки. Для суто механічних та електромеханічних колісних платформ розглянуто приклади використання запропонованих підходів для визначення допустимих керувань, що задовольняють обмеження на ривки при розганянні зі стану спокою. Отримано нерівність щодо визначення миттєво поданої ведучої механічної пари, яка забезпечить допустимі ривки руху колісної платформи, що прискорюється зі стану спокою. Показано, що тертя кочення та в'язкий опір є основними причинами ривків колісних платформ при розганянні зі стану спокою. Отримано нерівність, яка визначає електричну напругу, що миттєво подається на приводні електродвигуни та забезпечує допустимі ривки руху електромеханічної колісної платформи, що прискорюються зі стану спокою. Завдяки цьому показано, що запропоновані загальні підходи підходять також для дослідження колісних платформ різного типу. Розглядається комп'ютерне моделювання процесів розганяння зі стану спокою електромеханічних колісних платформ щоб мати підтвердження можливості використання запропонованих моделей та проілюструвати виконання обмежень на ривки під час рухів. Отримані результати комп'ютерного моделювання повністю узгоджуються з відомою фундаментальною властивістю, притаманною колісним платформам. Результати для ривків показують, що максимальне значення ривка дійсно є в початковий момент часу, як було запропоновано раніше, і показано, що значення ривків у початковий момент часу, отримані за допомогою комп'ютерного моделювання, повністю узгоджуються з відповідними значеннями, точно визначеними теоретично. Великі значення, отримані для ривків розглянутої електромеханічної колісної платформи, зумовлені миттєвою подачею напруги на приводні електродвигуни в початковий момент часу, і, зрозуміло, що обмеження величини миттєво поданої напруги не може забезпечити будь-яких бажаних невеликих ривків. Для забезпечення будь-яких невеликих бажаних ривків електромеханічних колісних платформ необхідно мати плавну залежить від часу напруг, що подають на електродвигуни приводу.
Ключові слова: **керування; рух; ривок; колісна платформа; математичне моделювання.**

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Homogeneous approximation for minimal realizations of series of iterated integrals

In the paper, realizable series of iterated integrals with scalar coefficients are considered and an algebraic approach to the homogeneous approximation problem for nonlinear control systems with output is developed. In the first section we recall the concept of the homogeneous approximation of a nonlinear control system which is linear w.r.t. the control and the concept of the series of iterated integrals. In the second section the statement of the realizability problem is given, a criterion for realizability and a method for constructing a minimal realization of the series are recalled. Also we recall some ideas of the algebraic approach to the description of the homogeneous approximation: the free graded associative algebra, which is isomorphic to the algebra of iterated integrals, the free Lie algebra, the Poincaré-Birkhoff-Witt basis, the dual basis and its construction by use of the shuffle product, the definition of the core Lie subalgebra, which defines the homogeneous approximation of a control system. In the third section we show how to find the core Lie subalgebra of the systems that is a realization of the one-dimensional series of iterated integrals without finding the system itself. The result obtained is illustrated by the example, in which we demonstrate two methods for finding the core Lie subalgebra of the realizing system. In the last section it is shown that for any graded Lie subalgebra of finite codimension there exists a one-dimensional homogeneous series such that this Lie subalgebra is the core Lie subalgebra for its minimal realization. The proof is constructive: we give a method of finding such a series; we use the dual basis to the Poincaré-Birkhoff-Witt basis of the free associative algebra, which is built by the core Lie subalgebra, and the shuffle product in this algebra. As a consequence, we get a classification of all possible homogeneous approximations of systems that are realizations of one-dimensional series of iterated integrals.

Keywords: **homogeneous approximation; series of iterated integrals; minimal realization; core Lie subalgebra.**

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1. Introduction

The homogeneous approximation problem has attracted great attention of experts in the control theory for several decades. We briefly recall the definition. In this paper we restrict ourselves to the class of control systems, which are linear w.r.t. the control, of the form

$$\dot{x} = \sum_{i=1}^{m} X_i(x)u_i, \tag{1}$$

where $X_1(x), \dots, X_m(x)$ are real analytic vector fields in a neighborhood of some point x^0. Under *homogeneous system* from this class we mean a system of the polynomial form

$$\dot{x}_k = \sum_{i=1}^{m} \sum \alpha^{ik}_{q_1 \cdots q_{k-1}} x_1^{q_1} \cdots x_{k-1}^{q_{k-1}} u_i, \quad \alpha^{ik}_{q_1 \cdots q_{k-1}} \in \mathbb{R}, \quad k = 1, \dots, n, \tag{2}$$

where the inner sum in the right hand side of (2) is taken over all integers $q_1, \dots, q_{k-1} \geq 0$ such that

$$q_1 w_1 + \cdots + q_{k-1} w_{k-1} + 1 = w_k,$$

and $1 \leq w_1 \leq \cdots \leq w_n$ are some integers called *weights of the coordinates* $x_1, \dots, x_n$. We note that a homogeneous system is *feedforward*, hence, if the controls $u_i(t)$ are known, then the components of the trajectory $x_k(t)$ can be found one by one by integrating known functions, without solving differential equations. It is convenient to deal with a coordinate-free definition. So, we say that *a system is homogeneous if it takes the form (2) after some change of variables.*

The concept of a homogeneous approximation can be introduced by different ways. Using coordinates, we can explain the definition as follows. Let us denote by $x(t;u)$ and $\widehat{x}(t;u)$ the trajectories of the systems (1) and (2) starting at x^0 and at the origin respectively and corresponding to the same control $u(t) = (u_1(t), \dots, u_m(t))$. We denote

$$U(1) = \{u(t) = (u_1(t), \dots, u_m(t)) : |u_i(t)| \leq 1,\ i = 1, \dots, m,\ t \in [0, 1]\}.$$

Finally, for any $u \in U(1)$, we denote by $u^{1/\theta}(t)$ the function $u^{1/\theta}(t) = u(t/\theta)$, $t \in [0, \theta]$ (i.e., $u^{1/\theta}(t)$ is obtained from $u(t)$ by "shrinking" its domain $[0, 1]$ to $[0, \theta]$).

We say that a system of the form (2) is *a homogeneous approximation* of the system (1) if there exists a change of variables $y = Q(x)$ such that $Q(x^0) = 0$ and for any $u(t) \in U(1)$

$$\theta^{-w_k}\left((Q(x(\theta; u^{1/\theta})))_k - \widehat{x}_k(\theta; u^{1/\theta})\right) \to 0 \quad \text{as} \quad \theta \to 0, \quad k = 1, \dots, n.$$

Informally, this means that after some change of variables trajectories of the initial system and of its approximation become equivalent at the origin for any fixed control.

Many results concerning homogeneous approximation exploited differential-geometric tools and language [3], [21], [1], [6], [2]; the results obtained within this approach were summarized in [10]. As an example of usage for a local analysis of a particular class of systems, we mention Goursat distributions [15].

Another fruitful way was initialized by M. Fliess [5]; it was based on interpreting control systems as formal series of noncommutative variables and used tools of free algebras [11], [13], [17], [18]; an overview can be found in [12]. Namely, instead of the system (1), one considers its trajectory as a *series of iterated integrals*

$$x(t; u) = x^0 + \sum \widetilde{c}_{i_1 \dots i_k} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{k-1}} u_{i_1}(\tau_1) \cdots u_{i_k}(\tau_k) d\tau_k \cdots d\tau_1,$$

where $\widetilde{c}_{i_1 \dots i_k} \in \mathbb{R}^n$ are expressed via values of the vector fields $X_i(x)$ and their derivatives at x^0. Therefore, $\widetilde{c}_{i_1 \dots i_k}$ are constant vectors. Iterated integrals are linearly independent functionals of u_i and, therefore, can be interpreted as a basis for a free associative algebra. We give more detailed explanations in the next section.

In [7], [19] a complete classification of homogeneous approximations was obtained. It turned out that a homogeneous approximation is defined by some Lie subalgebra in the free Lie algebra with m generators called *a core Lie subalgebra*, which is defined by the system. As an important benefit of the algebraic way of finding homogeneous approximations, we mention its convenience for computer realization [20].

In the present paper we study an algebraic description of homogeneous approximations for nonlinear control systems *with output*. More specifically, we consider series of iterated integrals with scalar coefficients

$$y(t; u) = y^0 + \sum c_{i_1 \dots i_k} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{k-1}} u_{i_1}(\tau_1) \cdots u_{i_k}(\tau_k) d\tau_k \cdots d\tau_1, \qquad (3)$$

where $c_{i_1 \dots i_k} \in \mathbb{R}$. The series (3) is called *realizable* if these exists a system of the form (1) and a function $y = h(x)$ such that $y(t; u) = h(x(t; u))$ admits the representation (3); it is known that the realization of the minimal possible dimension is unique up to a change of variables [9], [4], [8].

The main results of the paper can be outlined as follows. In Section 3 we show that the core Lie subalgebra of the minimal realization can be found without finding the realization itself, i.e. directly from the series (3). In Section 4 we prove the following classification theorem: any graded Lie subalgebra of finite nonzero codimension can serve as a core Lie subalgebra of a realizing system of a (homogeneous) series of the form (3).

2. Background

2.1. Realizability problem. The realizability problem for systems with output is well known. This problem deals with a description of the output behavior for analytic nonlinear control systems. Systems are represented as differential equations of the form (1) defined in some neighborhood of a point x^0, i.e., the vector fields $X_1(x), \dots, X_m(x)$ are defined and are analytic in a neighborhood of x^0. Let us consider also a function $y = h(x)$ that is defined in a neighborhood of x^0 and is analytic there.

We recall some basic concepts of the realizability theory. First we introduce some notation.

Below we denote by M the set of multi-indices

$$M = \{I = (i_1, \dots, i_k) : k \geq 1, \ \ 1 \leq i_1, \dots, i_k \leq m\}.$$

One of the most important concepts in this theory is the iterated integral, which is defined as follows

$$\eta_I(\theta, u) = \int_0^\theta \int_0^{\tau_1} \cdots \int_0^{\tau_{k-1}} u_{i_1}(\tau_1) \dots u_{i_k}(\tau_k) d\tau_k \dots d\tau_1.$$

It can be shown [5] that for any $\theta > 0$ iterated integrals are linearly independent as functionals on the set

$$U(\theta) = \{u(t) = (u_1(t), \dots, u_m(t)) : |u_i(t)| \leq 1, \ i = 1, \dots, m, \ t \in [0, \theta]\}.$$

We consider the set $\{\eta_I(\theta, u) : I \in M\}$ for an arbitrary fixed $\theta > 0$. Since the functionals $\eta_I(\theta, u)$ are linearly independent, they form a basis of some linear space. Then their linear span is a free associative algebra with the concatenation operation

$$\eta_{I_1}(\theta, u)\, \eta_{I_2}(\theta, u) = \eta_{I_1 I_2}(\theta, u);$$

we denote this algebra by $\mathcal{F}_\theta$. Note that for all $\theta > 0$ the algebras $\mathcal{F}_\theta$ are isomorphic to each other. Therefore, instead of the algebras $\mathcal{F}_\theta$, it is convenient to consider an abstract free algebra $\mathcal{F}$ isomorphic to all of them, which is generated by abstract independent elements $\eta_1, \dots, \eta_m$. Also let us consider the free Lie algebra $\mathcal{L}$ generated by $\eta_1, \dots, \eta_m$ with the bracket operation defined by $[a, b] = ab - ba$.

Below we use a unitary algebra $\mathcal{F}^e = \mathcal{F} + \mathbb{R}$ assuming that 1 is the unit in $\mathcal{F}^e$. In order to write elements from $\mathcal{F}$ and $\mathcal{F}^e$ in the same way, we complement M by the “empty index”,

$$M_0 = M \cup \{\varnothing\}$$

and assume that $\eta_\varnothing = 1$.

Now we can formulate the realizability problem from a formal point of view. Consider an arbitrary linear map

$$c : \mathcal{F}^e \to \mathbb{R}.$$

This map corresponds to a formal series S with scalar coefficients $c_I = c(\eta_I)$

$$S = \sum_{I \in M_0} c_I \eta_I. \tag{4}$$

Below we assume that *the map c is nontrivial*, i.e., $c(\mathcal{F}) \neq \{0\}$; then the series S has at least one nonzero term except a constant.

