Вісник Харківського національного університету імені В.Н. Каразіна
Серія "Математика, прикладна математика і механіка"
Том 84, 2016, с.61–92

Visnyk of V.N.Karazin Kharkiv National University
Ser. “Mathematics, Applied Mathematics and Mechanics”
Vol. 84, 2016, p. 61–92
DOI: 10.26565/2221-5646-2016-84-06

УДК 517

Approximation properties of generalized Fup-functions

I. V. Brysina, V. A. Makarichev

N.Ye. Zhukovsky National Aerospace University "Kharkiv Aviation Institute"
Chkalov Str. 17, 61070 Kharkiv, Ukraine
iryna.brysina@gmail.com, victor.makarichev@gmail.com

Generalized Fup-functions are considered. Almost-trigonometric basis theorem is proved. Spaces of linear combinations of shifts of the generalized Fup-functions are constructed and an upper estimate of the best approximation of classes of periodic differentiable functions by these spaces in the norm of $L_2[-\pi,\pi]$ is obtained.
Keywords: function with a compact support, approximation of periodic functions, up-function, Kolmogorov width, best approximation, generalized Fup-function.

Брисіна І. В., Макарічев В. О. **Апроксимаційні властивості узагальнених Fup-функцій.** Розглянуто узагальнені Fup-функції. Доведено теорему про майже-тригонометричний базис. Побудовано простори лінійних комбінацій зсувів узагальнених Fup-функцій і отримано верхню оцінку найкращого наближення цими просторами класів періодичних диференційованих функцій за нормою $L_2[-\pi,\pi]$.
Ключові слова: фінітна функція, наближення періодичних функцій, up-функція, поперечник за Колмогоровим, найкраще наближення, узагальнені Fup-функції.

Брысина И. В., Макаричев В. А. **Аппроксимационные свойства обобщенных Fup-функций.** Рассмотрены обобщенные Fup-функции. Доказана теорема о почти-тригонометрическом базисе. Построены пространства линейных комбинаций сдвигов обобщенных Fup-функций и получена верхняя оценка наилучшего приближения этими пространствами классов периодических дифференцируемых функций по норме $L_2[-\pi,\pi]$.
Ключевые слова: финитная функция, приближение периодических функций, up-функция, поперечник по Колмогорову, наилучшее приближение, обобщенные Fup-функции.

2000 Mathematics Subject Classification 41A30, 41A50

Introduction

Construction and investigation of compactly supported functions such as splines and wavelets is an intensively developing area of mathematics. Various systems of functions with a compact support are widely used in numerical methods, mathematical physics, approximation theory, digital signal, image processing etc. In particular these systems are used for numerical solution of differential equations. Notice that the function, which is a solution of some equation, is often infinitely differentiable or has high degree of smoothness. Hence, the problem of construction of the function space L that combines the following properties is of interest:

(i) all functions from L are infinitely differentiable (this property is important for approximation of smooth functions);

(ii) in the space L there exists a basis that consists of compactly supported functions (for example, this property makes it possible to construct effective algorithms of solution of some differential equations);

(iii) the space L has good approximation properties.

Consider in detail the last property. Let X be a linear space supplied with a norm $\|\cdot\|_X$. Denote by A some subset of X. Let L be a subspace of X such that $\dim L = N$. By

$$E_X(A, L) = \sup_{\varphi \in A} \inf_{f \in L} \|\varphi - f\|_X$$

we denote the best approximation of the set A by the linear space L in the norm of X. It can be said that L has good approximation properties, if there exists small $\varepsilon > 0$ such that $E_X(A, L) < \varepsilon$. At the same time it is interesting, if there exists some other linear space $V \subset X$ such that $\dim V = N$ and $E_X(A, V) < E_X(A, L)$ (this means that V has better approximation properties than L). Therefore the value of

$$d_N(A, X) = \inf_{\dim V = N} E_X(A, V)$$

is of interest. We note that $d_N(A, X)$ is the Kolmogorov width [1].

Let $\{N_k\}_{k=1}^{\infty}$ be a sequence of positive integer numbers.

Definition 1 *The sequence of spaces $\{L_k\}_{k=1}^{\infty}$ is extremal for approximation of a set A in the norm of X, if $\dim L_k = N_k$ and $E_X(A, L_k) = d_{N_k}(A, X)$ for any $k \in \mathbb{N}$.*

Definition 2 *The sequence of spaces $\{L_k\}_{k=1}^{\infty}$ is asymptotically extremal for approximation of a set A in the norm of X, if $\dim L_k = N_k$ for any $k \in \mathbb{N}$ and*

$$\lim_{k \to \infty} \frac{E_X(A, L_k)}{d_{N_k}(A, X)} = 1.$$

We say that spaces $\{L_k\}$ have good approximation properties, if the sequence $\{L_k\}$ is extremal or asymptotically extremal for approximation of A in the norm of X.

In [2, 3, 4], the spaces, which satisfy properties (i) – (iii), were introduced. Consider in detail main results of these publications.

Consider the function

$$mup_s(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{itx} \prod_{k=1}^{\infty} \frac{\sin^2\left(st(2s)^{-k}\right)}{s^2 t(2s)^{-k} \sin\left(t(2s)^{-k}\right)} dt,$$

where $s \in \mathbb{N}$.

For the case $s = 1$ the function $mup_s(x)$ is equal to well-known Rvachev function

$$up(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{itx} \prod_{k=1}^{\infty} \frac{\sin\left(t2^{-k}\right)}{t2^{-k}} dt,$$

which is a solution with a compact support of the functional differential equation

$$y'(x) = 2(y(2x+1) - y(2x-1)).$$

As a solution of this equation the function $up(x)$ was introduced in [5] (see aslo [2] and [3]).

The function $mup_s(x)$ for the case $s \geq 2$ was constructed in [6].

For any $s \in \mathbb{N}$ the function $mup_s(x)$ combines the following properties:

1) $supp\ mup_s(x) = [-1, 1]$;

2) $mup_s(x) \in C^{\infty}(\mathbb{R})$;

3) the function $mup_s(x)$ is not analytic at any $x \in [-1, 1]$;

4) $\int_{-\infty}^{\infty} mup_s(x) dx = 1$;

5) $mup_s(x)$ is a solution of the equation

$$y'(x) = \sum_{k=1}^{s} (y(2sx + 2s - 2k + 1) - y(2sx - 2k + 1));$$

6) $\left\| mup_s^{(n)}(x) \right\|_{C[-1,1]} = 2^n (2s)^{n(n-1)/2}$ for any $n = 0, 1, 2, \ldots$.

For the case $s = 1$ the properties 1) – 6) were proved in [5] (see also [2]). For the general case these properties were proved in [6] (see also [4, 7]).

Let $MUP_{s,n}$ be the space of functions $\psi(x)$ of the form

$$\psi(x) = \sum_k c_k \cdot mup_{s,n}\left(x - \frac{k}{(2s)^n}\right), x \in [-1, 1]$$

and $\widetilde{MUP}_{s,n}$ be the space of functions $\varphi(x)$ such that

$$\varphi(x) = \sum_k c_k \cdot mup_s \left(\frac{x}{\pi} - \frac{k}{(2s)^n} \right), x \in [-\pi, \pi]$$

and $\varphi^{(j)}(-\pi) = \varphi^{(j)}(\pi)$ for any $j = 0, 1, 2, \ldots$.

Theorem 1 ([4]) *For any $n = 0, 1, 2, \ldots$ there exists the set of coefficients $\{v_j\}_{j \in \mathbb{Z}}$ such that*

$$\sum_{j \in \mathbb{Z}} v_j \cdot mup_s \left(x - \frac{j}{(2s)^n} \right) \equiv x^n.$$

This means that $MUP_{s,n}$ contains all polynomials of order not greater than n.

Further, consider the function

$$Fmup_{s,n}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} \left(\frac{\sin\left(\frac{t}{2(2s)^n}\right)}{\frac{t}{2(2s)^n}} \right)^n F_s \left(\frac{t}{(2s)^n} \right) dt,$$

where $s \in \mathbb{N}$, $n = 0, 1, 2, \ldots$ and $F_s(t)$ is the Fourier transform of the function $mup_s(x)$.

It was shown in [4] that $Fmup_{s,n}(x) \in C^\infty(\mathbb{R})$ and

$$Fmup_{s,n}(x) = 0 \text{ for } |x| > \frac{n+2}{2(2s)^n}$$

(for the case $s = 1$ these properties were obtained in [3]).

Theorem 2 ([4]) *The system of functions*

$$\left\{ Fmup_{s,n} \left(x - \frac{j}{(2s)^n} + 1 + \frac{n+2}{2(2s)^n} \right) \right\}_{j=1}^{2(2s)^n+n+1}$$

is a basis of the space $MUP_{s,n}$.

Combining this theorem with theorem 1, we see that any polynomial of degree at most n can be expressed as a linear combination of shifts of the function $Fmup_{s,n}(x)$.

Theorem 3 ([4]) *The system of functions $\{\psi_1(x), \psi_2(x), \ldots, \psi_{2(2s)^n}(x)\}$ constitutes a basis of the space $\widetilde{MUP}_{s,n}$, where*

$$\psi_k(x) = Fmup_{s,n} \left(\frac{x}{\pi} - \frac{k}{(2s)^n} + \frac{n+2}{2(2s)^n} - 1 \right) +$$
$$+ Fmup_{s,n} \left(\frac{x}{\pi} - \frac{k}{(2s)^n} + \frac{n+2}{2(2s)^n} + 1 \right), k = 1, \ldots, n+1,$$
$$\psi_k(x) = Fmup_{s,n} \left(\frac{x}{\pi} - \frac{k}{(2s)^n} + \frac{n+2}{2(2s)^n} + 1 \right), k = n+2, \ldots, 2(2s)^n.$$

It follows from this theorem that dimension of $\widetilde{MUP}_{s,n}$ equals $2(2s)^n$.