Definition 1. *The series (4) is called realizable if there exist vector fields $X_1(x), \ldots, X_m(x)$ and a function $h(x)$, which are analytic in some neighborhood of some point x^0, such that the functional $y(\theta; u) = h(x(\theta; u))$ where $x(\theta; u)$ is a solution of the Cauchy problem*

$$\dot{x} = \sum_{i=1}^{m} X_i(x) u_i(t), \; x(0) = x^0,$$

satisfies the equality

$$y(\theta; u) = \sum_{I \in M_0} c_I \eta_I(\theta, u).$$

In this sense, (1) is *a realizing system* for (4).

To formulate a realizability criterion, we recall the following definition.

Definition 2 ([4], [8]). *Let $\mathfrak{V}$ denote the linear space of formal series of the form (4). Consider the map $F_c : \mathcal{L} \to \mathfrak{V}$ of the form*

$$F_c(\ell) = \sum_{I \in M_0} c(\eta_I \ell) \eta_I, \quad \ell \in \mathcal{L}. \tag{5}$$

The Lie rank of a series S is defined by the equality

$$\rho_L(c) = \dim \{F_c(\ell) : \ell \in \mathcal{L}\}.$$

Now we are ready to recall the following criterion of realizability.

Theorem 1 ([4], [8]). *Suppose that the series $S = \sum\limits_{I \in M_0} c(\eta_I)\eta_I$ satisfies the following growth conditions,*

$$|c_I| \leq C_1 |I|! C^{|I|} \tag{6}$$

with some $C, C_1 > 0$, where by $|I|$ we denote the length of the multi-index I. The series S is realizable if and only if $\rho_L(c) < \infty$. In this case $n = \rho_L(c)$ is the minimal dimension of a realizing system. Moreover, a minimal realization (i.e., a realization of the minimal dimension) is unique up to a change of variables.

In the language of the associative algebra $\mathcal{F}$, the realizability condition can be formulated as follows.

Theorem 2. *Consider the free associative algebra $\mathcal{F}$ and the corresponding Lie algebra $\mathcal{L}$. A formal series S satisfying the growth condition (6) is realizable if and only if there exist a natural number n and elements $\ell_1, \dots, \ell_n \in \mathcal{L}$ satisfying the following condition: for any element $\ell \in \mathcal{L}$ there exist coefficients $\alpha_1, \dots, \alpha_n$ such that*

$$c(a(\ell - \sum_{i=1}^{n} \alpha_i \ell_i)) = 0$$

for any element $a \in \mathcal{F}^e$.

One of the ways to construct a minimal realizing system for a given series S is as follows [8]. Since the Lie rank is n, there exist n linearly independent elements $\ell_1, \dots, \ell_n \in \mathcal{L}$ for which the series $F_c(\ell_1), \dots, F_c(\ell_n)$ are linearly independent. Consider the coefficients of all possible elements of the form $\eta_I \ell_j$. As the series $F_c(\ell_j)$ are linearly independent, there exist n multi-indices $I_1, \dots, I_n \in M_0$ for which the matrix

$$\{c(\eta_{I_i} \ell_j)\}_{i,j=1}^{n} \tag{7}$$

is non-singular. We define the linear map $\widetilde{c} : \mathcal{F}^e \to \mathbb{R}^n$ by the equality

$$\widetilde{c}(\eta_I) = \begin{pmatrix} c(\eta_{I_1} \eta_I) \\ \cdots \\ c(\eta_{I_n} \eta_I) \end{pmatrix} \tag{8}$$

and consider the corresponding series

$$\widetilde{S} = \sum_{I \in M_0} \widetilde{c}(\eta_I) \eta_I \tag{9}$$

with n-dimensional coefficients. The unique system constructed by this series is a minimal realization of the series S.

2.2. Grading in the algebra $\mathcal{F}$ and homogeneous approximations of control systems. The free associative algebra $\mathcal{F}$ is *graded*, namely, it admits the following representation

$$\mathcal{F} = \sum_{k=1}^{\infty} \mathcal{F}^k, \quad \mathcal{F}^k = \mathrm{Lin}\{\eta_I : I \in M, \ |I| = k\}.$$

This grading is justified by the following observation, which concerns iterated integrals:

$$\int_0^\theta \int_0^{\tau_1} \cdots \int_0^{\tau_{k-1}} u_{i_1}(\tau_1) \cdots u_{i_k}(\tau_k) d\tau_k \cdots d\tau_1 =$$
$$= \theta^k \int_0^1 \int_0^{\tau_1} \cdots \int_0^{\tau_{k-1}} u_{i_1}(\tau_1 \theta) \cdots u_{i_k}(\tau_k \theta) d\tau_k \cdots d\tau_1.$$

Thus, $\eta_I(\theta, u^{1/\theta}) = \theta^{|I|}\eta_I(1,u)$, where $u^{1/\theta}(t) = u(t/\theta)$, $t \in [0,\theta]$. In this sense, $|I|$ denotes the order of $\eta_I(\theta, u^{1/\theta})$ as a function of θ as $\theta \to 0$.

The Lie algebra $\mathcal{L}$ inherits this grading,

$$\mathcal{L} = \sum_{k=1}^{\infty} \mathcal{L}^k, \quad \mathcal{L}^k = \mathcal{F}^k \cap \mathcal{L}.$$

Below we say that $a \in \mathcal{F}^k$ is *homogeneous* and k is its *order*; in this case we write $\operatorname{ord}(a) = k$.

Let us consider a series with vector coefficients of the form

$$\widetilde{S} = \sum_{I \in M_0} \widetilde{c}_I \eta_I, \tag{10}$$

where $\widetilde{c}_I \in \mathbb{R}^n$; it defines a linear map $\widetilde{c} : \mathcal{F}^e \to \mathbb{R}^n$ by $\widetilde{c}(\eta_I) = \widetilde{c}_I$. Assume that this map satisfies the Rashevsky-Chow condition

$$\widetilde{c}(\mathcal{L}) = \mathbb{R}^n. \tag{11}$$

Suppose also that the series $\widetilde{S}$ is realizable, that is, there exists a system of the form (1) such that its trajectory $x(\theta; u)$ is represented as $x(\theta; u) = \sum\limits_{I \in M_0} \widetilde{c}_I \eta_I(\theta, u)$. It can be shown that this system is unique, and the condition (11) means that the realizing system is locally controllable, i.e., the initial point x^0 belongs to the interior of the set of all points that are reachable from x^0 in a time $\theta > 0$.

The following definition takes into account the grading introduced above.

Definition 3 ([7],[19]). *Suppose the series (10) corresponds to the system (1). Let us define the subspaces*

$$\widetilde{\mathcal{P}}^1 = \{\ell \in \mathcal{L}^1 : \widetilde{c}(\ell) = 0\}, \quad \widetilde{\mathcal{P}}^k = \{\ell \in \mathcal{L}^k : \widetilde{c}(\ell) \in \widetilde{c}(\mathcal{L}^1 + \cdots + \mathcal{L}^{k-1})\}, \ k \geq 2,$$

and

$$\mathcal{L}_{X_1,\ldots,X_m} = \sum_{k=1}^{\infty} \widetilde{\mathcal{P}}^k.$$

Then $\mathcal{L}_{X_1,\ldots,X_m}$ is a graded Lie subalgebra; it is called a core Lie subalgebra of the system (1).

It can be shown that the core Lie subalgebra is of codimension n (in $\mathcal{L}$) and that it is invariant w.r.t. changes of variables in the system.

It turns out that the core Lie subalgebra is responsible for the homogeneous approximation of the system [7], [19]. Namely, two control systems of the form (1) have the same homogeneous approximation if and only if their core Lie subalgebras coincide. Moreover, any graded Lie subalgebra of codimension n is a core Lie subalgebra for some locally controllable system of the form (1).

In Section 3 we describe the core Lie subalgebra for a realizing system of a series of the form (4).

2.3. Basis in the algebra $\mathcal{F}$. Suppose $\{\ell_i\}_{i=1}^{\infty}$ is a (homogeneous) basis of $\mathcal{L}$. Then, due to the Poincaré-Birkhoff-Witt Theorem [16], the set

$$\{\ell_{i_1}^{q_1}\cdots\ell_{i_k}^{q_k} : k\ge 1,\ 1\le i_1<\cdots<i_k,\ q_1,\ldots,q_k\ge 1\} \tag{12}$$

is a (homogeneous) basis of $\mathcal{F}$, where $\ell^q=\ell\cdots\ell$ (q times).

Let us introduce the *inner product* in $\mathcal{F}$ assuming the basis $\{\eta_I : I\in M\}$ is orthonormed. Also, let us introduce the *shuffle product* in $\mathcal{F}$ by the recursive formula

$$\eta_i ш \eta_j=\eta_{ij}+\eta_{ji},$$

$$\eta_{i_1I_1} ш \eta_j=\eta_j ш \eta_{i_1I_1}=\eta_{i_1}(\eta_{I_1} ш \eta_j)+\eta_{ji_1I_1},$$

$$\eta_{i_1I_1} ш \eta_{i_2I_2}=\eta_{i_1}(\eta_{I_1} ш \eta_{i_2I_2})+\eta_{i_2}(\eta_{i_1I_1} ш \eta_{I_2})$$

for any $I_1,I_2\in M$. Denote by

$$\{d_{i_1\ldots i_k}^{q_1\ldots q_k} : k\ge 1,\ 1\le i_1<\cdots<i_k,\ q_1,\ldots,q_k\ge 1\} \tag{13}$$

a dual basis for (12) in the sense of the inner product introduced above. It can be shown [14] that

$$d_{i_1\ldots i_k}^{q_1\ldots q_k}=\frac{1}{q_1!\cdots q_k!}d_{i_1}^{шq_1} ш \cdots ш d_{i_k}^{шq_k},$$

where $d^{ш q}=d ш \cdots ш d$ (q times); here the notation $d_i=d_i^1$ is used for brevity. Therefore, we can rewrite the series S in the basis (13)

$$S=c(1)+\sum\frac{1}{q_1!\cdots q_k!}c(\ell_{i_1}^{q_1}\cdots\ell_{i_k}^{q_k})d_{i_1}^{шq_1} ш \cdots ш d_{i_k}^{шq_k},$$

where the sum is taken over all $k\ge 1$ and $1\le i_1<\cdots<i_k$, $q_1,\ldots,q_k\ge 1$. In Section 4 we apply an analogous representation to the series $F_c(\ell)$.

3. Description of the core Lie subalgebras of realizing systems

In this section we show that the core Lie subalgebra of a realizing system (Definition 3) can be found without finding the realizing system itself.

Theorem 3. *Let S be a realizable series of the form (4) and an n-dimensional system (1) be its minimal realization. Then the core Lie subalgebra $\mathcal{L}_{X_1,\ldots,X_m}$ of this minimal realization can be found in the following way:*

$$\mathcal{L}_{X_1,\ldots,X_m}=\sum_{k=1}^{\infty}\mathcal{P}^k,$$

where

$$\mathcal{P}^1=\left\{\ell\in\mathcal{L}^1 : c(a\ell)=0 \text{ for any } a\in\mathcal{F}^e\right\},$$

$$\mathcal{P}^k=\big\{\ell\in\mathcal{L}^k:\ \text{there exists } \ell'\in\mathcal{L}^1+\cdots+\mathcal{L}^{k-1} \text{ such that } c(a(\ell-\ell'))=0 \text{ for any } a\in\mathcal{F}^e\big\},\quad k\ge 2. \tag{14}$$

Proof. Take an element ℓ from the subspace $\mathcal{P}^k$. It suffices to show that this element also belongs to the subspace $\widetilde{\mathcal{P}}^k$. Let $\ell \in \mathcal{P}^k$, then by formula (14) there exists an element ℓ' belonging to the sum of subspaces $\mathcal{L}^1 + \cdots + \mathcal{L}^{k-1}$ such that the equality

$$c(a(\ell - \ell')) = 0 \tag{15}$$

holds for any element a from $\mathcal{F}^e$. As an element a, we take those elements η_{I_i} for which the matrix (7) is nonsingular. Since equality (15) holds for any element a then it is true that

$$c(\eta_{I_i}(\ell - \ell')) = 0, \quad i = 1, \ldots, n. \tag{16}$$

Consider the n-dimensional mapping (8), then

$$\widetilde{c}(\ell - \ell') = \begin{pmatrix} c(\eta_{I_1}(\ell - \ell')) \\ \ldots \\ c(\eta_{I_n}(\ell - \ell')) \end{pmatrix}.$$

Since the condition (16) holds for any row, then $\widetilde{c}(\ell - \ell') = 0$. This means that the element ℓ belongs to the subspace $\widetilde{\mathcal{P}}^k$.

Take an element ℓ from the subspace $\widetilde{\mathcal{P}}^k$. It suffices to show that this element also belongs to the subspace $\mathcal{P}^k$. By definition, $\widetilde{c}(\ell) \in \widetilde{c}(\mathcal{L}^1 + \cdots + \mathcal{L}^{k-1})$, therefore, there exists an element $\ell' \in \mathcal{L}^1 + \cdots + \mathcal{L}^{k-1}$ such that $\widetilde{c}(\ell - \ell') = 0$. This means that $c(\eta_{I_i}(\ell - \ell')) = 0$ for $i = 1, \ldots, n$. Since the series $F_c(\ell - \ell')$ is a linear combination of the series $F_c(\ell_1), \ldots, F_c(\ell_n)$, there exist the numbers $\alpha_1, \ldots, \alpha_n$ such that for any $I \in M_0$

$$c(\eta_I(\ell - \ell')) = \sum_{j=1}^{n} \alpha_j c(\eta_I \ell_j).$$

In particular, substituting $I = I_i$, for which the matrix (7) is nonsingular, we obtain the following equality

$$\begin{pmatrix} c(\eta_{I_1}(\ell - \ell')) \\ \ldots \\ c(\eta_{I_n}(\ell - \ell')) \end{pmatrix} = \sum_{j=1}^{n} \alpha_j \begin{pmatrix} c(\eta_{I_1}\ell_j) \\ \ldots \\ c(\eta_{I_n}\ell_j) \end{pmatrix} = 0.$$

Since the matrix (7) is non-singular, the vectors $(c(\eta_{I_1}\ell_j), \ldots, c(\eta_{I_n}\ell_j))^\top$ are linearly independent. Hence, all coefficients α_j are equal to zero. This means that

$$c(a(\ell - \ell')) = 0$$

for any element $a \in \mathcal{F}^e$, therefore, $\ell \in \mathcal{P}^k$. The theorem is proved.

Example. Let a one-dimensional series

$$S = \eta_1 + \eta_{21} + \eta_{211}$$

be given. Let us show that the Lie rank of this series is equal to 2. To do this, we write down all the nonzero series of the form (5):

$$\begin{aligned} F_c(\eta_1) &= 1+\eta_2+\eta_{21}, \\ F_c([\eta_1,\eta_2]) &= -1, \\ F_c([\eta_1,[\eta_1,\eta_2]]) &= 1. \end{aligned}$$

Since two of them are linearly independent, the Lie rank of S equals 2. We can choose $\ell_1 = \eta_1$, $\ell_2 = [\eta_1,\eta_2]$ and $I_1 = (\varnothing)$, $I_2 = (2)$, then the matrix (7) is nonsingular. Then we get n-dimensional series of the form (9)

$$\widetilde{S} = \begin{pmatrix} \eta_1+\eta_{21}+\eta_{211} \\ \eta_1+\eta_{11} \end{pmatrix}.$$

Using Definition 3, let us find the core Lie subalgebra for a realization of the n-dimensional series $\widetilde{S}$. Consider the subspace

$$\widetilde{\mathcal{P}}^1 = \left\{\ell \in \mathcal{L}^1 : \widetilde{c}(\ell) = 0\right\}.$$

We have $\mathcal{L}^1 = \mathrm{Lin}\{\eta_1,\eta_2\}$. For the elements η_1,η_2 we write down their coefficients

$$\widetilde{c}(\eta_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \widetilde{c}(\eta_2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{17}$$

Then, obviously, the space $\widetilde{\mathcal{P}}^1$ is a linear span of only one element η_2

$$\widetilde{\mathcal{P}}^1 = \mathrm{Lin}\{\eta_2\}.$$

For $k=2$ we get

$$\widetilde{\mathcal{P}}^2 = \left\{\ell \in \mathcal{L}^2 : \widetilde{c}(\ell) \in \widetilde{c}(\mathcal{L}^1)\right\}$$

and $\mathcal{L}^2 = \mathrm{Lin}\{[\eta_1,\eta_2]\}$. For the element $\ell = [\eta_1,\eta_2]$ we find

$$\widetilde{c}([\eta_1,\eta_2]) = \widetilde{c}(\eta_{12}-\eta_{21}) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Taking into account the form of the coefficients (17), we see that $\widetilde{c}(\ell) \notin \widetilde{c}(\mathcal{L}^1)$. That is, $\widetilde{\mathcal{P}}^2 = \{0\}$.

Therefore, $\dim(\widetilde{c}(\mathcal{L}^1+\mathcal{L}^2)) = 2$, which means that $\widetilde{\mathcal{P}}^k = \mathcal{L}^k$ for all $k \geq 3$. Thus, we have found the core Lie subalgebra for the n-dimensional series $\widetilde{S}$:

$$\mathcal{L}_{X_1,X_2} = \mathrm{Lin}\{\eta_2\} + \sum_{k=3}^{\infty} \mathcal{L}^k. \tag{18}$$

Now we show how to use Theorem 3 and find this core Lie subalgebra using only the one-dimensional series S. We write down the non-zero coefficients of this series:

$$c(\eta_1) = 1, \ c(\eta_{21}) = 1, \ c(\eta_{211}) = 1.$$

Consider the subspaces (14). For $k=1$ we have

$$\mathcal{P}^1 = \left\{\ell \in \mathcal{L}^1 : c(a\ell) = 0 \text{ for any } a \in \mathcal{F}^e\right\}.$$

First, as an element ℓ we take η_1. In particular, for $a=1$ we get $c(a\eta_1) = c(\eta_1) = 1$, hence, $\eta_1 \notin \mathcal{P}^1$. Now we choose $\ell = \eta_2$, then $c(a\eta_2) = 0$ for all $a \in \mathcal{F}$. This means that $\mathcal{P}^1 = \mathrm{Lin}\{\eta_2\}$. Now consider the subspace

$$\mathcal{P}^2 = \left\{\ell \in \mathcal{L}^2 : \text{there exists } \ell' \in \mathcal{L}^1 \text{ such that } c(a(\ell - \ell')) = 0 \text{ for any } a \in \mathcal{F}^e\right\}.$$

As an element ℓ, we take the bracket $[\eta_1, \eta_2] = \eta_{12} - \eta_{21}$, and $\ell' \in \mathcal{L}^1$ is a linear combination $\alpha\eta_1 + \beta\eta_2$, where α, β are numbers. In the definition (14) for $k=2$, we first take $a=1$. Then

$$c(a(\ell - \ell')) = c(\eta_{12} - \eta_{21} - \alpha\eta_1 - \beta\eta_2) = -1 - \alpha = 0,$$

which means that $\alpha = -1$. Now we choose $a = \eta_2$, which gives

$$c(a(\ell - \ell')) = c(\eta_{212} - \eta_{221} - \alpha\eta_{21} - \beta\eta_{22}) = -\alpha = 0,$$

hence, $\alpha = 0$. We have got a contradiction, therefore, $[\eta_1, \eta_2] \notin \mathcal{P}^2$. This means that $\mathcal{P}^2 = \{0\}$. Finally, we consider the subspace

$$\mathcal{P}^3 = \left\{\ell \in \mathcal{L}^3 : \text{there exists } \ell' \in \mathcal{L}^1 + \mathcal{L}^2 \text{ such that } c(a(\ell - \ell')) = 0 \text{ for any } a \in \mathcal{F}^e\right\}$$

and take into account that $\mathcal{L}^3 = \mathrm{Lin}\{[\eta_1, [\eta_1, \eta_2]], [\eta_2, [\eta_1, \eta_2]]\}$. First we take $\ell = [\eta_1, [\eta_1, \eta_2]] = \eta_{112} - 2\eta_{121} + \eta_{211}$ and $\ell' = \alpha\eta_1 + \beta(\eta_{12} - \eta_{21})$. Then for $a = 1$ we get

$$c(a(\ell - \ell')) = c(\eta_{112} - 2\eta_{121} + \eta_{211} - \alpha\eta_1 - \beta\eta_{12} + \beta\eta_{21}) = 1 - \alpha + \beta = 0$$

while for $a = \eta_2$ we get

$$c(a(\ell - \ell')) = c(\eta_{2112} - 2\eta_{2121} + \eta_{2211} - \alpha\eta_{21} - \beta\eta_{212} + \beta\eta_{221}) = -\alpha = 0.$$

This gives $\alpha = 0$ and $\beta = -1$, that is, $\ell' = -\eta_{12} + \eta_{21}$. One easily checks that $c(a\ell) = c(a\ell')$ for any $a \in \mathcal{F}^e$, hence, $[\eta_1, [\eta_1, \eta_2]] \in \mathcal{P}^3$. Using similar reasoning for the element $\ell = [\eta_2, [\eta_1, \eta_2]]$, we see that $c(a\ell) = 0$ for any $a \in \mathcal{F}^e$. This means that $\mathcal{P}^3 = \mathcal{L}^3$. Since $c(\mathcal{L}^k) = 0$ for $k \geq 4$, we get $\mathcal{P}^k = \mathcal{L}^k$. Thus, we have obtained the core Lie subalgebra (18) using only the initial one-dimensional series S.

One can check that a realization of the series S in a neighborhood of the point $x^0 = 0$ can be chosen in the following form

$$\begin{aligned}
&\dot{x}_1 = u_1 + x_2 u_2, \\
&\dot{x}_2 = \sqrt{1 + 2x_2}\, u_1, \\
&y = x_1,
\end{aligned}$$

that is, $X_1(x) = (1, \sqrt{1+2x_2})^\top$, $X_2(x) = (x_2, 0)^\top$, $h(x) = x_1$. As a homogeneous approximation for this system, we can choose a homogeneous system with the same core Lie subalgebra

$$\dot{x}_1 = u_1,$$
$$\dot{x}_2 = x_1 u_2.$$

We observe that manipulating with the series for finding the core Lie subalgebra is more convenient than with vector fields directly.

4. Description of all possible homogeneous approximations of realizing systems

In this section we show that any graded Lie subalgebra of finite nonzero codimension is the core Lie subalgebra of a realizing system of some series (3). We introduce such a series using the dual basis (13); the corresponding linear map is defined by formula (19) below. The following lemma describes one property of this map.

Lemma 1. *Suppose $\{\ell_i\}_{i=1}^\infty$ is a homogeneous basis of the Lie algebra $\mathcal{L}$. Let a linear map $c : \mathcal{F} \to \mathbb{R}$ be defined on the elements of the corresponding Poincaré-Birkhoff-Witt basis (12) as follows: for any $k \geq 1$ and any $1 \leq i_1 \leq \cdots \leq i_k$*

$$c(\ell_{i_1} \cdots \ell_{i_k}) = \begin{cases} 1 & \text{if } \ k = n \ \text{ and } \ (i_1, \ldots, i_n) = (1, \ldots, n), \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Consider any k-tuple $(j_1, \ldots, j_k)$ of natural numbers, where $1 \leq k \leq n$. Then

$$c(\ell_{j_1} \cdots \ell_{j_k}) = 0 \quad \text{if } 1 \leq k \leq n-1, \tag{20}$$

$$c(\ell_{j_1} \cdots \ell_{j_n}) = \begin{cases} 1 & \text{if } (j_1, \ldots, j_n) \text{ is a permutation of } \{1, \ldots, n\}, \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

Proof. Let us denote by $\mathrm{inv}(j_1, \ldots, j_k)$ the number of inversions in the tuple $(j_1, \ldots, j_k)$, i.e., the number of pairs (s', s'') such that $s' < s''$ and $j_{s'} > j_{s''}$. If $\mathrm{inv}(j_1, \ldots, j_k) = r$, then sorting the tuple in non-decreasing order requires r adjacent transpositions. Below we use the notation

$$N_{k,r} = \{(j_1, \ldots, j_k) : \mathrm{inv}(j_1, \ldots, j_k) = r\}, \quad k \geq 1, \ r \geq 0.$$

For any k the maximal possible number of inversions is $\frac{1}{2}k(k-1)$ (this number of inversions is achieved when the numbers in the tuple strictly decrease). Therefore, if $r > \frac{1}{2}k(k-1)$, then $N_{k,r} = \varnothing$. Hence, the set of all tuples of natural numbers can be represented as a union of the sets $N_{k,r}$ where $k \geq 1$, $0 \leq r \leq \frac{1}{2}k(k-1)$. We are interested in k such that $1 \leq k \leq n$.