Let us remark that for the case $s = 1$ theorems 1 – 3 were obtained in [3].

Further, we need some notations. By $\widetilde{W}^r_\infty$ denote the class of functions $g \in C^r_{[-\pi,\pi]}$ such that $g^{(k)}(-\pi) = g^{(k)}(\pi)$ for any $k = 0, 1, \dots, r-1$ and $\|g^{(r)}\|_{C([-\pi,\pi])} \le 1$. Let $\widetilde{W}^r_2$ be the class of functions $g \in C^{r-1}_{[-\pi,\pi]}$ such that the equality $g^{(k)}(-\pi) = g^{(k)}(\pi)$ holds for any $k = 0, 1, \dots, r-1$, $g^{(r-1)}(x)$ is absolutely continuous and $\|g^{(r)}\|_{L_2[-\pi,\pi]}$.

Theorem 4 ([3], see also [2]) *We have*

$$\lim_{n\to\infty} \frac{E_{C([-\pi,\pi])}(\widetilde{W}^r_\infty, \widetilde{MUP}_{1,n})}{d_{2^{n+1}}(\widetilde{W}^r_\infty, C([-\pi,\pi]))} = 1.$$

In other words, spaces $\widetilde{UP}_n = \widetilde{MUP}_{1,n}$ are asymptotically extremal for approximation of $\widetilde{W}^r_\infty$ in the norm of $C([-\pi,\pi])$.

Theorem 5 ([2]) *There exists $n(r)$ such that*

$$E_{L_2[-\pi,\pi]}(\widetilde{W}^r_2, \widetilde{MUP}_{1,n}) = d_{2^{n+1}}(\widetilde{W}^r_2, L_2[-\pi,\pi])$$

for any $n \ge n(r)$.

Therefore $\widetilde{UP}_n$ is extremal for approximation of functions from the class $\widetilde{W}^r_2$ in the norm of the space $L_2[-\pi,\pi]$.

Theorem 6 ([4]) *For any $s \ge 2$ the following equality holds:*

$$\lim_{n\to\infty} \frac{E_{L_2[-\pi,\pi]}(\widetilde{W}^r_2, \widetilde{MUP}_{s,n})}{d_{2(2s)^n}(\widetilde{W}^r_2, L_2[-\pi,\pi])} = 1.$$

This means that spaces $\widetilde{MUP}_{s,n}$ are asymptotically extremal for approximation of $\widetilde{W}^r_2$ in the norm of $L_2[-\pi,\pi]$.

We see that the functions $mup_s(x)$ (in particular, the function $up(x)$) and $Fmup_{s,n}(x)$ have a number of convenient properties. Therefore these functions have applications to wavelet theory [8, 9, 10, 11, 12, 13, 14], digital signal processing [15, 16], numerical methods and numerical modeling [17, 18, 19, 20] (note that a comprehensive survey also can be found in chapter 2 of [21]), the theory of generalized Taylor series [2, 7, 11, 12, 22, 23, 24, 25] etc.

We note also that the growth rate of the dimension of $MUP_{s,n}$ and $\widetilde{MUP}_{s,n}$ is a disadvantage of these spaces. Indeed, for any $s \in \mathbb{N}$ the value of $\dim \widetilde{MUP}_{s,n} = 2(2s)^n$ increases exponentially. Our goal is to construct the space of functions that does not have this disadvantage and has all advantages of $\widetilde{MUP}_{s,n}$.

Let $f(x) \in L_2(\mathbb{R})$ be a function such that

1) *supp* $f(x) = [-1, 1]$,

2) $f(x)$ is an even function,

3) $f(x) \geq 0$ for any $x \in [-1, 1]$,

4) $\int_{-\infty}^{\infty} f(x)dx = 1$.

By $F(t)$ denote the Fourier transform of this function.

Definition 3 ([26]) *The function*

$$f_{N,m}(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{itx} \left(\frac{\sin(t/N)}{t/N} \right)^{m+1} F(t/N)\, dt,$$

where $F(t)$ is the Fourier transform of $f(x)$, $N \neq 0$ and $m \in \mathbb{N}$ is called a generalized Fup-function and f(x) is called its mother function.

Here we use the term "mother function"just as the term "mother wavelet"is used in the theory of wavelets.

It can be seen that the generalized Fup-function is a generalization of the function $Fmup_{s,n}(x)$.

The aim of this paper is to investigate the best approximation of the class $\widetilde{W}_2^r$ by the spaces of linear combinations of shifts of the generalized Fup-functions in the norm of the space $L_2[-\pi, \pi]$.

This paper is organized as follows. In section 2, we introduce and prove the almost-trigonometric basis theorem. In section 3, we construct the spaces of shifts of the generalized Fup-functions and using the almost-trigonometric basis theorem, we obtain an upper estimate of the best approximation of $\widetilde{W}_2^r$ by these spaces in the norm of $L_2[-\pi, \pi]$. In the last section, we analyze the results of this paper and consider some open problems.

Actually, in this paper we introduce a new method of construction of locally supported functions with good approximation properties.

Further, we assume that

$$\|f\| = \|f\|_{L_2[-\pi,\pi]} \text{ and } (f, g) = \int\limits_{-\pi}^{\pi} f(x)g(x)dx.$$

Almost-trigonometric basis theorem

Let N be an arbitrary even natural number.

Denote by V_N the space of functions $f(x) \in L_2[-\pi, \pi]$ such that

$$f(x) = \sum_{p=0}^{N/2-1} \left(a_p \cdot v_{N,p}(x) + b_p \cdot w_{N,p}(x) \right),$$

where

$$v_{N,0}(x) \equiv 1,$$

$$v_{N,p}(x) = \sum_{k=0}^{\infty} \left(r_{N,p,k} \cdot \cos((p+kN)x) + q_{N,p,k} \cdot \cos((N(k+1)-p)x)\right) \tag{1}$$

and

$$w_{N,p}(x) = \sum_{k=0}^{\infty} \left(s_{N,p,k} \cdot \sin((p+kN)x) + t_{N,p,k} \cdot \sin((N(k+1)-p)x)\right) \tag{2}$$

for $p = 1, 2, \ldots, N/2 - 1$,

$$w_{N,0}(x) = \sum_{k=0}^{\infty} \left(y_{N,k} \cdot \cos\left(\frac{N}{2}(2k+1)x\right) + z_{N,k} \cdot \sin\left(\frac{N}{2}(2k+1)x\right)\right), \tag{3}$$

$r_{N,p,0} = s_{N,p,0} = 1$ for any $p = 1, 2, \ldots, N/2 - 1$. We assume that series in (1) – (3) are convergent in $L_2[-\pi, \pi]$.

We shall say that the system of functions $\{v_{N,p}, w_{N,p}\}_{p=0}^{N/2-1}$ is an almost-trigonometric basis of the space V_N.

In this section, we obtain an upper estimate of the best approximation of the class $\widetilde{W}_2^r$ by the space V_N in the norm of $L_2[-\pi, \pi]$.

Theorem 7 *If there exists $m \in \mathbb{N}$, $M \geq 0$ and functions φ_1, φ_2 such that*

(i) φ_1, φ_2 *are positive, increasing and differentiable on* $[0, 1/2]$*;*

(ii) $\varphi_1(1/2) \leq 1$, $\varphi_2(1/2) \leq 1$*;*

(iii) $m + 1 \geq r$*;*

(iv) for any $p = 1, 2, \ldots, N/2 - 1$ *the following conditions hold:*

$$q_{N,p,0}^2 = \left(\frac{p}{N-p}\right)^{2(m+1)} \varphi_1^2\left(\frac{p}{N}\right), \tag{4}$$

$$t_{N,p,0}^2 = \left(\frac{p}{N-p}\right)^{2(m+1)} \varphi_2^2\left(\frac{p}{N}\right); \tag{5}$$

(v)

$$\sum_{k=1}^{\infty} \left(r_{N,p,k}^2 + q_{N,p,k}^2\right) \leq \left(\frac{p}{N}\right)^{2(m+1)} M, \tag{6}$$

$$\sum_{k=1}^{\infty} \left(s_{N,p,k}^2 + t_{N,p,k}^2\right) \leq \left(\frac{p}{N}\right)^{2(m+1)} M \tag{7}$$

for any $p = 1, 2, \ldots, N/2 - 1$*;*

then

$$E_{L_2[-\pi,\pi]}\left(\widetilde{W}_2^r, V_N\right) \le \left(\frac{N}{2}\right)^{-r} \sqrt{1+\varepsilon}, \tag{8}$$

where

$$\varepsilon = \frac{M^2}{2^{4m+3}} + \frac{M}{2^{2m+1}} + \frac{\sqrt{2M}}{2^{m+r+1}}.$$

Proof. To prove this statement, it is sufficient to obtain the following inequality:

$$\inf_{\omega\in V_N} \|f-\omega\| \le \left(\frac{N}{2}\right)^{-r} \sqrt{1+\frac{M^2}{2^{4m+3}} + \frac{M}{2^{2m+1}} + \frac{\sqrt{2M}}{2^{m+r+1}}} \tag{9}$$

for any $f \in \widetilde{W}_2^r$.

Let $f(x)$ be an arbitrary function from the class $\widetilde{W}_2^r$.

By construction, the space V_n contains all constant functions.