We use induction on the set of pairs (k, r) such that $k \geq 1$, $0 \leq r \leq \frac{1}{2}k(k-1)$ ordered lexicographically. Namely, we assume

$$(k', r') < (k'', r'') \ \text{ if } \ k' < k'' \text{ or } k' = k'' \text{ and } r' < r''.$$

If $k = 1$, then the required equalities (20), (21) follow from (19).

If $2 \le k \le n$ and $(j_1, \dots, j_k) \in N_{k,0}$, then $j_1 \le \dots \le j_k$. Therefore, $\ell_{j_1} \cdots \ell_{j_k}$ belongs to the Poincaré-Birkhoff-Witt basis. Hence, equalities (20), (21) follow from (19).

Let us consider any pair (k, r) such that $2 \le k \le n$ and $1 \le r \le \frac{1}{2}k(k-1)$ and suppose that the equalities (20), (21) hold for any element $\ell_{q_1} \cdots \ell_{q_{k'}}$ where $(q_1, \dots, q_{k'}) \in N_{k',r'}$ and $(k', r') < (k, r)$. This means that $c(\ell_{q_1} \cdots \ell_{q_{k'}}) = 0$ except the case when $(k', r') = (n, r')$ and $\{q_1, \dots, q_{k'}\} = \{1, \dots, n\}$; in this case $c(\ell_{q_1} \cdots \ell_{q_{k'}}) = 1$.

Consider any $(j_1, \dots, j_k) \in N_{k,r}$. Since $r \ge 1$, there exists $1 \le s \le k-1$ such that $j_s > j_{s+1}$. Since

$$\ell_{j_s}\ell_{j_{s+1}} = [\ell_{j_s}, \ell_{j_{s+1}}] + \ell_{j_{s+1}}\ell_{j_s},$$

we can express

$$\ell_{j_1} \cdots \ell_{j_k} = a_1 + a_2,$$

where

$$a_1 = \ell_{j_1} \cdots \ell_{j_{s-1}}[\ell_{j_s}, \ell_{j_{s+1}}]\ell_{j_{s+2}} \cdots \ell_{j_k},$$

$$a_2 = \ell_{j_1} \cdots \ell_{j_{s-1}}\ell_{j_{s+1}}\ell_{j_s}\ell_{j_{s+2}} \cdots \ell_{j_k}.$$

First we consider a_1. Since the element $[\ell_{j_s}, \ell_{j_{s+1}}]$ belongs to the Lie algebra $\mathcal{L}$, it equals a linear combination of basis elements, $[\ell_{j_s}, \ell_{j_{s+1}}] = \sum \alpha_p \ell_p$, where $\alpha_p \in \mathbb{R}$. Then

$$a_1 = \sum \alpha_p \ell_{j_1} \cdots \ell_{j_{s-1}} \ell_p \ell_{j_{s+2}} \cdots \ell_{j_k},$$

where $(j_1, \dots, j_{s-1}, j_p, j_{s+2}, \dots, j_k) \in N_{k-1,r'}$ for some r'. Since $(k-1, r') < (k, r)$, we get $c(a_1) = 0$ by the induction supposition (we take into account that $k \le n$).

Therefore, $c(\ell_{j_1} \cdots \ell_{j_k}) = c(a_2)$. Obviously, $a_2 \in N_{k,r-1}$ and $(k, r-1) < (k, r)$. Hence, the equalities (20), (21) hold for the element $\ell_{j_1} \cdots \ell_{j_k}$ since, due to the induction supposition, they hold for a_2. This completes the proof of Lemma 1.

The following theorem is the main result of this section.

Theorem 4. *Let $\mathcal{L}'$ be a graded Lie subalgebra of codimension $n \ge 1$. Then there exists a one-dimensional homogeneous series of Lie rank n such that $\mathcal{L}'$ is a core Lie subalgebra of its (minimal) realization.*

Proof. Since $\mathcal{L}'$ is a graded Lie subalgebra of codimension n, we can choose homogeneous elements $\ell_1, \dots, \ell_n \in \mathcal{L}$ such that $\mathcal{L}' + \mathrm{Lin}\{\ell_1, \dots, \ell_n\} = \mathcal{L}$. Without loss of generality we assume $\mathrm{ord}(\ell_i) \le \mathrm{ord}(\ell_j)$ if $i < j$. Then choose a homogeneous basis $\{\ell_i\}_{i=n+1}^{\infty}$ of $\mathcal{L}'$ and consider the corresponding Poincaré-Birkhoff-Witt basis (12) and its dual basis (13). Introduce the series

$$S = d_1 ш \cdots ш d_n. \tag{22}$$

We note that this series corresponds to a linear map $c : \mathcal{F}^e \to \mathbb{R}$ defined by (19) and such that $c(1) = 0$.

We show that the series (22) is of Lie rank n. In fact, its Lie rank is not greater than n since the series has an n-dimensional realization, namely, the n-dimensional system corresponding to the series

$$\widetilde{S} = \begin{pmatrix} d_1 \\ \dots \\ d_n \end{pmatrix}$$

with the output $y = h(x) = x_1 \cdots x_n$. Such a system can be explicitly found as is described in [19]. It satisfies the Rashevsky-Chow condition (11) since $\widetilde{c}(\ell_i) = e_i$, $i = 1, \dots, n$. Obviously, $\widetilde{c}(\mathcal{L}') = 0$, hence, the core Lie subalgebra of this system equals $\mathcal{L}'$. Now we show that this realization is minimal.

To this end, we show that the Lie rank of the series (22) is not less than n. By definition, the Lie rank equals the dimension of the set of series of the form (5). It is convenient to re-expand the series w.r.t. the dual basis (13). Thus, the Lie rank equals the dimension of the set of series of the form

$$F_c(\ell) = c(\ell) + \sum_{j_1 < \cdots < j_k} \frac{1}{q_1! \dots q_k!} c(\ell_{j_1}^{q_1} \cdots \ell_{j_k}^{q_k} \ell) d_{j_1}^{\sqcup\!\sqcup q_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup d_{j_k}^{\sqcup\!\sqcup q_k}.$$

Now we show that the series $F_c(\ell_1), \dots, F_c(\ell_n)$ are linearly independent. For $n = 1$, there is nothing to prove. Suppose $n \geq 2$. Let us introduce the notation

$$\overline{d_1} = d_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup d_n, \quad \overline{d_n} = d_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup d_{n-1},$$
$$\overline{d_r} = d_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup d_{r-1} \sqcup\!\sqcup d_{r+1} \cdots \sqcup\!\sqcup d_n, \quad r = 2, \dots, n-1.$$

In other words, $\overline{d_r}$ is the shuffle product of all elements $d_1, \dots, d_n$ except d_r. Analogously, define

$$\overline{\ell_1} = \ell_2 \cdots \ell_n, \quad \overline{\ell_n} = \ell_1 \cdots \ell_{n-1},$$
$$\overline{\ell_r} = \ell_1 \cdots \ell_{r-1} \ell_{r+1} \cdots \ell_n, \quad r = 2, \dots, n-1.$$

Then the coefficient of $\overline{d_r}$ in the series $F_c(\ell_i)$ equals $c(\overline{\ell_r}\ell_i)$. Due to Lemma 1,

$$c(\overline{\ell_r}\ell_i) = \begin{cases} 1 & \text{if } \ i = r, \\ 0 & \text{otherwise.} \end{cases}$$

This means that the matrix $n \times n$ formed by the coefficients of elements $\overline{d_1}, \dots, \overline{d_n}$ in the series $F_c(\ell_1), \dots, F_c(\ell_n)$ is identity. Hence, series $F_c(\ell_1), \dots, F_c(\ell_n)$ are linearly independent, and therefore, the Lie rank of the series (22) is not less than n.

Thus, the series (22) is of Lie rank n, therefore, its minimal realization is of dimension n. As was mentioned before, this series has a realization with the core Lie subalgebra $\mathcal{L}'$. Since the minimal realization is unique up to a change of variables, the mentioned realization is minimal. The theorem is proved.

Theorem 4 has the following classification corollary close to [19].

Corollary 1. *Any graded Lie subalgebra of a finite (nonzero) codimension is a core Lie subalgebra of the minimal realization of some one-dimensional (nontrivial) series, and the dimension of this realization equals the codimension of the Lie subalgebra.*

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Однорідна апроксимація мінімальних реалізацій рядів ітерованих інтегралів

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У статті розглядаються реалізовні ряди ітерованих інтегралів зі скалярними коефіцієнтами і розвивається алгебраїчний підхід до задачі однорідної апроксимації

нелінійних керованих систем з виходом. У першому розділі ми нагадуємо поняття однорідної апроксимації нелінійної керованої системи, лінійної за керуванням, та поняття ряду ітерованих інтегралів. У другому розділі наведено постановку задачі реалізовності, нагадано критерій реалізовності ряду ітерованих інтегралів та спосіб побудови мінімальної реалізації ряду. Також ми нагадуємо деякі ідеї алгебраїчного підходу до опису однорідної апроксимації: вільна градуйована асоціативна алгебра, що ізоморфна алгебрі ітерованих інтегралів, вільна алгебра Лі, базис Пуанкаре-Біркгофа-Вітта, біортогональний базис і його побудова за допомогою тасуючого добутку, означення кореневої підалгебри Лі, яка визначає однорідну апроксимацію керованої системи. У третьому розділі ми показуємо, як можна знайти кореневу підалгебру Лі системи, яка є реалізацією одновимірного ряду ітерованих інтегралів, не знаходячи самої системи. Отриманий результат проілюстровано прикладом, в якому продемонстровано два способи знаходження кореневої підалгебри Лі реалізуючої системи. В останньому розділі показано, що для будь-якої градуйованої підалгебри Лі скінченної ковимірності існує такий одновимірний однорідний ряд, що ця підалгебра Лі є кореневою підалгеброю Лі його мінімальної реалізації. Доведення є конструктивним: ми наводимо спосіб побудови такого ряду, в якому використовується біортогональний базис до базису Пуанкаре-Біркгофа-Вітта вільної асоціативної алгебри, побудований за кореневою підалгеброю Лі, і тасуючий добуток в цій алгебрі. Як наслідок, отримуємо класифікацію всіх можливих однорідних апроксимацій систем, які є реалізаціями одновимірних рядів ітерованих інтегралів.
Ключові слова: **однорідна апроксимація; ряд ітерованих інтегралів; мінімальна реалізація; коренева підалгебра Лі.**

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The explicit form of the switching surface in admissible synthesis problem

In this article we consider the problem related to positional synthesis and controllability function method and more precisely to admissible maximum principle. Unlike the more common approach the admissible maximum principle method gives discontinuous solutions to the positional synthesis problem. Let us consider the canonical system of linear equations $\dot{x}_i = x_{i+1}, i = \overline{1, n-1}, \dot{x}_n = u$ with constraints $|u| \le d$. The problem for an arbitrary linear system $\dot{x} = Ax + bu$ can be simplified to this problem for the canonical system. A controllability function $\Theta(x)$ is given as a unique positive solution of some equation $\Phi(x, \Theta) = 0$. The control is chosen to minimize derivative of the function $\Theta(x)$ and can be written as $u(x) = -d\ \mathrm{sign}(s(x, \Theta(x)))$. The set of points $s(x, \Theta(x)) = 0$ is called the switching surface, and it determines the points where control changes its sign. Normally it contains the variable Θ which is given implicitly as the solution of equation $\Phi(x, \Theta) = 0$. Our aim in this paper is to find a representation of the switching surface that does not depend on the function $\Theta(x)$. We call this representation the explicit form. In our case the expressions $\Phi(x, \Theta)$ and $s(x, \Theta)$ are both polynomials with respect to Θ, so this problem is related to the problem of finding conditions when two polynomials have a common positive root. Earlier the solution for the 2-dimensional case was known. But during the exploration it was found out that for systems of higher dimensions there exist certain difficulties. In this article the switching surface for the three dimensional case is presented and researched. It is shown that this switching surface is a sliding surface (according to Filippov's definition). Also the other ways of constructing the switching surface using the interpolation and approximation are proposed and used for finding the trajectories of concrete points.

Keywords: **controllability; controllability function method; admissible maximum principle; switching surface.**

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1. Introduction

Let us consider the system of differential equations

$$\dot{x} = f(x, u), x \in \mathbb{R}^n, u \in \Omega \subset \mathbb{R}^r, \tag{1}$$

and let Q be a neighbourhood of the origin. Our aim is to construct a control $u = u(x), u \in \Omega$, such that the trajectory of the system

$$\dot{x} = f(x, u(x)), \tag{2}$$

starting at an arbitrary point $x_0 \in Q$, transfers into the origin in a some finite time $T = T(x_0)$. This problem is called the admissible positional synthesis problem.