Let f be a non-constant function. It follows from the definition of the class $\widetilde{W}_2^r$ that

$$0 < \|f^{(r)}\| \le 1.$$

Consider the Fourier series expansion of the function f:

$$f(x) = \widetilde{a}_0 + \sum_{n=1}^{\infty}\left(\widetilde{a}_n \cdot \cos(nx) + \widetilde{b}_n \cdot \sin(nx)\right).$$

Let $g(x) = f(x) - \widetilde{a}_0$. By the above

$$\begin{aligned}\inf_{\omega\in V_N} \|f-\omega\| &= \inf_{\omega\in V_N} \|g-\omega\| \\ = \|f^{(r)}\| \cdot \inf_{\omega\in V_N} \left\|\frac{1}{\|f^{(r)}\|}g - \omega\right\| &\le \inf_{\omega\in V_n} \|\zeta-\omega\|,\end{aligned} \tag{10}$$

where $\zeta(x) = \frac{1}{\|f^{(r)}\|}g(x)$.

Notice that

$$\|\zeta^{(r)}\| = 1. \tag{11}$$

The function ζ can be represented in the following form:

$$\zeta(x) = \sum_{p=1}^{N/2}\left(\theta_{N,p}(x) + \mu_{N,p}(x)\right),$$

where

$$\theta_{N,p}(x) = \sum_{k=0}^{\infty}\left(\frac{\widetilde{a}_{p+kN}}{\|f^{(r)}\|} \cdot \cos((p+kN)x) + \frac{\widetilde{a}_{(k+1)N-p}}{\|f^{(r)}\|} \cdot \cos((N(k+1)-p)x)\right)$$

and

$$\mu_{N,p}(x)=\sum_{k=0}^{\infty}\left(\frac{\widetilde{b}_{p+kN}}{\|f^{(r)}\|}\cdot\sin((p+kN)x)+\frac{\widetilde{b}_{(k+1)N-p}}{\|f^{(r)}\|}\cdot\sin((N(k+1)-p)x)\right)$$

for $p=1,2,\ldots,N/2-1$,

$$\theta_{N,N/2}(x)=\sum_{k=0}^{\infty}\frac{\widetilde{a}_{(k+1)N/2}}{\|f^{(r)}\|}\cdot\cos\left(\frac{N}{2}(k+1)x\right),$$

$$\mu_{N,N/2}(x)=\sum_{k=0}^{\infty}\frac{\widetilde{b}_{(k+1)N/2}}{\|f^{(r)}\|}\cdot\sin\left(\frac{N}{2}(k+1)x\right).$$

Further we need some notations.

Let $c_p=\|\theta_{N,p}^{(r)}\|$, $d_p=\|\mu_{N,p}^{(r)}\|$,

$$f_p(x)=\begin{cases}\frac{1}{c_p}\theta_{N,p}(x), \text{ if } c_p\neq 0,\\ \theta_{N,p}(x), \text{ if } c_p=0\end{cases} \quad\text{and}\quad g_p(x)=\begin{cases}\frac{1}{d_p}\mu_{N,p}(x), \text{ if } d_p\neq 0,\\ \mu_{N,p}(x), \text{ if } d_p=0\end{cases}$$

for $p=1,2,\ldots,N/2$.

In these notations,

$$\zeta(x)=\sum_{p=1}^{N/2}\left(c_pf_p(x)+d_pg_p(x)\right) \text{ and } \|f_p^{(r)}\|=\|g_p^{(r)}\|=1.$$

In addition, from (11) it follows that

$$\sum_{p=1}^{N/2}\left(c_p^2+d_p^2\right)=1.$$

Therefore, to prove inequality (9) for any non-constant function $f\in\widetilde{W}_2^r$, it is sufficient to obtain an upper estimate of $\inf\limits_{\omega\in V_N}\|\zeta-\omega\|$, where

$$\zeta(x)=\sum_{p=1}^{N/2}(c_pf_p(x)+d_pg_p(x)),$$

$$f_p(x)=\sum_{k=0}^{\infty}\left(a_{p+kN}\cdot\cos((p+kN)x)+a_{(k+1)N-p}\cdot\cos((N(k+1)-p)x)\right)$$

and

$$g_p(x)=\sum_{k=0}^{\infty}\left(b_{p+kN}\cdot\sin((p+kN)x)+b_{(k+1)N-p}\cdot\sin((N(k+1)-p)x)\right)$$

for $p = 1, 2, \ldots, N/2-1$,

$$f_{N/2}(x) = \sum_{k=0}^{\infty} a_{(k+1)N/2} \cdot \cos\left(\frac{N}{2}(k+1)x\right),$$

$$g_{N/2}(x) = \sum_{k=0}^{\infty} b_{(k+1)N/2} \cdot \sin\left(\frac{N}{2}(k+1)x\right),$$

$$\left\|f_p^{(r)}\right\| = \left\|g_p^{(r)}\right\| = 1 \tag{12}$$

for $p = 1, 2, \ldots, N/2$,

$$\sum_{p=1}^{N/2} \left(c_p^2 + d_p^2\right) = 1.$$

Let us introduce some notations.

For any $p = 1, 2, \ldots, N/2-1$ let

$$\alpha_{p,1} = \left\|f_p - \frac{(f_p, v_{N,p})}{(v_{N,p}, v_{N,p})} v_{N,p}\right\|, \ \alpha_{p,2} = \left\|g_p - \frac{(g_p, w_{N,p})}{(w_{N,p}, w_{N,p})} w_{N,p}\right\|$$

and $\alpha_{N/2,1} = \|f_{N/2}\|$, $\alpha_{N/2,2} = \|g_{N/2}\|$.

It is easily shown that

$$\inf_{\omega \in V_N} \|\zeta - \omega\| \leq \max_{p=1,\ldots,N/2, j=1,2} \alpha_{p,j}. \tag{13}$$

Consider $\alpha_{p,j}$ for $p = 1, 2, \ldots, N/2$ and $j = 1, 2$.

1. For the case $p = N/2$, we have

$$\alpha_{N/2,1}^2 = \pi \sum_{k=0}^{\infty} a_{(k+1)N/2}^2 = \left(\frac{N}{2}\right)^{-2r} \pi \sum_{k=0}^{\infty} a_{(k+1)N/2}^2 \left(\frac{N}{2}\right)^{2r}$$
$$\leq \left(\frac{N}{2}\right)^{-2r} \pi \sum_{k=0}^{\infty} \left((k+1)\frac{N}{2}\right)^{2r} a_{(k+1)N/2}^2 = \left(\frac{N}{2}\right)^{-2r} \left\|f_{N/2}^{(r)}\right\|^2.$$

If we combine this with (12), we get $\alpha_{N/2,1}^2 \leq (N/2)^{-2r}$.

By the same argument, $\alpha_{N/2,2}^2 \leq (N/2)^{-2r}$.

Hence,

$$\alpha_{N/2,j}^2 \leq \left(\frac{N}{2}\right)^{-2r} \tag{14}$$

for $j = 1, 2$.

2. Let $p = 1, 2, \ldots, N/2-1$ and $j = 1, 2$.

Consider the functions

$$\ell_{p,j}(x) = \begin{cases} a_p \cdot \cos(px) + a_{N-p} \cdot \cos((N-p)x), & \text{if } j = 1, \\ b_p \cdot \sin(px) + b_{N-p} \cdot \sin((N-p)x), & \text{if } j = 2, \end{cases}$$

$$h_{p,j}(x) = \begin{cases} f_p(x) - \ell_{p,j}(x), & \text{if } j = 1, \\ g_p(x) - \ell_{p,j}(x), & \text{if } j = 2, \end{cases}$$

$$\zeta_{p,j}(x) = \begin{cases} \cos(px) + q_{N,p,0} \cdot \cos((N-p)x), & \text{if } j = 1, \\ \sin(px) + t_{N,p,0} \cdot \sin((N-p)x), & \text{if } j = 2, \end{cases}$$

$$\epsilon_{p,j}(x) = \begin{cases} v_{N,p}(x) - \zeta_{p,j}(x), & \text{if } j = 1, \\ w_{N,p}(x) - \zeta_{p,j}(x), & \text{if } j = 2. \end{cases}$$

Then

$$\left\| (\ell_{p,j} + h_{p,j}) - \frac{(\ell_{p,j} + h_{p,j}, \zeta_{p,j} + \epsilon_{p,j})}{(\zeta_{p,j} + \epsilon_{p,j}, \zeta_{p,j} + \epsilon_{p,j})} (\zeta_{p,j} + \epsilon_{p,j}) \right\|^2.$$

It is not hard to prove that $\ell_{p,j}$, $\zeta_{p,j}$ are orthogonal to the functions $h_{p,j}$, $\epsilon_{p,j}$, i. e.

$$(\ell_{p,j}, h_{p,j}) = (\ell_{p,j}, \epsilon_{p,j}) = (\zeta_{p,j}, h_{p,j}) = (\zeta_{p,j}, \epsilon_{p,j}) = 0. \tag{15}$$

This implies that

$$\alpha^2_{p,j} = \left\| \ell_{p,j} + h_{p,j} - \frac{(\ell_{p,j}, \zeta_{p,j}) + (h_{p,j}, \epsilon_{p,j})}{\|\zeta_{p,j}\|^2 + \|\epsilon_{p,j}\|^2} (\zeta_{p,j} + \epsilon_{p,j}) \right\|^2 = A_1 + A_2 + A_3,$$

where

$$A_1 = \left\| \ell_{p,j} - \frac{(\ell_{p,j}, \zeta_{p,j})}{\|\zeta_{p,j}\|^2 + \|\epsilon_{p,j}\|^2} (\zeta_{p,j} + \epsilon_{p,j}) \right\|^2,$$

$$A_2 = \left\| h_{p,j} - \frac{(h_{p,j}, \epsilon_{p,j})}{\|\zeta_{p,j}\|^2 + \|\epsilon_{p,j}\|^2} (\zeta_{p,j} + \epsilon_{p,j}) \right\|^2 \tag{16}$$

and

$$A_3 = 2 \left(\ell_{p,j} - \frac{(\ell_{p,j}, \zeta_{p,j}) \cdot (\zeta_{p,j} + \epsilon_{p,j})}{\|\zeta_{p,j}\|^2 + \|\epsilon_{p,j}\|^2} \ , \ h_{p,j} - \frac{(h_{p,j}, \epsilon_{p,j}) \cdot (\zeta_{p,j} + \epsilon_{p,j})}{\|\zeta_{p,j}\|^2 + \|\epsilon_{p,j}\|^2} \right). \tag{17}$$

Further, $A_1 = A_{1,1} + A_{1,2} - 2 \cdot A_{1,3}$, where

$$A_{1,1} = \left\| \ell_{p,j} - \frac{(\ell_{p,j}, \zeta_{p,j})}{\|\zeta_{p,j}\|^2 + \|\epsilon_{p,j}\|^2} \zeta_{p,j} \right\|^2,$$

$$A_{1,2} = \frac{(\ell_{p,j}, \zeta_{p,j})^2}{\left(\|\zeta_{p,j}\|^2 + \|\epsilon_{p,j}\|^2\right)^2} \|\epsilon_{p,j}\|^2 \tag{18}$$

and

$$A_{1,3} = \left(\ell_{p,j} - \frac{(\ell_{p,j}, \zeta_{p,j})}{\|\zeta_{p,j}\|^2 + \|\epsilon_{p,j}\|^2} \zeta_{p,j} \ , \ \frac{(\ell_{p,j}, \zeta_{p,j})}{\|\zeta_{p,j}\|^2 + \|\epsilon_{p,j}\|^2} \epsilon_{p,j} \right).$$

Using (15), we get $A_{1,3} = 0$. Also, we see that

$$A_{1,1} = \left\| \ell_{p,j} - (\ell_{p,j}, \zeta_{p,j}) \left(\frac{1}{\|\zeta_{p,j}\|^2} + \frac{1}{\|\zeta_{p,j}\|^2 + \|\epsilon_{p,j}\|^2} - \frac{1}{\|\zeta_{p,j}\|^2} \right) \zeta_{p,j} \right\|^2 =$$

$$= A_{1,1,1} + A_{1,1,2} + A_{1,1,3},$$

where

$$A_{1,1,1} = \left\| \ell_{p,j} - \frac{(\ell_{p,j}, \zeta_{p,j})}{\|\zeta_{p,j}\|^2} \zeta_{p,j} \right\|^2, \tag{19}$$

$$A_{1,1,2} = (\ell_{p,j}, \zeta_{p,j})^2 \left(\frac{1}{\|\zeta_{p,j}\|^2 + \|\epsilon_{p,j}\|^2} - \frac{1}{\|\zeta_{p,j}\|^2} \right)^2 \|\zeta_{p,j}\|^2 \tag{20}$$

and

$$A_{1,1,3} = \left(\ell_{p,j} - \frac{(\ell_{p,j}, \zeta_{p,j})}{\|\zeta_{p,j}\|^2} \zeta_{p,j} \ , \ \zeta_{p,j} \right) \cdot \left(\frac{1}{\|\zeta_{p,j}\|^2 + \|\epsilon_{p,j}\|^2} - \frac{1}{\|\zeta_{p,j}\|} \right).$$

It follows from (15) that $A_{1,1,3} = 0$.

Consequently

$$\alpha_{p,j}^2 = A_{1,1,1} + A_{1,1,2} + A_{1,2} + A_2 + A_3. \tag{21}$$

By $\gamma_{p,j}$ denote $\left\| \ell_{p,j}^{(r)} \right\|^2$. Using (12), we get

$$\left\| h_{p,j}^{(r)} \right\|^2 = 1 - \gamma_{p,j} \tag{22}$$

and

$$0 \leq \gamma_{p,j} \leq 1. \tag{23}$$

Let us prove that

$$A_{1,1,1} \leq \gamma_{p,j} \left(\frac{N}{2} \right)^{-2r}. \tag{24}$$

By construction

$$(\ell_{p,j}, \zeta_{p,j}) = \begin{cases} \pi \left(a_p + a_{N-p} \cdot q_{N,p,0} \right), \ \text{if } j = 1, \\ \pi \left(b_p + b_{N-p} \cdot t_{N,p,0} \right), \ \text{if } j = 2 \end{cases} \tag{25}$$

and

$$\|\zeta_{p,j}\|^2 = \begin{cases} \pi \left(1 + q_{N,p,0}^2 \right), \ \text{if } j = 1, \\ \pi \left(1 + t_{N,p,0}^2 \right), \ \text{if } j = 2. \end{cases} \tag{26}$$

If we combine these equalities with (19), we get

$$A_{1,1,1} = \pi \left(\left(a_p - \frac{a_p + a_{N-p} q_{N,p,0}}{1 + q_{N,p,0}^2} \right)^2 + \left(a_{N-p} - \frac{a_p + a_{N-p} q_{N,p,0}}{1 + q_{N,p,0}^2} q_{N,p,0} \right)^2 \right)$$

$$= \pi \frac{(a_p \cdot q_{N,p,0}^2 - a_{N-p} \cdot q_{N,p,0})^2 + (a_{N-p} - a_p \cdot q_{N,p,0})^2}{\left(1 + q_{N,p,0}^2\right)^2} = \pi \frac{(a_{N-p} - a_p \cdot q_{N,p,0})^2}{1 + q_{N,p,0}^2}$$

for the case $j = 1$.

Similarly,

$$A_{1,1,1} = \pi \frac{(b_{N-p} - b_p \cdot t_{N,p,0})^2}{1 + t_{N,p,0}^2}$$

for the case $j = 2$.

Also, we see that

$$\gamma_{p,j} = \begin{cases} \pi \left(p^{2r} \cdot a_p^2 + (N-p)^2 \cdot a_{N-p}^2 \right), \text{ if } j = 1 \\ \pi \left(p^{2r} \cdot b_p^2 + (N-p)^{2r} \cdot b_{N-p}^2 \right), \text{ if } j = 2. \end{cases} \tag{27}$$

It is not hard to prove that

$$\max_{(x,y)\in D} (y - c \cdot x)^2 = \frac{\gamma_{p,j}}{\pi} \cdot \frac{p^{2r} + c^2 \cdot (N-p)^{2r}}{p^{2r} \cdot (N-p)^{2r}},$$

where

$$D = \left\{ (x,y) \in \mathbb{R}^2 : \ x^2 \cdot p^{2r} + y^2 \cdot (N-p)^{2r} = \frac{\gamma_{p,j}}{\pi} \right\}.$$

By setting $c = a_p$ and $c = b_p$, we obtain

$$A_{1,1,1} \le \begin{cases} \frac{\pi}{1+q_{N,p,0}^2} \cdot \frac{\gamma_{p,j}}{\pi} \cdot \frac{p^{2r}+(N-p)^{2r} \cdot q_{N,p,0}^2}{p^{2r} \cdot (N-p)^{2r}}, \text{ if } j = 1, \\ \frac{\pi}{1+t_{N,p,0}^2} \cdot \frac{\gamma_{p,j}}{\pi} \cdot \frac{p^{2r}+(N-p)^{2r} \cdot t_{N,p,0}^2}{p^{2r} \cdot (N-p)^{2r}}, \text{ if } j = 2. \end{cases}$$

By (4) and (5), it follows that

$$A_{1,1,1} \le \gamma_{p,j} \cdot \frac{p^{2r} + (N-p)^{2r} \cdot \frac{p^{2(m+1)}}{(N-p)^{2(m+1)}} \cdot \varphi_j^2 \left(\frac{p}{N}\right)}{p^{2r} \cdot (N-p)^{2r} \cdot \left(1 + \frac{p^{2(m+1)}}{(N-p)^{2(m+1)}} \cdot \varphi_j^2 \left(\frac{p}{N}\right)\right)}$$

$$= \gamma_{p,j} \cdot \frac{N^{2(m+1-r)}}{N^{2(m+1)}} \cdot \frac{\left(1 - \frac{p}{N}\right)^{2(m+1-r)} + \left(\frac{p}{N}\right)^{2(m+1-r)} \cdot \varphi_j^2 \left(\frac{p}{N}\right)}{\left(1 - \frac{p}{N}\right)^{2(m+1)} + \left(\frac{p}{N}\right)^{2(m+1)} \cdot \varphi_j^2 \left(\frac{p}{N}\right)}.$$

Hence,

$$A_{1,1,1} \le \gamma_{p,j} \cdot N^{-2r} \cdot \eta_j \left(\frac{p}{N}\right), \tag{28}$$

where

$$\eta_j(x) = \frac{(1-x)^{2(m+1-r)} + x^{2(m+1-r)} \cdot \varphi_j^2(x)}{(1-x)^{2(m+1)} + x^{2(m+1)} \cdot \varphi_j^2(x)}.$$

Let us prove that $\eta_j(p/N) \le 2^{2r}$.