One of the ways to solve it is the admissible maximum principle [6]. We consider constraints $|u| \le d$ and the linear canonical system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \quad \ldots \\ \dot{x}_{n-1} = x_n, \\ \dot{x}_n = u. \end{cases} \tag{3}$$

In this case, the obtained control is discontinuous and takes only values $u = \pm\, d$, with trajectories of points sliding along the switching surface. The solution to this problem is known, but it is interesting to consider a problem of finding explicit form of the switching surface. It was earlier solved for the two-dimensional system[7], and in this work we extend it to the three-dimensional case.

The conditions for reaching the equilibrium point are important problems of mechanics and differential equations. Important results in this area were obtained by O. M. Lyapunov, and subsequently they became a part of the foundation of the mathematical theory of control.

The contributions to the development of the control theory were made by L. S. Pontryagin, V. G. Boltayanskii, R. V. Gamkrelidze, E. F. Mishchenko, R. Kalman, R. Bellman and many others. In particular, R. Bellman obtained the equation that must be satisfied by the solution of the optimal synthesis problem (finding the control that transfers an arbitrary point to the origin in the shortest time):

$$\min_{u \in \Omega} \left(\sum_{i=1}^{n} \frac{\partial T(t, x)}{\partial x_i} f_i(x, u) \right) = -1, \tag{4}$$

where $T(t,x)$ is a cost function and also a time needed to reach the origin.

In many cases, finding a control that is a solution to this equation is quite difficult. This is one of the reasons why V. I. Korobov introduced the problem of admissible positional synthesis. Admissibility means that the chosen control does not necessarily provide the given or the shortest time, but ensures its finiteness.

The solution of the admissible positional synthesis problem, called the controllability function method, was proposed by V. I. Korobov in [4] and later developed in many other works. This method is based on the construction of the control $u(x)$, such that for the system (2) there exists a function $\Theta(x)$ which is an analogue of the Lyapunov function in the stability theory, but also satisfies a condition which ensures finiteness of the time. More precisely the following theorem holds.

Theorem 1 ([4]). *Suppose that in the system (1) at any set of points $K_1(\rho_1,\rho_2)=\{(x,u): 0<\rho_1\le ||x||\le\rho_2,\ u\in\Omega\}$ the vector function $f(x,u)$ satisfies the Lipschitz continuity condition:*

$$\|f(x',u')-f(x'',u'')\|\le L_1(\rho_1,\rho_2)(\|x''-x'\|+\|u''-u'\|),$$

for any $(x',u'),(x'',u'')\in K_1(\rho_1,\rho_2)$.

And suppose that there exists a function $\Theta(x)$, such that the following conditions hold:

1. *$\Theta(x)\ge 0$ if $x\ne 0$ and $\Theta(0)=0$;*
2. *$\Theta(x)$ is continuous everywhere and continuously differentiable at any point except, perhaps, the point $x=0$;*
3. *there exists a number $c>0$ such that the set $Q=\{x:\Theta(x)\le c\}$ is bounded and there exists $R>0$ such that $Q\subset\{x:\|x\|<R\}$;*
4. *there exists a function $u(x): Q\to\Omega$, that satisfies the inequality*

$$\dot{\Theta}=\sum_{i=1}^{n}\frac{\partial\Theta(x)}{\partial x_i}f_i(x,u(x))\le -\beta\Theta^{1-\frac{1}{\alpha}}(x)$$

for some $\alpha>0$, $\beta>0$. And $u(x)$ is Lipschitz continuous at any point of the set $K(\rho_1,\rho_2)=\{x\in Q: 0<\rho_1\le\|x\|\le\rho_2\}$, that is

$$\|u(x'')-u(x')\|\le L_2(\rho_1,\rho_2)\|x''-x'\|,$$

for any $x',x''\in K(\rho_1,\rho_2)$.

Then the trajectory $x(t)$ of the system $\dot{x}=f(x,u(x))$, which starts at an arbitrary point $x\in Q$, ends at the point $x_1=0$ at a certain finite moment of time (which depends on x_0) $T(x_0)\le(\alpha/\beta)\Theta^{\frac{1}{\alpha}}(x_0)$. Moreover if $\alpha=\infty$, then $x(t)\to 0$ as $t\to\infty$.

The function $\Theta(x)$ is called the controllability function. The conditions 1-3 of this theorem coincide with the conditions of Lyapunov theorem on asymptotic stability, and the condition 4 ensures the finiteness of the time for an arbitrary point to reach the origin. In the case where $\alpha = \infty$ the function $\Theta(x)$ is a Lyapunov function for the obtained system.

Also in the case when $\alpha = \beta = 1$, and instead of inequality, equality is fulfilled, i.e.

$$\sum_{i=1}^{n} \frac{\partial \Theta(x)}{\partial x_i} f_i(x, u(x)) = -1, \tag{5}$$

the controllability function is also a motion time from an arbitrary point to the origin. If, in addition, the Bellman equation is satisfied:

$$\min_{u \in \Omega} \left(\sum_{i=1}^{n} \frac{\partial \Theta(x)}{\partial x_i} f_i(x, u) \right) = \left(\sum_{i=1}^{n} \frac{\partial \Theta(x)}{\partial x_i} f_i(x, u(x)) \right) = -1, \tag{6}$$

the function $\Theta(x)$ is also an optimal time.

The function $\Theta(x)$ is naturally constructed implicitly as a solution of some equation $\Phi(x, \Theta) = 0$. It makes it different from the Lyapunov function which is constructed in explicit form. On the other hand, in the linear optimal control problem, the motion time is also found implicitly[5].

Let us consider the canonical system:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \quad \dots \\ \dot{x}_{n-1} = x_n, \\ \dot{x}_n = u, \end{cases} \tag{7}$$

with the constraint on control $|u| \le d$. It is a linear system $\dot{x} = A_0 x + b_0 u$, where

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad b_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \dots \\ 1 \end{pmatrix}. \tag{8}$$

An admissible position synthesis problem for an arbitrary linear system $\dot{x} = Ax + bu$ can be simplified to this problem for the canonical system[4].

Let us describe the algorithm of constructing the control using the admissible maximum principle described in [7]. We determine the controllability function $\Theta(x)$ at an arbitrary point x as a positive root of the equation

$$\Phi(x, \Theta) = 2a_0 \Theta - (D(\Theta) F D(\Theta) x, x) = 0, \tag{9}$$

(it can be proved that this root is unique at every point [7]), where F is a positive definite matrix,

$$D(\Theta) = \operatorname{diag}\left(\Theta^{-\frac{m+n-2i+1}{2\alpha}}\right)_{i=1}^{n}, \tag{10}$$

and numbers $m \in \mathbb{N}, \alpha \geq 1$ are chosen so that the matrix

$$F^{\alpha} = \left(\left(1 + \frac{m+n-i-j+1}{\alpha}\right) f_{ij}\right)_{i,j=1}^{n}$$

is positive definite. In particular, we will consider $m = n$, $\alpha = 1$. The number a_0 is chosen to satisfy the constraint on control.

The derivative $\dot{\Theta}$ of the function $\Theta(x)$ can be written in the following form:

$$\dot{\Theta} = \frac{\Theta((FA_0 + A_0^* F)y(x,\Theta), y(x,\Theta)) + 2u\Theta(D(\Theta)FD(\Theta)x, b_0)}{(F^{\alpha}y(x,\Theta), y(x,\Theta))}, \tag{11}$$

where and $y(x,\Theta) = D(\Theta)x$. Let us denote

$$s(x, \Theta(x)) = (D(\Theta(x))FD(\Theta(x))x, b_0), \tag{12}$$

that is,

$$\begin{gathered} s(x_1, x_2, \ldots, x_n, \Theta(x_1, x_2, \ldots, x_n)) = \\ = f_{n1}x_1 + f_{n2}\Theta(x_1, x_2, \ldots x_n)x_2 + \ldots + f_{nn}\Theta^{n-1}(x_1, x_2, \ldots, x_n)x_n. \end{gathered} \tag{13}$$

We choose the control as $u(x) = -d\,\mathrm{sign}(s(x,\Theta(x)))$ and call the set of points satisfying the equation

$$s(x, \Theta(x)) = 0 \tag{14}$$

the switching surface S.

This control gives the minimum value of the derivative $\dot{\Theta}$ of the function $\Theta(x)$ that can be obtained under given constraints. We note that this control is not continuous. It takes only boundary values and has discontinuity at points of the surface (14).

After substitution of the control to the system (7) we obtain:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \quad \ldots \\ \dot{x}_{n-1} = x_n, \\ \dot{x}_n = -d\,\mathrm{sign}\, s(x_1, x_2, ..., x_n, \Theta(x_1, x_2, ..., x_n)). \end{cases} \tag{15}$$

Algorithm of finding the concrete trajectory from the point x_0 to the point $x_1 = 0$ in the case when the switching surface is given by the equation (14) is the following. At the point x_0 we find a unique positive solution Θ_0 of the equation

(9) and add the equation (11) to the system (15). After that we find the trajectory $(x_1(t), x_2(t), \dots x_n(t))$ as the solution of the Cauchy problem:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \quad \dots \\ \dot{x}_{n-1} = x_n, \\ \dot{x}_n = -d \operatorname{sign} s(x_1, x_2, ..., x_n, \Theta), \\ \dot{\Theta} = \frac{2\Theta(F(\Theta)x, A_0 x) - 2d\Theta|(D(\Theta)FD(\Theta)x, b_0)|}{(F^1 y(\Theta, x), y(\Theta, x))}, \end{cases} \tag{16}$$

$$x_1(0) = x_{10}, x_2(0) = x_{20}, \dots, x_n(0) = x_{n0}, \Theta(0) = \Theta_0. \tag{17}$$

2. The explicit form of the switching surface

The formula $s(x, \Theta(x)) = 0$ gives the implicit form of the switching surface, that is, it contains the function $\Theta(x)$ as an implicit solution of the equation (9). We are considering the problem of finding the switching surface in the explicit form. Hence, we need to exclude the variable Θ from the equation for the surface.

For this let us write the equation (9) and the formula for the switching surface in the following form:

$$\Phi(x, \Theta) = 2a_0 \Theta^{2n} - \sum_{i,j=1}^{n} f_{ij} \Theta^{i+j-2} x_i x_j = 0, \tag{18}$$

$$s(x, \Theta) = f_{n1} x_1 + f_{n2} \Theta x_2 + ... + f_{nn} \Theta^{n-1} x_n = 0. \tag{19}$$

One way to remove a common factor from two equations is to use the resultant. Let $x \in S$, $x \neq 0$ be a fixed point, then $\Phi(x, \Theta)$, $s(x, \Theta)$ are the polynomials of variable Θ. If $\Phi(x, \Theta)$ and $s(x, \Theta)$ have a common root, then their resultant $R(\Phi, s)$ is equal to zero. Hence, the set of all points where they have a common root can be given by the equation:

$$R(\Phi, s) = 0. \tag{20}$$

But the surface given by equation (20) is larger than the switching surface, because it also contains points where $\Phi(x, \Theta)$, $s(x, \Theta)$ have common negative root, or this root equals zero. Instead, the switching surface contains only those points where a common root $\Theta > 0$. Therefore, we have certain difficulties related to the fact that we need to find a way to separate the points where $\Theta(x) > 0$ from the entire set. Hence, further we will use the resultant only for obtaining this wider set.

As an example let us consider the process of finding switching surface for the case $n = 2$ described in [7].

Let us determine Θ with the equation

$$\Phi(\Theta, x) = \frac{2}{9}\Theta^4 - \Theta^2 x_2^2 - 2\Theta x_1 x_2 - 3x_1^2 = 0 \tag{21}$$

(the algorithm for finding such equations is described in [7]). Then the switching surface has the equation:

$$s(\Theta, x) = x_1 + \Theta x_2 = 0. \tag{22}$$

Using the formula (20) we obtain the surface given by resultant:

$$R(\Phi, s) = \begin{vmatrix} \frac{2}{9} & 0 & -x_2^2 & -2x_1x_2 & -3x_1^2 \\ x_2 & x_1 & 0 & 0 & 0 \\ 0 & x_2 & x_1 & 0 & 0 \\ 0 & 0 & x_2 & x_1 & 0 \\ 0 & 0 & 0 & x_2 & x_1 \end{vmatrix} = \frac{1}{9}x_1^4 - x_1^2 x_2^4 = 0. \tag{23}$$

To separate points where the common root of equations (21) and (22) is positive we use the fact that the equation (22) has only one root $\Theta = -\frac{x_1}{x_2}$, which is positive only when $x_1 x_2 < 0$. The part of surface (23) that satisfies this condition can be written in the form:

$$x_1 = -3x_2|x_2|. \tag{24}$$

This formula gives the equation of the switching surface. But for systems of higher dimensions, overcoming such difficulties can be more complicated. Now we give the explicit form of the switching surface in the case $n = 3$.