We get

$$\eta_j'(x) = \frac{\xi_j(x)}{\left((1-x)^{2(m+1)} + x^{2(m+1)} \cdot \varphi_j^2(x)\right)^2},$$

where

$$\zeta_j(x) = \Big((1-x)^{2(m+1-r)} + x^{2(m+1-r)} \cdot \varphi_j^2(x)\Big)'$$
$$\times \Big((1-x)^{2(m+1)} + x^{2(m+1)} \cdot \varphi_j^2(x)\Big)$$
$$- \Big((1-x)^{2(m+1-r)} + x^{2(m+1-r)} \cdot \varphi_j^2(x)\Big)$$
$$\times \Big((1-x)^{2(m+1)} + x^{2(m+1)} \cdot \varphi_j^2(x)\Big)'.$$

It can be easily checked that if $m+1=r$, then

$$\xi_j(x) = 2\varphi_j(x)\varphi_j'(x)\left((1-x)^{2r} - x^{2r}\right) + 2r\varphi_j^2(x)\left((1-x)^{2r} - x^{2r}\right) +$$
$$+2r\left((1-x)^{2r-1} - x^{2r-1}\varphi_j^4(x)\right).$$

By the assumption of the theorem φ_j is an increasing differentiable function and $0 \le \varphi_j(x) \le 1$. Thus $\xi_j(x) \ge 0$. Hence, $\eta_j'(x) \ge 0$ for any $x \in [0, 1/2]$.

It is not hard to prove that if $m+1 > r$, then

$$\xi_j(x) = 2r\left((1-x)^{4m-2r+3} - x^{4m-2r+3}\varphi_j^4(x)\right)$$
$$+2\varphi_j(x)\varphi_j'(x)(1-x)^{2(m+1-r)}x^{2(m+1-r)}\left((1-x)^{2r} - x^{2r}\right)$$
$$+2\varphi_j^2(x)((m+1-r)x^{2(m-r)+1}(1-x)^{2(m-r)+1}((1-x)^{2r+1} - x^{2r+1})$$
$$+(m+1)x^{2(m+1-r)}(1-x)^{2(m+1-r)}((1-x)^{2r-1} - x^{2r-1})).$$

By the same argument, $\xi_j(x) \ge 0$ and $\eta_j'(x) \ge 0$ for any $x \in [0, 1/2]$. Therefore the function $\eta_j(x)$ increases on the segment $[0, 1/2]$.

Since $p \le N/2 - 1$, we see that $\eta_j(p/N) \le \eta_j(1/2 - 1/N)$. By construction,

$$\eta_j\left(\frac{1}{2} - \frac{1}{N}\right) = \frac{\left(\frac{1}{2} + \frac{1}{N}\right)^{2(m+1-r)} + \left(\frac{1}{2} - \frac{1}{N}\right)^{2(m+1-r)} \cdot \varphi_j^2\left(\frac{1}{2} - \frac{1}{N}\right)}{\left(\frac{1}{2} + \frac{1}{N}\right)^{2(m+1)} + \left(\frac{1}{2} - \frac{1}{N}\right)^{2(m+1)} \cdot \varphi_j^2\left(\frac{1}{2} - \frac{1}{N}\right)}$$

$$= 2^{2r} \cdot \frac{\sum\limits_{k=0}^{2(m+1-r)} \binom{2(m+1-r)}{k} \left(1 + (-1)^k \cdot \varphi_j^2\left(\frac{1}{2} - \frac{1}{N}\right)\right)\left(\frac{2}{N}\right)^k}{\sum\limits_{k=0}^{2(m+1)} \binom{2(m+1)}{k} \left(1 + (-1)^k \cdot \varphi_j^2\left(\frac{1}{2} - \frac{1}{N}\right)\right)\left(\frac{2}{N}\right)^k}.$$

It follows from the properties of the function $\varphi_j(x)$ that

$$0 \le \varphi_j\left(\frac{1}{2} - \frac{1}{N}\right) \le \varphi_j\left(\frac{1}{2}\right) \le 1. \tag{29}$$

If we combine this with

$$\binom{2(m+1-r)}{k} \le \binom{2(m+1)}{k},$$

we get

$$\eta_j\left(\frac{1}{2}-\frac{1}{N}\right)\leq 2^{2r}\cdot\frac{\sum\limits_{k=0}^{2(m+1-r)}\left(\begin{array}{c}2(m+1)\\k\end{array}\right)\left(1+(-1)^k\cdot\varphi_j^2\left(\frac{1}{2}-\frac{1}{N}\right)\right)\left(\frac{2}{N}\right)^k}{\sum\limits_{k=0}^{2(m+1)}\left(\begin{array}{c}2(m+1)\\k\end{array}\right)\left(1+(-1)^k\cdot\varphi_j^2\left(\frac{1}{2}-\frac{1}{N}\right)\right)\left(\frac{2}{N}\right)^k}.$$

Obviously,

$$\eta_j\left(\frac{1}{2}-\frac{1}{N}\right)\leq 2^{2r}.$$

Hence, $\eta_j(p/N)\leq 2^{2r}$. Combining this inequality with (28), we obtain (24).

Let us prove that

$$A_{1,1,2}\leq\gamma_{p,j}\cdot M^2\cdot\left(\frac{N}{2}\right)^{-2r}\cdot\frac{1}{2^{4m+3}}. \tag{30}$$

Using (20), we obtain

$$A_{1,1,2}=(\ell_{p,j},\zeta_{p,j})^2\cdot\frac{\|\epsilon_{p,j}\|^4}{\|\zeta_{p,j}\|^2\cdot(\|\zeta_{p,j}\|^2+\|\epsilon_{p,j}\|^2)^2}.$$

It follows from (25) that $\|\zeta_{p,j}\|^2\geq\pi$ and

$$\|\zeta_{p,j}\|^2+\|\epsilon_{p,j}\|^2\geq\pi. \tag{31}$$

Therefore,

$$A_{1,1,2}\leq(\ell_{p,j},\zeta_{p,j})^2\cdot\|\epsilon_{p,j}\|^4\cdot\pi^{-3}.$$

From (4), (5) and (29) it follows that $q_{N,p,0}^2\leq 1$ and $t_{N,p,0}^2\leq 1$. If we combine these inequalities with (25) and (27), we get

$$(\ell_{p,1},\zeta_{p,1})^2\leq 2\pi^2\left(a_p^2+a_{N-p}^2\cdot q_{N,p,0}^2\right)\leq 2\pi^2\left(a_p^2+a_{N-p}^2\right)$$

$$\leq 2\pi^2\cdot p^{-2r}\cdot\left(a_p^2\cdot p^{2r}+a_{N-p}^2\cdot(N-p)^{2r}\right)=2\pi\cdot p^{-2r}\cdot\gamma_{p,1}.$$

Similarly, $(\ell_{p,2},\zeta_{p,2})^2\leq 2\pi\cdot p^{-2r}\cdot\gamma_{p,2}$.

Hence,

$$(\ell_{p,j},\zeta_{p,j})^2\leq 2\pi\cdot p^{-2r}\cdot\gamma_{p,j} \tag{32}$$

for $j=1,2$.

By construction,

$$\epsilon_{p,j}(x)=\begin{cases}\sum\limits_{k=1}^{\infty}(r_{N,p,k}\cdot\cos((p+kN)x)+\\ \qquad +q_{N,p,k}\cdot\cos((N(k+1)-p)x)),\ \text{if } j=1,\\ \sum\limits_{k=1}^{\infty}(s_{N,p,k}\cdot\sin((p+kN)x)+\\ \qquad +t_{N,p,k}\cdot\sin((N(k+1)-p)x)),\ \text{if } j=2.\end{cases}$$

This implies that

$$\|\epsilon_{p,j}\|^2 = \begin{cases} \pi \sum\limits_{k=1}^{\infty} \left(r_{N,p,k}^2 + q_{N,p,k}^2 \right), \text{ if } j = 1, \\ \pi \sum\limits_{k=1}^{\infty} \left(s_{N,p,k}^2 + t_{N,p,k}^2 \right), \text{ if } j = 2. \end{cases}$$

Using (6) and (7), we get

$$\|\epsilon_{p,j}\|^2 \leq M \cdot \pi \cdot \left(\frac{p}{N} \right)^{2(m+1)}. \tag{33}$$

If we combine this inequality with (32), we obtain

$$A_{1,1,2} \leq 2M^2 \cdot \gamma_{p,j} \cdot \frac{p^{4(m+1)-2r}}{N^{4(m+1)}}.$$

Since $p < N/2$, it can easily be checked that (30) holds.
Let us show that

$$A_{1,2} \leq \gamma_{p,j} \cdot \left(\frac{N}{2} \right)^{-2r} \cdot \frac{M}{2^{2m+1}}. \tag{34}$$

It follows from (18), (31), (32) and (33) that

$$A_{1,2} \leq 2\pi p^{-2r} \gamma_{p,j} \frac{1}{\pi^2} \left(\frac{p}{N} \right)^{2(m+1)} \cdot M\pi = \gamma_{p,j} \cdot 2M \cdot \frac{p^{2(m+1-r)}}{N^{2(m+1)}}.$$

By assumption, $m + 1 \geq r$ and $p < N/2$. Therefore,

$$A_{1,2} \leq \gamma_{p,j} \cdot 2M \cdot \left(\frac{N}{2} \right)^{2(m+1-r)} \cdot \frac{1}{N^{2(m+1)}} = \gamma_{p,j} \left(\frac{N}{2} \right)^{-2r} \frac{M}{2^{2m+1}}.$$

Let us prove that

$$A_2 \leq \left(\frac{N}{2} \right)^{-2r} \cdot \frac{1 - \gamma_{p,j}}{2^{2r}} \tag{35}$$

Using (16), we get

$$A_2 = \|h_{p,j}\|^2 + \frac{(h_{p,j}, \epsilon_{p,j})^2}{(\|\zeta_{p,j}\|^2 + \|\epsilon_{p,j}\|^2)^2} \cdot (\zeta_{p,j} + \epsilon_{p,j} \ , \ \zeta_{p,j} + \epsilon_{p,j})$$

$$-2 \cdot \frac{(h_{p,j}, \epsilon_{p,j})}{\|\zeta_{p,j}\|^2 + \|\epsilon_{p,j}\|^2} \cdot (h_{p,j} \ , \ \zeta_{p,j} + \epsilon_{p,j}).$$

If we combine this with (15), we obtain

$$A_2 = \|h_{p,j}\|^2 + \frac{(h_{p,j}, \epsilon_{p,j})^2}{\|\zeta_{p,j}\|^2 + \|\epsilon_{p,j}\|^2} - 2 \cdot \frac{(h_{p,j}, \epsilon_{p,j})^2}{\|\zeta_{p,j}\|^2 + \|\epsilon_{p,j}\|^2} \leq \|h_{p,j}\|^2.$$

By construction, we have

$$\|h_{p,1}\|^2 = \pi \sum_{k=1}^{\infty} \left(a_{p+kN}^2 + a_{(k+1)N-p}^2 \right)$$

$$\leq \pi (p+N)^{-2r} \sum_{k=1}^{\infty} \left((p+kN)^{2r} a_{p+kN}^2 + ((k+1)N-p)^{2r} a_{(k+1)N-p}^2 \right) =$$

$$= (p+N)^{-2r} \cdot \left\| h_{p,1}^{(r)} \right\|^2 .$$

Similarly, $\|h_{p,2}\|^2 \leq (p+N)^{-2r} \cdot \left\| h_{p,2}^{(r)} \right\|^2$. Thus

$$\|h_{p,j}\|^2 \leq (p+N)^{-2r} \cdot \left\| h_{p,j}^{(r)} \right\|^2 , j = 1, 2$$

Combining this with (22) and $p < N/2$, we get

$$\|h_{p,j}\|^2 \leq \frac{1-\gamma_{p,j}}{(p+N)^{2r}} = \frac{1-\gamma_{p,j}}{N^{2r}\left(1+\frac{p}{N}\right)^{2r}} \leq \frac{1-\gamma_{p,j}}{N^{2r}} = \frac{1-\gamma_{p,j}}{2^{2r}} \cdot \left(\frac{N}{2}\right)^{-2r} . \quad (36)$$

Inequality (35) follows.

Let us show that

$$|A_3| \leq \sqrt{2M \cdot \gamma_{p,j} \cdot (1-\gamma_{p,j})} \cdot \left(\frac{N}{2}\right)^{-2r} \cdot \frac{1}{2^{m+r}} . \quad (37)$$

From (15) and (17) it follows that

$$A_3 = 2(\ell_{p,j}, h_{p,j}) - 2\frac{(h_{p,j}, \epsilon_{p,j})}{\|\zeta_{p,j}\|^2 + \|\epsilon_{p,j}\|^2} (\ell_{p,j}, \zeta_{p,j} + \epsilon_{p,j})$$

$$-2\frac{(\ell_{p,j}, \zeta_{p,j})}{\|\zeta_{p,j}\|^2 + \|\epsilon_{p,j}\|^2} (h_{p,j}, \zeta_{p,j} + \epsilon_{p,j}) +$$

$$+2\frac{(\ell_{p,j}, \zeta_{p,j}) \cdot (h_{p,j}, \epsilon_{p,j})}{(\|\zeta_{p,j}\|^2 + \|\epsilon_{p,j}\|^2)^2} \left(\|\zeta_{p,j}\|^2 + \|\epsilon_{p,j}\|^2\right) = -2\frac{(\ell_{p,j}, \zeta_{p,j}) \cdot (h_{p,j}, \epsilon_{p,j})}{\|\zeta_{p,j}\|^2 + \|\epsilon_{p,j}\|^2} .$$

Therefore,

$$|A_3| \leq 2\frac{|(\ell_{p,j}, \zeta_{p,j})| \cdot |(h_{p,j}, \epsilon_{p,j})|}{\|\zeta_{p,j}\|^2 + \|\epsilon_{p,j}\|^2} .$$

If we combine this inequality with (31), (32), (33), (36) and $p < N/2$, we obtain

$$|A_3| \leq 2 \cdot \frac{1}{\pi} \cdot \sqrt{2\pi \cdot \gamma_{p,j}} \cdot p^{-r} \cdot \frac{\sqrt{1-\gamma_{p,j}}}{(p+N)^r} \cdot \sqrt{\pi M} \cdot \left(\frac{p}{N}\right)^{m+1}$$

$$= 2\sqrt{2M\gamma_{p,j}(1-\gamma_{p,j})} \frac{p^{m+1-r}}{N^{m+1}(p+N)^r}$$

$$= 2\sqrt{2M\gamma_{p,j}(1-\gamma_{p,j})}\frac{p^{m+1-r}}{N^{m+1+r}\left(1+\frac{p}{N}\right)^r} \le$$

$$\le 2\sqrt{2M\gamma_{p,j}(1-\gamma_{p,j})}\cdot\frac{(N/2)^{m+1-r}}{N^{m+1+r}}$$

$$= \sqrt{2M\gamma_{p,j}(1-\gamma_{p,j})}\left(\frac{N}{2}\right)^{-2r}\frac{1}{2^{m+r}}.$$

Inequality (37) is proved.

It follows from (21), (24), (30), (34), (35) and (37) that

$$\alpha_{p,j}^2 \le \left(\frac{N}{2}\right)^{-2r}\left(\gamma_{p,j}+\frac{1-\gamma_{p,j}}{2^{2r}}+\frac{M^2\gamma_{p,j}}{2^{4m+3}}+\frac{M\gamma_{p,j}}{2^{2m+1}}+\frac{\sqrt{2M\gamma_{p,j}(1-\gamma_{p,j})}}{2^{m+r}}\right).$$

Since $\gamma_{p,j}\in[0,1]$ (see (23)), we have

$$\gamma_{p,j}+\frac{1-\gamma_{p,j}}{2^{2r}}\le 1 \text{ and } \sqrt{\gamma_{p,j}(1-\gamma_{p,j})}\le\frac{1}{2}.$$

Hence,

$$\alpha_{p,j}^2 \le \left(\frac{N}{2}\right)^{-2r}\left(1+\frac{\sqrt{2M}}{2^{m+r+1}}+\frac{M}{2^{2m+1}}+\frac{M^2}{2^{4m+3}}\right),$$

where $p=1,2,\ldots,N/2-1$ and $j=1,2$. If we combine this with (13) and (14), we get

$$\inf_{\omega\in V_N}\|\zeta-\omega\| \le \left(\frac{N}{2}\right)^{-2r}\left(1+\frac{\sqrt{2M}}{2^{m+r+1}}+\frac{M}{2^{2m+1}}+\frac{M^2}{2^{4m+3}}\right).$$

Using (10), we see that inequality (9) is satisfied for any non-constant function $f\in\widetilde{W}_2^r$.

This completes the proof.

Remark In [1], A.N. Kolmogorov proved that

$$d_N\left(\widetilde{W}_2^r, L_2[-\pi,\pi]\right) = \left(\frac{N}{2}\right)^{-r}. \qquad (38)$$

Therefore the estimate (8) can be expressed as follows:

$$E_{L_2[-\pi,\pi]}\left(\widetilde{W}_2^r, V_N\right) \le d_N\left(\widetilde{W}_2^r, L_2[-\pi,\pi]\right)\sqrt{1+\frac{M^2}{2^{4m+3}}+\frac{M}{2^{2m+1}}+\frac{\sqrt{2M}}{2^{m+r+1}}}.$$

This means that V_N is an almost best linear space for approximation of $\widetilde{W}_2^r$ in the norm of $L_2[-\pi,\pi]$.

Approximation properties of generalized Fup-functions

Consider the generalized Fup-function

$$f_{N,m}(x)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}e^{itx}\left(\frac{\sin(t/N)}{t/N}\right)^{m+1}F(t/N)\,dt,$$

where N is an even natural number, $m\in\mathbb{N}$, $m\le N-2$ and $F(t)$ is the Fourier transform of the mother function $f(x)\in L_2(\mathbb{R})$ such that

(i) *supp* $f(x)=[-1,1]$,

(ii) $f(x)$ is an even function,

(iii) $f(x)\ge 0$ for any $x\in[-1,1]$,

(iv) $\int_{-\infty}^{\infty}f(x)dx=1$,

(v) $F(\pi)\ge 0$.

Using Paley–Wiener theorem, we get

$$\mathit{supp}\ f_{N,m}\subseteq\left[-\frac{m+2}{N},\frac{m+2}{N}\right]. \tag{39}$$

Let

$$f_{N,m,k}(x)=f_{N,m}\left(\frac{x}{\pi}-\frac{2k-m-2+N}{N}\right)+f_{N,m}\left(\frac{x}{\pi}-\frac{2k-m-2-N}{N}\right)$$

for any $k=1,2,\ldots,m+1$ and

$$f_{N,m,k}(x)=f_{N,m}\left(\frac{x}{\pi}-\frac{2k-m-2-N}{N}\right)$$

for $k=m+2,\ldots,N$.

Denote by $L_{N,m}$ the space of functions φ such that

$$\varphi(x)=\sum_{k=1}^{N}c_k\cdot f_{N,m,k}(x),\ \ x\in[-\pi,\pi].$$

The aim of this section is to prove the following result.

Theorem 8 *If $m+1\ge r$, then there exists $M\ge 0$ such that*

$$E_{L_2[-\pi,\pi]}\left(\widetilde{W}_2^r,L_{N,m}\right)\le\left(\frac{N}{2}\right)^{-r}\sqrt{1+\varepsilon}, \tag{40}$$

where

$$\varepsilon=\frac{M^2}{2^{4m+3}}+\frac{M}{2^{2m+1}}+\frac{\sqrt{2M}}{2^{m+r+1}}$$

Proof.

First we shall show that there exists an almost-trigonometric basis of the space $L_{N,m}$.