Let us determine the controllability function by the equation:

$$\Phi(x, \Theta) = \frac{9}{1625}\Theta^6 - 38x_1^2 - 30\frac{4}{5}x_1x_2\Theta - 4x_1x_3\Theta^2 - 6\frac{4}{5}x_2\Theta^2 - 2x_2x_3\Theta^3 - \frac{1}{5}x_3^2\Theta^4 = 0. \tag{25}$$

Then the switching surface has the form:

$$s(x, \Theta) = 10x_1 + 5\Theta x_2 + \Theta^2 x_3 = 0, \tag{26}$$

and equation defined by the resultant is as follows:

$$\begin{aligned} R(\Phi, s) = x_1^2\big(&160x_1^4 - 1625x_2^6 + 5200x_1x_2^4x_3 - 4940x_1^2x_2^2x_3^2 + 1040x_1^3x_3^2 + \\ &+ 845x_2^4x_3^4 - 2366x_1x_2^2x_3^5 + 1690x_1^2x_3^6\big) = 0. \end{aligned} \tag{27}$$

We are searching for the points where there exists a common root $\Theta > 0$. Let us show that the factor x_1^2 can be discarded. Indeed, if $x_1 = 0$ then

$$\Phi(x, \Theta) = \frac{9}{1625}\Theta^6 - 6\frac{4}{5}x_2\Theta^2 - 2x_2x_3\Theta^3 - \frac{1}{5}x_3^2\Theta^4 = 0, \tag{28}$$

$$s(x, \Theta) = 5\Theta x_2 + \Theta^2 x_3 = 0. \tag{29}$$

These polynomials always have a common root $\Theta = 0$. The second root $\Theta = \frac{-5x_2}{x_3}$ of equation (29) is also a root for (28) if:

$$\frac{1125x_2^6}{13x_3^6} - \frac{45x_2^4}{x_3^2} = 0. \tag{30}$$

That is,

$$x_2^2 = \frac{13}{25}x_3^4. \tag{31}$$

But the points $\{x_1 = 0, x_2^2 = \frac{13}{25}x_3^4\}$ are also solutions for the equation

$$\begin{gathered} 160x_1^4 - 1625x_2^6 + 5200x_1x_2^4x_3 - 4940x_1^2x_2^2x_3^2 + 1040x_1^3x_3^2 + \\ + 845x_2^4x_3^4 - 2366x_1x_2^2x_3^5 + 1690x_1^2x_3^6 = 0. \end{gathered} \tag{32}$$

Hence the factor x_1^2 does not add any non-zero roots to the equation (27) compared to (32). There is also a case when $\{x_1 = 0,\ x_2 = 0,\ x_3 \neq 0\}$. Then

$$\Phi(x, \Theta) = \frac{9}{1625}\Theta^6 - \frac{1}{5}x_3^2\Theta^4 = 0, \tag{33}$$

$$s(x, \Theta) = \Theta^2 x_3 = 0. \tag{34}$$

In this case $\Phi(x, \Theta)$ and $s(x, \Theta)$ have common root $\Theta = 0$ and we do not consider it. The surface that show all other solutions for equation (32) is shown in Figure 1.

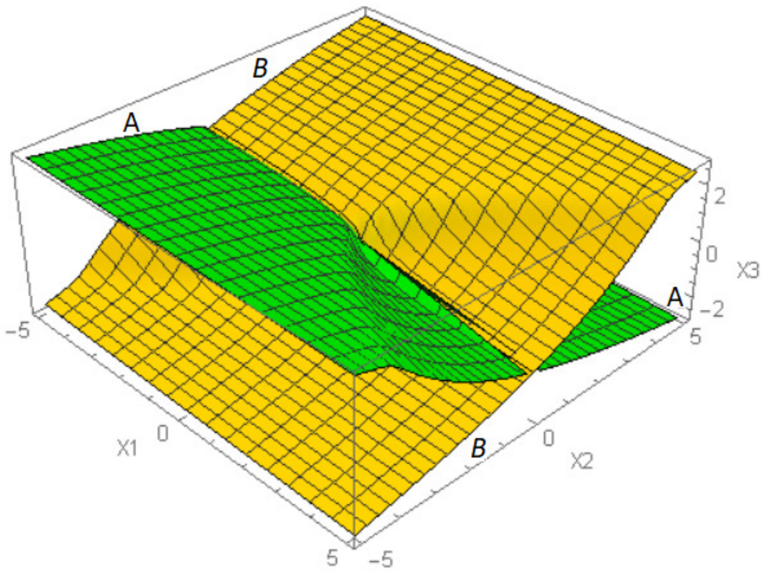

Fig. 1. Surface given by equation (32)

This surface consists of two parts. One of them (Part A) includes points where the common root $\Theta(x)$ of equations (25) and (26) is positive, and the other (Part B) includes points corresponding to the negative root and this part should be excluded.

If we find the switching surface we obtain the system of differential equations with a discontinuous right-hand side. The control $u(x)$ equals -1 above the switching surface and $+1$ below it.

Let us find the switching surface by examining the roots of the polynomial $s(x,\Theta) = 10x_1 + 5\Theta x_2 + \Theta^2 x_3$.

First, we consider the case when $x_3 = 0$. Then

$$\Phi(x,\Theta) = \frac{9\Theta^6}{1625} - \frac{34x_2^2\Theta^2}{5} - \frac{154x_1x_2\Theta}{5} - 38x_1^2, \tag{35}$$

$$s(x,\Theta) = x_2\Theta + 10x_1. \tag{36}$$

Then

$$R(\Phi, s) = x_1^2(160x_1^4 - 1625x_2^6) = 0. \tag{37}$$

The equation (37) has solutions $x_1 = 0$ and $x_1 = \pm\left(\frac{1}{2}\sqrt{5}\left(\frac{13}{2}\right)^{\frac{1}{4}}\sqrt{|x_2|^3}\right)$. Using the fact that $s(x,\Theta)$ has a positive root only when $x_1x_2 < 0$ we obtain the curve:

$$\begin{cases} x_1 + \left(\frac{1}{2}\sqrt{5}\left(\frac{13}{2}\right)^{\frac{1}{4}}\sqrt{|x_2|^3}\right)\operatorname{sign}(x_2) = 0, \\ x_3 = 0. \end{cases} \tag{38}$$

If $x_3 \neq 0$, then $s(x,\Theta)$ is a quadratic polynomial, if $5x_2^2 - 8x_1x_3 > 0$ then it has two roots $\Theta_{1,2} = \frac{-5x_2 \pm \sqrt{5}\sqrt{5x_2^2 - 8x_1x_3}}{2x_3}$. Now we are using the fact that $\Phi(x,\Theta)$ always has exactly one positive root Θ, hence, any point on the switching surface corresponds either to root Θ_1 or to root Θ_2 and we can construct parts of switching surface for this roots separately and then unite them.

By substituting the root $\Theta_1 = \frac{-5x_2 + \sqrt{5}\sqrt{5x_2^2 - 8x_1x_3}}{2x_3}$ into (25) we obtain the surface given by equation:

$$\begin{gathered} 1125x_2^6 - 2700x_1x_2^4x_3 + 1620x_1^2x_2^2x_3^2 - 144x_1^3x_3^3 - 585x_2^4x_3^4 + \\ + 1170x_1x_2^2x_3^5 - 468x_1^2x_3^6 + \sqrt{5x_2^2 - 8x_1x_3}\Big(-225\sqrt{5} + 360\sqrt{5}x_1x_2^3x_3 - \\ -108\sqrt{5}x_1^2x_2x_3^2 + 117\sqrt{5}x_2^3x_3^4 - \frac{702x_1x_2x_3^5}{\sqrt{5}}\Big) = 0. \end{gathered} \tag{39}$$

The root Θ_1 is positive when $\frac{-5x_2 + \sqrt{5}\sqrt{5x_2^2 - 8x_1x_3}}{2x_3} > 0$. We can rewrite this as:

$$\begin{gathered} \text{if } x_3 > 0 \text{ then } \left(\left(x_2 < 0 \text{ and } x_1 < \frac{5x_2^2}{8x_3}\right) \text{ or } x_1 < 0\right), \\ \text{if } x_3 < 0 \text{ then } \left(x_2 > 0 \text{ and } \frac{5x_2^2}{8x_3} \leq x_1 < 0\right). \end{gathered} \tag{40}$$

By constructing (39) only at points where these conditions hold we obtain the part A_1 of the switching surface. Similarly, considering the case of the root $\Theta_2 = \frac{-5x_2-\sqrt{5}\sqrt{5x_2^2-8x_1x_3}}{2x_3} > 0$, with conditions

$$\begin{aligned} &\text{if } x_3 > 0 \text{ then } \left(x_2 < 0 \text{ and } 0 < x_1 \leq \frac{5x_2^2}{8x_3}\right), \\ &\text{if } x_3 < 0 \text{ then } \left(\left(x_2 > 0 \text{ and } x_1 > \frac{5x_2^2}{8x_3}\right) \text{ or } x_1 > 0\right), \end{aligned} \tag{41}$$

we obtain the part A_2. By combining the parts A_1, A_2, the curve (38), (purple line in Figure 2) and the point $(0,0,0)$ we get the graph of the switching surface. It also can be shown that in the neighborhood of the curve (38) the root Θ remains continuous, hence we can consider that switching surface consists of two parts, each corresponding to a separate root.

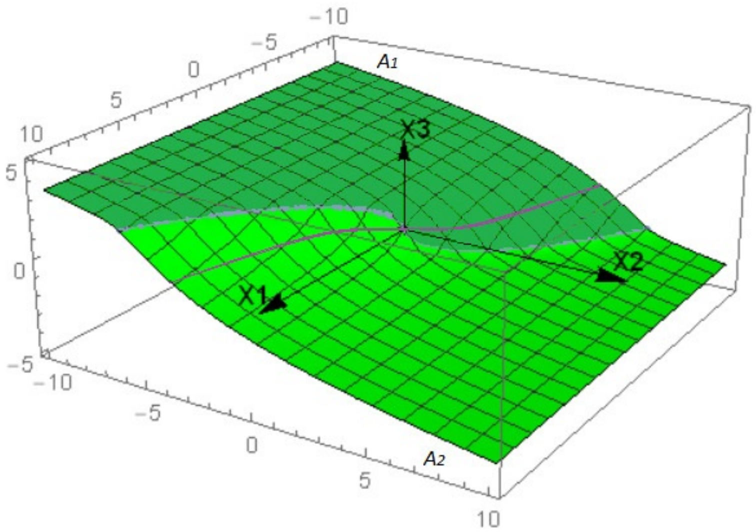

Fig. 2. Switching surface

The line separating these parts (blue line in Figure 2) consists of points where $\Theta_1 = \Theta_2 = \frac{-5x_2}{2x_3}$ and can be found explicitly. By substituting root $\Theta = \frac{-5x_2}{2x_3}$ into $\Phi(x,\Theta)$ and by using the fact that in this case $5x_2^2 - 8x_1x_3 = 0$, we can write as follows:

$$\begin{cases} 1125x_2^6 - 15860x_2^4x_3^4 + 43264x_1x_2^2x_3^5 - 31616x_1^2x_3^6 = 0, \\ 5x_2^2 - 8x_1x_3 = 0. \end{cases}$$

The solutions of the form $x_1 = 0, x_2 = 0, x_3 \neq 0$ belong to case when the common root $\Theta = 0$, all other solutions can be written as

$$x_1 = -\operatorname{sign}(x_2)\sqrt[4]{\frac{325}{2048}}\sqrt{|x_2|^3}, \quad x_3 = -\operatorname{sign}(x_2)\sqrt[4]{\frac{25}{26}}\sqrt{|x_2|}. \tag{42}$$

Now let us denote:

$$P_1(x_1, x_2, x_3) = 1125x_2^6 - 2700x_1x_2^4x_3 + 1620x_1^2x_2^2x_3^2 - 144x_1^3x_3^3 - 585x_2^4x_3^4 + \\ + 1170x_1x_2^2x_3^5 - 468x_1^2x_3^6,$$

$$P_2(x_1, x_2, x_3) = -225\sqrt{5} + 360\sqrt{5}x_1x_2^3x_3 - 108\sqrt{5}x_1^2x_2x_3^2 + 117\sqrt{5}x_2^3x_3^4 - \\ - \frac{702x_1x_2x_3^5}{\sqrt{5}}.$$

Hence, the switching surface is written in the form $s(x_1, x_2, x_3) = 0$, where:

$$s(x_1, x_2, x_3) = x_1 + \left(\tfrac{1}{2}\sqrt{5}\left(\tfrac{13}{2}\right)^{\frac{1}{4}}\sqrt{|x_2|^3}\right)\operatorname{sign}(x_2), \text{ if } x_3 = 0,$$

$$\begin{gathered} s(x_1, x_2, x_3) = P_1(x_1, x_2, x_3) + \sqrt{5x_2^2 - 8x_1x_3}P_2(x_1, x_2, x_3), \\ \text{if } x_1 < -\operatorname{sign}(x_2)\sqrt[4]{\tfrac{325}{2048}}\sqrt{|x_2|^3} \text{ and } (40), \\ s(x_1, x_2, x_3) = P_1(x_1, x_2, x_3) - \sqrt{5x_2^2 - 8x_1x_3}P_2(x_1, x_2, x_3), \\ \text{if } x_1 \geq -\operatorname{sign}(x_2)\sqrt[4]{\tfrac{325}{2048}}\sqrt{|x_2|^3} \text{ and } (41). \end{gathered} \tag{43}$$

Now we show graphically that S is a sliding surface [8]. Consider an arbitrary point x on the surface S and its velocity vectors f^+ and f^- when it approaches the switching surface from above and from below respectively. And let α be a tangent plane to the surface S at the point x (Fig. 3).

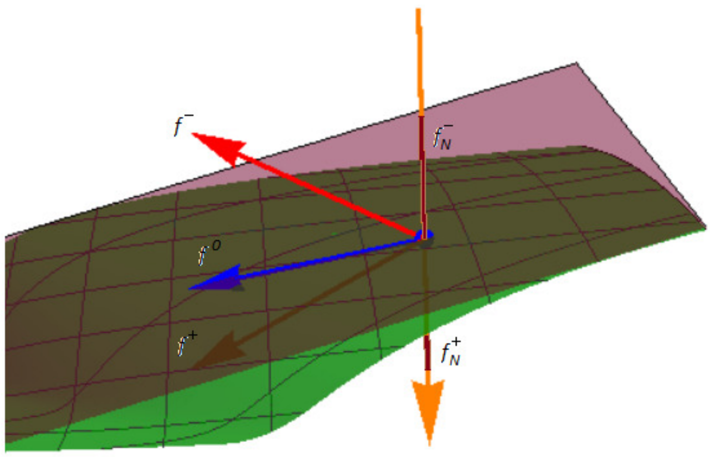

Fig. 3. Velocity vector on the switching surface

We consider

$$f_N^- = \frac{\langle \nabla s, f^- \rangle}{|\nabla s|}, \quad f_N^+ = \frac{\langle \nabla s, f^+ \rangle}{|\nabla s|}, \tag{44}$$

and build the graphs of $\tilde{f}_N^- = \langle \nabla s, f^- \rangle$ and $\tilde{f}_N^+ = \langle \nabla s, f^+ \rangle$ (Figures 4 and 5 respectively). We see that $f_N^- \leq 0$ and $f_N^+ \geq 0$ (and $f_N^- = 0$ if and only if $f_N^+ = 0$) for an arbitrary point $x \in S$. This means that at any point the velocity vectors are located on different sides of the plane α and, therefore, the resulting vector always lies in this plane.

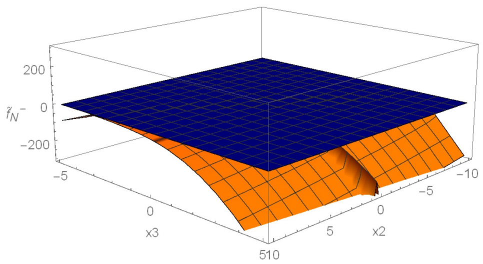

Fig. 4. Graph of $\tilde{f}_N^-$

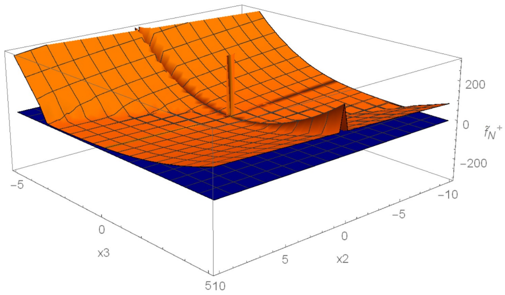

Fig. 5. Graph of $\tilde{f}_N^+$

3. Approximation of the surface

To find specific trajectories, we propose to use an approximate surface that has a simpler shape. One of the methods can be a construction with an interpolation polynomial in the form $x_3 = L(x_1, x_2)$. By substituting numbers instead of x_1, x_2 in equation $s(x_1, x_2, x_3) = 0$ and finding the solution for x_3, we can get any number of points on the switching surface. For interpolation, we select the points in such a way that they form a rectangular grid in the x_1x_2 plane. Then the interpolation polynomial is given by the formula

$$L(x_1, x_2) = \sum_{n=1}^{N} \sum_{m=1}^{M} \left(x_3(x_{1i}, x_{2j}) \prod_{i=1, i \neq n}^{N} \frac{x_1 - x_{1i}}{x_{1n} - x_{1i}} \prod_{j=1, j \neq m}^{M} \frac{x_2 - x_{2j}}{x_{2m} - x_{2j}} \right). \tag{45}$$

The approximated control $u(x)$ is given in the form: $u(x) = -\text{sign}(x_3 - L(x_1, x_2))$. The surface obtained by interpolation and the trajectory of the point $(-1, 2.5, 1)$ are shown in Figure 6.

Another method of approximation that can be used is the least-squares approximation. As an example, we choose multiples with maximal power 3 for x_1, x_2 and construct the approximating surface in the following form:

$$x_3 = w(x_1, x_2) = a_1x_1 + a_2x_1^2 + a_3x_1^3 + a_4x_2 + a_5x_1x_2 + \cdots + a_{15}x_1^3x_2^3, \tag{46}$$

where $a_1, a_2, \ldots, a_{15}$ are unknown coefficients.

In this case, the points do not necessarily have to form a rectangular grid, so the interpolating surface can be constructed for both parts of the surface S separately (Fig. 7). In addition, if we take symmetrically located points, then the resulting parts will also be symmetrical.

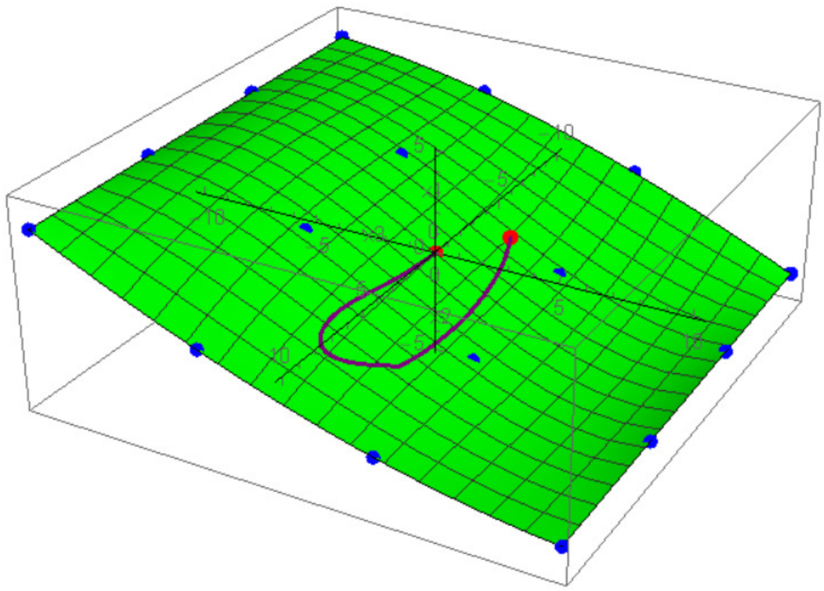

Fig. 6. Interpolating surface and the trajectory

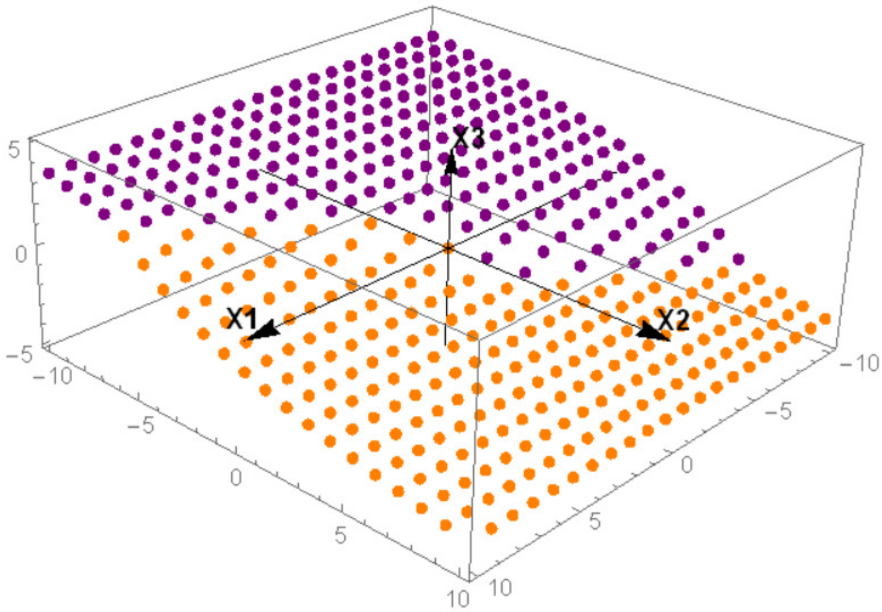

Fig. 7. Points for approximation

Numbers $a_1, a_2, \dots, a_{15}$ are chosen to minimize the function

$$J(a_1, a_2, \dots, a_{15}) = \sum_{i=1}^{k} \left(x_{3i} - L(x_{1i}, x_{2i})\right)^2. \tag{47}$$

Then in our case we have:

$$w(x_1,x_2) \approx \begin{cases} -0.433897x_1 - 0.05253x_1^2 - 0.00240945x_1^3 - \\ -0.994791x_2 + 0.170404x_1x_2 - 0.0174874x_1^2x_2 - \\ -0.000655178x_1^3x_2 - 0.118976x_2^2 - 0.0222263x_1x_2^2 - \\ -0.00191042x_1^2x_2^2 - 0.0000592797x_1^3x_2^2 - 0.00572649x_2^3 - \\ -0.000934956x_1x_2^3 - 0.000068384x_1^2x_2^3 - 1.8162\cdot 10^{-6}x_1^3x_2^3 \\ \text{if } x_1 \geq -\mathrm{sign}(x_2)\sqrt[4]{\frac{325}{2048}}\sqrt{|x_2|^3}, \\ \\ -0.433897x_1 + 0.05253x_1^2 - 0.00240945x_1^3 - \\ -0.994791x_2 - 0.170404x_1x_2 - 0.0174874x_1^2x_2 + \\ +0.000655178x_1^3x_2 + 0.118976x_2^2 - 0.0222263x_1x_2^2 + \\ +0.00191042x_1^2x_2^2 - 0.0000592797x_1^3x_2^2 - 0.00572649x_2^3 + \\ +0.000934956x_1x_2^3 - 0.000068384x_1^2x_2^3 + 1.8162\cdot 10^{-6}x_1^3x_2^3 \\ \text{if } x_1 < -\mathrm{sign}(x_2)\sqrt[4]{\frac{325}{2048}}\sqrt{|x_2|^3}. \end{cases} \tag{48}$$

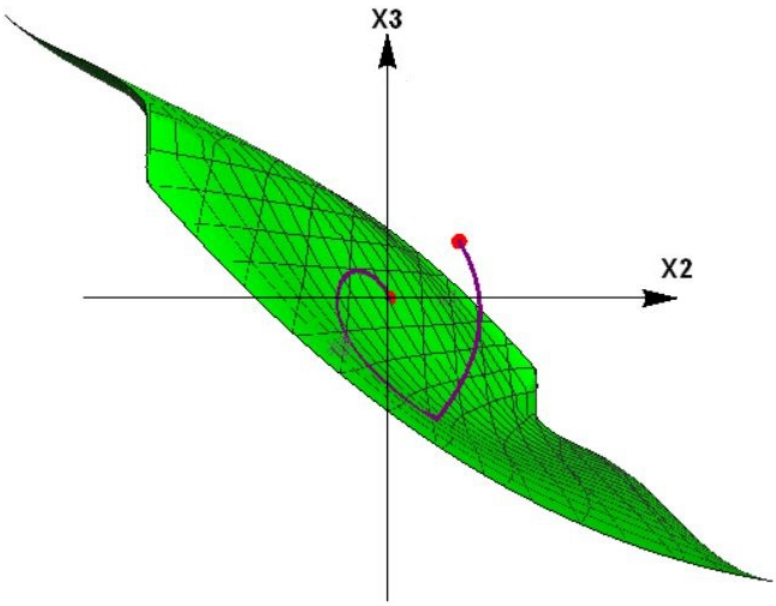

Fig. 8. Approximating surface and the trajectory

The trajectory starting at the point $(-1, 2.5, 1)$ is shown in Figure 8. We note that the question whether the concrete obtained approximating or interpolating surface is a sliding surface can be checked in the same way as for the surface S and in general this can be not true. The problem which can be considered is how to choose the interpolation nodes to obtain the sliding surface and to ensure that the trajectories reach the origin in a finite time, and if so, how much can time increase comparing to the original surface.