Let us expand the function $f_{N,m,k}$ in the Fourier series:

$$f_{N,m,k}(x) = \sum_{j=-\infty}^{\infty} c_j \cdot e^{ijx}, \quad \text{where} \quad c_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_{N,m,k}(x) \cdot e^{-ijx} dx.$$

For the case $k = 1, 2, \ldots, m+1$ we get $c_j = I_1 + I_2$, where

$$I_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_{N,m} \left(\frac{x}{\pi} - \frac{2k-m-2+N}{N} \right) \cdot e^{-ijx} dx$$

and

$$I_2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_{N,m} \left(\frac{x}{\pi} - \frac{2k-m-2-N}{N} \right) \cdot e^{-ijx} dx.$$

After the change of variables we obtain

$$I_1 = \frac{1}{2} \exp\left(-ij\pi \cdot \frac{2k-m-2-N}{N} \right) \cdot \int_{-2-\frac{2k-m-2}{N}}^{-\frac{2k-m-2}{N}} f_{N,m}(z) \cdot e^{-ij\pi z} dz,$$

$$I_2 = \frac{1}{2} \exp\left(-ij\pi \cdot \frac{2k-m-2+N}{N} \right) \cdot \int_{-\frac{2k-m-2}{N}}^{2-\frac{2k-m-2}{N}} f_{N,m}(z) \cdot e^{-ij\pi z} dz.$$

Since

$$\exp\left(-ij\pi \cdot \frac{2k-m-2-N}{N} \right) = \exp\left(-ij\pi \cdot \frac{2k-m-2}{N} \right) \cdot \exp(ij\pi)$$

$$= \exp\left(-ij\pi \cdot \frac{2k-m-2}{N} \right) \cdot \exp(-ij\pi) = \exp\left(-ij\pi \cdot \frac{2k-m-2+N}{N} \right),$$

we get

$$c_j = \frac{1}{2} \exp\left(-ij\pi \cdot \frac{2k-m-2-N}{N} \right) \cdot \int_{-2-\frac{2k-m-2}{N}}^{2-\frac{2k-m-2}{N}} f_{N,m}(z) \cdot e^{-ijz} dz.$$

Using $1 \le k \le m+1$ and $m \le N-2$, we have

$$-2 - \frac{2k-m-2}{N} \le -1 \le -\frac{m+2}{N} \quad \text{and} \quad 2 - \frac{2k-m-2}{N} \ge 1 \ge \frac{m+2}{N}.$$

If we combine these inequalities with (39), we obtain

$$c_j = \frac{1}{2}\exp\left(-ij\pi\cdot\frac{2k-m-2-N}{N}\right)\cdot\int\limits_{-\infty}^{\infty} f_{N,m}(z)\cdot e^{-ij\pi z}dz.$$

Therefore

$$c_j = \frac{1}{2}\exp\left(-ij\pi\cdot\frac{2k-m-2-N}{N}\right)\cdot F_{N,m}(j\pi). \tag{41}$$

Similarly we can obtain (41) for $k = m+2,\dots,N$.

Consequently

$$f_{N,m,k}(x) = \frac{1}{2}\sum_{j=-\infty}^{\infty}\exp\left(-ij\pi\cdot\frac{2k-m-2-N}{N}\right)\cdot F_{N,m}(j\pi)\cdot e^{ijx} \tag{42}$$

holds for every $k = 1,2,\dots,N$.

Consider the functions

$$\zeta_{N,m,p}(x) = \frac{2}{N}\sum_{k=1}^{N} f_{N,m,k}(x)\cdot\exp\left(ip\pi\cdot\frac{2k-m-2-N}{N}\right)$$

for $p = 1,2,\dots,N$.

By construction,

$$\zeta_{N,m,p}(x) = \frac{1}{N}\sum_{j=-\infty}^{\infty} a_{p,j}\cdot F_{N,m}(j\pi)\cdot e^{ijx},$$

where

$$a_{p,j} = \sum_{k=1}^{N}\exp\left(i\pi(p-j)\cdot\frac{2k-m-2-N}{N}\right).$$

We have

$$a_{p,j} = \exp\left(i\pi(p-j)\left(-1-\frac{m}{N}\right)\right)\cdot\sum_{k=1}^{N}\left(\exp\left(i\pi\cdot\frac{2(p-j)}{N}\right)\right)^{k-1}$$

$$= \begin{cases} N\cdot(-1)^{mq}, \text{ if } j = p+Nq, \text{ where } q\in\mathbb{Z}, \\ 0, \text{ otherwise.}\end{cases}$$

Hence

$$\zeta_{N,m,p}(x) = \sum_{q=-\infty}^{\infty} F_{N,m}(\pi(p+Nq))\cdot(-1)^{mq}\cdot e^{i(p+Nq)x} \tag{43}$$

for $p = 1,2,\dots,N$.

Consider several cases.

1. Let m be an even number. By (43), we obtain

$$\zeta_{N,m,p}(x) = \sum_{q=-\infty}^{\infty} F_{N,m}(\pi(p+Nq)) \cdot e^{i(p+Nq)x}. \tag{44}$$

By definition,

$$F_{N,m}(\pi N(q+1)) = \left(\frac{\sin(\pi(q+1))}{\pi(q+1)}\right)^{m+1} F(\pi(q+1)) = \begin{cases} 0, & \text{if } q \neq -1, \\ F(0), & \text{if } q = -1. \end{cases}$$

From property (iv) of the function f it follows that

$$F_{N,m}(\pi N(q+1)) = \begin{cases} 0, & \text{if } q \neq -1, \\ 1, & \text{if } q = -1. \end{cases} \tag{45}$$

Therefore

$$\zeta_{N,m,N}(x) \equiv 1.$$

Let

$$\psi_{N,m,p}(x) = \begin{cases} \frac{1}{2}(\zeta_{N,m,p}(x) + \zeta_{N,m,N-p}(x)), & \text{if } p = 1, 2, \ldots, N/2, \\ \frac{1}{2i}(\zeta_{N,m,p-N/2}(x) - \zeta_{N,m,3N/2-p}(x)), & \\ \qquad\qquad \text{if } p = N/2+1, \ldots, N-1, & \\ \zeta_{N,m,N}(x), & \text{if } p = N. \end{cases}$$

By the above

$$\psi_{N,m,N}(x) = 1. \tag{46}$$

Using (44), we obtain

$$\zeta_{N,m,N-p}(x) = \sum_{q=-\infty}^{\infty} F_{N,m}(-\pi(p - N(q+1))) \cdot e^{-i(p-N(q+1))x}.$$

It follows from property (ii) of the function f that the function F is even. Therefore

$$\psi_{N,m,N-p}(x) = \sum_{q=-\infty}^{\infty} F_{N,m}(\pi(p - N(q+1))) \cdot e^{-i(p-N(q+1))x}$$

$$= \sum_{s=-\infty}^{\infty} F_{N,m}(\pi(p+Ns)) \cdot e^{-i(p+Ns)x}.$$

Hence, for any $p = 1, 2, \ldots, N/2$ we get

$$\psi_{N,m,p}(x) = \sum_{q=-\infty}^{\infty} F_{N,m}(\pi(p+Nq)) \cdot \cos((p+Nq)x).$$

Since F is even function, we obtain

$$\begin{aligned}\psi_{N,m,p}(x) = \sum_{q=0}^{\infty}(F_{N,m}(\pi(p+Nq)) \cdot \cos((p+Nq)x) \\ +F_{N,m}(\pi((q+1)N-p)) \cdot \cos(((q+1)N-p)x))\end{aligned} \tag{47}$$

for $p = 1, 2, \ldots, N/2-1$ and

$$\psi_{N,m,N/2}(x) = 2\sum_{q=0}^{\infty} F_{N,m}\left(\pi(2q+1)\frac{N}{2}\right) \cdot \cos\left((2q+1)\frac{N}{2}x\right). \tag{48}$$

In the same way,

$$\begin{aligned}\psi_{N,m,p}(x) = \sum_{q=0}^{\infty}(F_{N,m}(\pi(p-N/2+Nq)) \cdot \sin((p-N/2+Nq)x) \\ +F_{N,m}(\pi((q+1)N-p+N/2)) \cdot \sin(((q+1)N-p+N/2)x))\end{aligned} \tag{49}$$

for $p = N/2+1, \ldots, N-1$.