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Явний вигляд поверхні перемикання в задачі допустимого позиційного синтезу

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В цій статті розглядається проблема, пов'язана із задачею допустимого позиційного синтезу та методом функції керованості, а саме, з допустимим принципом максимуму. На відміну від більш звичого підходу, допустимий принцип максимуму дає розривний розв'язок задачі синтезу. Нехай задана канонічна керована система $\dot{x}_i = x_{i+1}, i = \overline{1, n-1}, \dot{x}_n = u$ з обмеженнями на керування вигляду $|u| \leq d$. Задача синтезу для довільної лінійної системи вигляду $\dot{x} = Ax + bu$ може бути зведена до канонічної. Функція керованості $\Theta(x)$ задана як єдиний додатний розв'язок деякого рівняння $\Phi(x, \Theta) = 0$. Керування обирається таким чином, щоб мінімізувати похідну функції $\Theta(x)$ за часом в кожній точці, і воно може бути записано у вигляді $u(x) = -d \operatorname{sign}(s(x, \Theta(x)))$. Множина точок, що задовольняє рівності $s(x, \Theta(x)) = 0$, називається поверхнею перемикання і визначає точки, де керування змінює свій знак. Зазвичай вона включає змінну Θ, що є неявним розв'язком рівняння $\Phi(x, \Theta) = 0$. В цій роботі ми шукаємо явне представлення поверхні перемикання, тобто таке, що не включає зміної Θ. В нашому випадку вирази $\Phi(x, \Theta)$ та $s(x, \Theta)$ є поліномами відносно Θ, тому задача пов'язана з задачею знаходження умов при яких два поліноми мають спільний додатний корінь. Раніше було відомо рішення для 2-вимірного випадку. Але в ході дослідження з'ясувалося, що для систем більшої розмірності існують певні труднощі. У цій статті представлено та досліджено поверхню перемикання для тривимірного випадку. Також показано, що ця поверхня перемикання є поверхнею ковзання (згідно з визначенням Філіппова). В роботі також запропоновані інші способи побудови поверхні перемикання за допомогою інтерполяції та апроксимації. Ці способи застосовано для знаходження траєкторій конкретних початкових точок.

Ключові слова: **керованість; метод функції керованості; допустимий принцип максимуму; поверхня перемикання.**

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ВЯЧЕСЛАВ ОЛЕКСІЙОВИЧ РЕЗУНЕНКО (некролог)

15.02.1941 – 26.08.2022

26 серпня 2022 року відійшов у засвіти доцент кафедри Вищої математики та інформатики Харківського національного університету імені В.Н. Каразіна Вячеслав Олексійович Резуненко. Народився він 15 лютого 1941 року в

м. Вороніж, де його батько, молодий офіцер, викладав у Вороніжському військовому училищі зв'язку. Батько 1920 р.н. українець, родом з Харківської області, мати родом з Вороніжської області. Батько пропав без вісті на фронті Другої світової війни в серпні 1941 року. Важкі воєнні роки, евакуація, післявоєнний голод, залишили свій відбиток на все життя. Вячеслав Олексійович після закінчення семирічної школи, у 1955 поступив без іспитів, як відмінник, в Харківський будівельний технікум (технікум зеленого будівництва). Технікум закінчив з відзнакою у 1959 році та за направленням поїхав працювати в м. Одесу. У 1962 - 1967 навчався на механіко-математичному факультеті Харківського державного університету. За направленням почав працювати в університеті з вересня 1967 року. Спочатку робота математиком-програмістом на Обчислювальному центрі (ОЦ) університету. Програмування та вирішення прикладних задач на різних ЕОМ: спочатку на Урал-1, потім на М-20, ЕС. Обчислював задачі, зокрема, для Турбінного заводу, Автодорожнього інституту, фізичного, радіофізичного факультетів нашого університету.

Одночасно з роботою на ОЦ університету був у 1969-1973 роки директором Заочної юнацької математичної школи (ЗЮМШ) при університеті (прообраз сучасного Малого Каразінського університету). ЗЮМШ організовувала разом з механіко-математичним, радіофізичним, фізичним і фізико-технічним факультетами шкільні олімпіади 7-10 класів з математики і фізики. В олімпіадах брали участь обласні команди школярів з більшості областей України, а також з Естонії, Білорусі, Росії. У деякі роки приїздило до 500 школярів та викладачів-керівників команд. Шкільна математика залишалась в полі зору В.О. все його життя, про що свідчать видані числені посібники з математики для школярів.

У 1973-1976 навчався в аспірантурі університету, в 1977 захистив кандидатську дисертацію з фізико-математичних наук на тему "Розсіяння електромагнітних хвиль зосереджених джерел на сфері з круговим отвором". Науковим керівником був випускник кафедри механіки нашого університету доктор фіз.-мат. наук член-кор. АН України В.П. Шестопалов. Після аспірантури за направленням працював 1 рік науковим співробітником на кафедрі теоретичної радіофізики університету. За запрошенням декана механіко-математичного факультету завідувача кафедри Вищої математики Гордевського Д.З., у 1978 прийшов і з того часу безперервно працював на мехматі, на кафедрі Вищої математики та інформатики. Спочатку викладач, потім з 1979 старший викладач, до останнього свого дня - доцент кафедри. У 1989 р. отримав вчене звання доцента.

30 років (1978-2008) відповідав за роботу Опорної кафедри математики (ОКМ) Харківського ВУЗівського центру (ХВЦ). ОКМ співпрацювала з кафедрами мех-мату, фіз-факу та фіз-теху університету та кафедрами вищої та прикладної математики більш ніж 20-ти ВУЗів ХВЦ.

Основні напрямки наукових досліджень: математична теорія дифракції електромагнітних та акустичних хвиль, електростатика, методи регуляризації інтегральних та суматорних рівнянь, обчислювальні методи в електродинаміці. Опубліковано 134 науково-методичні праці (orcid: 0000-0003-4577-4950). Був членом наукового товариства IEEE з 1997 року.

В напрямку дослідження *просторових розподілів електростатичних полів, створених складними системами електричних зарядів*, отримав наступні результати: a) із використанням методу регуляризації виділено й отримано головну частину оператора задачі електростатики для сферичного сегмента, зануреного у діелектричне заокруглення конусу та розв'язано задачу електростатики для сферичного сегмента, екранованого замкненими секційованими сферами; b) виділено й отримано сингулярну частину оператора задачі електростатики для сферичного сегмента та секційованого провідного конуса; c) із застосуванням методів інтегральних перетворень, регуляризації та виділення й обернення головної частини інтегральних і суматорних рівнянь отримано строгий розв'язок задачі про електростатичний потенціал сфери з коловим отвором та пакету горизонтальних диполів, екранованих сферою; d) отримано потенціали сферичного сегмента й електростатичного заряда у присутності конусу та секційованої сфери; e) із використанням методу обернення інтегрального оператора і напівобернення матричного оператора задачі побудовано чисельно-аналітичний алгоритм дослідження потенціалу сфери з коловим отвором і заряду, оточених стрічковими сферами.

В напрямку *розрахунку електромагнітних полів, створених складними системами електричних струмів*, отримав результати: a) із використанням методів регуляризації парних функціональних суматорних рівнянь, інтег-

ральних перетворень і виділення й обернення головних частин суматорних рівнянь досліджено електромагнітне поле, що створено витком радіального електричного струму, розсіяного спірально провідним сферичним диском;
b) за допомоги методу регуляризації оператора задачі розв'язано задачу про електромагнітне поле, створюване вертикально розташованим електричним диполем над спірально провідною незамкненою сферою.

В напрямку *дифракції хвиль на об'єктах складної форми*, отримав результати: a) із використанням інтегрального перетворення типу Абеля відшукано й обернено головну частину матричного оператора задачі дифракції плоскої акустичної хвилі на сфері з коловим отвором; b) із застосуванням методу регуляризації матричного оператора задачі досліджено потенціал швидкостей плоскої акустичної хвилі, що розсіяно сферою, складеною з м'якого та жорсткого колових сегментів та розв'язано задачу дифракції електромагнітного поля, створеного вертикально розташованим диполем біля спірально провідної сфери у присутності конусу.

Вячеслав Олексійович завжди був готовий допомогати іншим, часто, відкладаючи свої власні справи. В пам'яті багатьох він залишиться як активна, працелюбна, відкрита до спілкування, добра людина.

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1. В журналі публікуються статті, що мають результати математичних досліджень (англійською або українською мовами).

2. Поданням статті вважається отримання редакцією файлів статті оформлених у редакторі LATEX (версія 2e), анотацій, відомостей про авторів та архіва, що включає LATEX файли статті та файли малюнків. Файл-зразок оформлення статті можна знайти в редакції журналу та на веб-сторінці (http://vestnik-math.univer.kharkov.ua).

3. Стаття повинна починатися з розширеної анотації (обсягом **не менш ніж 1800 знаків**), в якій повинні бути чітко сформульовані мета та результати роботи. Анотація повинна бути тією мовою (англійською або українською), якою є основний текст статті. Закордонні автори можуть звернутися до редакції за допомогою з перекладом анотацій на українську мову. Повинні бути наведені прізвища, ініціали авторів, назва роботи, ключові слова та номер за міжнародною математичною класифікацією (Mathematics Subject Classification 2010). Анотація не повинна мати посилань на літературу чи малюнки. На першій сторінці вказується номер УДК класифікації. В кінці статті треба додати переклад анотації (обсягом **не менш ніж 1800 знаків**) на другу мову (англійську чи українську).

4. Список літератури повинен бути оформлений латинським шрифтом. Приклади оформлення списка літератури:

1. A.M. Lyapunov. A new case of integrability of differential equations of motion of a solid body in liquid, Rep. Kharkov Math. Soc., – 1893. – 2. V.**4**. – P. 81-85.
2. A.M. Lyapunov. The general problem of the stability of motion. Kharkov Mathematical Society, Kharkov. - 1892. - 251 p.

5. Кожний малюнок повинен бути пронумерований та представлений окремим файлом в одному з форматів: EPS, BMP, JPG. В файлі статті малюнок повинен бути вставлений автором. Під малюнком повинен бути підпис. Назви файлів малюнків повинні починатись з призища першого автора.

6. Відомості про авторів повинні містити: прізвища, імена, по батькові, службові адреси та номери телефонів, науковий ступінь, посаду, адреси електронних скриньок та інформацію про наукові профайли авторів (orcid.org, www.researcherid.com, www.scopus.com) з відповідними посиланями. Прохання також повідомити прізвище автора, з яким треба вести листування.

7. Рекомендуємо використовувати в якості зразка оформлення останні випуски журналу (vestnik-math.univer.kharkov.ua/currentv.htm).

8. У випадку порушення правил оформлення редакція не буде розглядати статтю.

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