2. Let m be an odd number. In this case, using (43), we get

$$\zeta_{N,m,p}(x) = \sum_{q=-\infty}^{\infty} (-1)^q \cdot F_{N,m}(\pi(p+qN)) \cdot e^{i(p+qN)x}.$$

This implies that

$$\zeta_{N,m,N}(x) = \sum_{q=-\infty}^{\infty} (-1)^q \cdot F_{N,m}(\pi N(q+1)) \cdot e^{iN(q+1)x}.$$

If we combine this with (45), we obtain

$$\zeta_{N,m,N}(x) \equiv -1.$$

Let

$$\psi_{N,m,p}(x) = \begin{cases} \frac{1}{2i}\left(\zeta_{N,m,p}(x) + \zeta_{N,m,N-p}(x)\right), & \text{if } p = 1, 2, \ldots, N/2, \\ \frac{1}{2}\left(\zeta_{N,m,p-N/2}(x) - \zeta_{N,m,3N/2-p}(x)\right), & \\ & \text{if } \ p = N/2+1, \ldots, N-1, \\ \zeta_{N,m,N}(x), \quad \text{if } \ p = N. & \end{cases}$$

As above, it can be proved that

$$\begin{aligned}\psi_{N,m,p}(x) = \sum_{q=0}^{\infty}(-1)^q(F_{N,m}(\pi(p+Nq)) \cdot \sin((p+qN)x) \\ +F_{N,m}(\pi((q+1)N-p)) \cdot \sin(((q+1)N-p)x))\end{aligned} \tag{50}$$

for $p = 1, 2, \dots, N/2 - 1$,

$$\psi_{N,m,N/2} = 2\sum_{q=0}^{\infty}(-1)^q F_{N,m}\left(\pi(2q+1)\frac{N}{2}\right)\sin\left((2q+1)\frac{N}{2}x\right), \tag{51}$$

$$\begin{aligned}\psi_{N,m,p}(x) = \sum_{q=0}^{\infty}(-1)^q (F_{N,m}(\pi(p-N/2+qN))\cdot\cos((p-N/2+qN)x)- \\ -F_{N,m}(\pi((q+1)N-p+N/2))\cdot\cos(((q+1)N-p+N/2)x))\end{aligned} \tag{52}$$

for $p = N/2+1, \dots, N-1$ and

$$\psi_{N,m,N}(x) \equiv -1. \tag{53}$$

Consider the functions

$$v_{N,m,0}(x) = (-1)^m\cdot\psi_{N,m,N}(x), \quad w_{N,m,0}(x) = \frac{1}{2F_{N,m}(\pi N/2)}\cdot\psi_{N,m,N/2}(x),$$

$$v_{N,m,p}(x) = \begin{cases}\psi_{N,m,p+N/2}(x)/F_{N,m}(\pi p), & \text{if } m \text{ is odd},\\ \psi_{N,m,p}(x)/F_{N,m}(\pi p), & \text{if } m \text{ is even}\end{cases}$$

and

$$w_{N,m,p}(x) = \begin{cases}\psi_{N,m,p}(x)/F_{N,m}(\pi p), & \text{if } m \text{ is odd},\\ \psi_{N,m,p+N/2}(x)/F_{N,m}(\pi p), & \text{if } m \text{ is even}\end{cases}$$

for $p = 1, 2, \dots, N/2 - 1$.

Let us remark that $F_{N,m}(\pi p) \neq 0$ for $p = 1, 2, \dots, N/2$. This statement will be proved later.

From (46) – (53) it follows that

$$v_{N,m,0}(x) \equiv 1,$$

$$w_{N,m,0}(x) = \sum_{k=0}^{\infty}\left(y_{N,m,k}\cos\left(\frac{N}{2}(2k+1)x\right) + z_{N,m,k}\sin\left(\frac{N}{2}(2k+1)x\right)\right),$$

$$v_{N,m,p}(x) = \sum_{k=0}^{\infty}(r_{N,m,p,k}\cos((p+kN)x) + q_{N,m,p,k}\cos(((k+1)N-p)x))$$

and

$$w_{N,m,p}(x) = \sum_{k=0}^{\infty}(s_{N,m,p,k}\sin((p+kN)x) + t_{N,m,p,k}\sin((N(k+1)-p)x))$$

for $p = 1, 2, \dots, N/2 - 1$, where

$$y_{N,m,k} = \frac{1+(-1)^m}{2}\cdot\frac{F_{N,m}\left(\pi\cdot\frac{N}{2}\cdot(2k+1)\right)}{F_{N,m}\left(\pi\cdot\frac{N}{2}\right)}, \tag{54}$$

$$z_{N,m,k} = (-1)^k \cdot \frac{1-(-1)^m}{2} \cdot \frac{F_{N,m}\left(\pi \cdot \frac{N}{2} \cdot (2k+1)\right)}{F_{N,m}\left(\pi \cdot \frac{N}{2}\right)}, \tag{55}$$

$$r_{N,m,p,k} = (-1)^{mk} \cdot \frac{F_{N,m}(\pi(p+kN))}{F_{N,m}(\pi p)}, \tag{56}$$

$$q_{N,m,p,k} = (-1)^{m(k+1)} \cdot \frac{F_{N,m}(\pi(N(k+1)-p))}{F_{N,m}(\pi p)}, \tag{57}$$

$$s_{N,m,p,k} = (-1)^{mk} \cdot \frac{F_{N,m}(\pi(p+kN))}{F_{N,m}(\pi p)}, \tag{58}$$

$$t_{N,m,p,k} = (-1)^{m(k+1)+1} \cdot \frac{F_{N,m}(\pi(N(k+1)-p))}{F_{N,m}(\pi p)}. \tag{59}$$

We see that the system $\{v_{N,m,p}, w_{N,m,p}\}_{p=0}^{N/2-1}$ constitutes an almost-trigonometric basis of the space $L_{N,m}$.

Now we show that all conditions of Theorem 7 are satisfied.

First note that condition (iii) of Theorem 7 is satisfied.

Secondly, from (57) and (59) it follows that

$$q^2_{N,m,p,0} = t^2_{N,m,p,0} = \left(\frac{F_{N,m}(\pi(N-p))}{F_{N,m}(\pi p)}\right)^2.$$

By construction,

$$F_{N,m}(\pi p) = \left(\frac{\sin(\pi p/N)}{p/N}\right)^{m+1} \cdot F(\pi p/N)$$

and

$$F_{N,m}(\pi(N-p)) = \left(\frac{\sin(\pi - \pi p/N)}{\pi(N-p)/N}\right)^{m+1} \cdot F\left(\frac{\pi(N-p)}{N}\right)$$
$$= \left(\frac{\sin(\pi p/N)}{\pi(N-p)/N}\right)^{m+1} \cdot F\left(\frac{\pi(N-p)}{N}\right).$$

This implies that

$$t^2_{N,m,p,0} = q^2_{N,m,p,0} = \left(\frac{p}{N-p}\right)^{2(m+1)} \cdot \varphi^2\left(\frac{p}{N}\right),$$

where

$$\varphi(t) = \frac{F(\pi(1-t))}{F(\pi t)}.$$

Let us prove that the function φ is positive, increasing and differentiable on $[0, 1/2]$.

It follows from properties (i) and (ii) of the function f that

$$F(\pi t) = 2\int_0^1 \cos(\pi t x) f(x)\,dx.$$

For any $x \in [0,1]$, $t_1 \in [0,1/2]$ and $t_2 \in [0,1/2]$ such that $t_1 < t_2$ the inequality $0 \leq \pi t_1 x \leq \pi t_2 x \leq \pi/2$ holds. Since the function $f(x)$ is positive (see property (iii)), we have $\cos(\pi t_1 x) f(x) \geq \cos(\pi t_2 x) f(x)$. Therefore $F(\pi t_1) \geq F(\pi t_2)$. This means that the function $F(\pi t)$ decreases on the segment $[0,1/2]$. Hence, $F(\pi t) \geq F(\pi/2)$ for any $t \in [0,1/2]$. Furthermore, $\cos(x\pi/2) > 0$ for every $x \in [0,1)$. This implies that the equality

$$\int_0^1 \cos(x\pi/2) f(x) dx = 0$$

holds if and only if $f(x) = 0$ for almost every $x \in [0,1]$. The last statement contradicts the properties of the function $f(x)$. We see that $F(\pi/2) > 0$. Therefore $F(\pi t) > 0$ for every $t \in [0,1/2]$.

Moreover, by construction,

$$F_{N,m}(\pi p) = \left(\frac{\sin\left(\frac{\pi p}{N}\right)}{\frac{\pi p}{N}} \right)^{m+1} \cdot F\left(\frac{\pi p}{N}\right) > 0$$

for $p = 1, 2, \ldots, N/2$. This implies that the functions $v_{N,m,p}$ and $w_{N,m,p}$ were defined correctly.

Further,

$$F(\pi(1-t)) = 2 \int_0^1 \cos(\pi(1-t)x) f(x) dx.$$

Consider $t_1, t_2 \in [0,1/2]$ such that the inequality $t_1 < t_2$ holds. We obtain $0 \leq \pi(1-t_2)x \leq \pi(1-t_1)x \leq \pi$ for all $x \in [0,1]$. If we combine this with property (iii) of the function f, we see that $\cos(\pi(1-t_1)x) f(x) \leq \cos(\pi(1-t_2)x) f(x)$. Equivalently, the function $F(\pi(1-t))$ increases on the segment $[0,1/2]$. Therefore $F(\pi(1-t)) \geq F(\pi)$ for $t \in [0,1/2]$. From property (v) of the function $f(x)$ it follows that the function $F(\pi(1-t)) \geq 0$ for $t \in [0,1/2]$.

Thus the function φ is positive and increasing on the segment $[0,1/2]$. In particular $\varphi(t) \leq \varphi(1/2) = 1$ for any $t \in [0,1/2]$.

By Paley-Wiener theorem, the function F is an entire function. Therefore the function φ is differentiable on $[0,1/2]$.

Finally we shall show that the last condition of Theorem 7 is satisfied.

By construction,

$$F_{N,m}(\pi p) = \left(\frac{\sin(\pi p/N)}{\pi p/N} \right)^{m+1} \cdot F\left(\frac{\pi p}{N}\right),$$

$$F_{N,m}(\pi(p+kN)) = \left(\frac{\sin(\pi k + \pi p/N)}{\pi(p+kN)/N} \right)^{m+1} \cdot F\left(\frac{\pi(p+kN)}{N}\right)$$

and

$$F_{N,m}(\pi((k+1)N-p)) = \left(\frac{\sin\left(\pi(k+1)-\pi p/N\right)}{\pi((k+1)N-p)/N}\right)^{m+1}$$
$$\times F\left(\frac{\pi((k+1)N-p)}{N}\right).$$

Hence, we obtain

$$\left(\frac{F_{N,m}(\pi(p+kN))}{F_{N,m}(\pi p)}\right)^2 = \left(\frac{p}{p+kN}\right)^{2(m+1)} \cdot \left(\frac{F\left(\pi(p+kN)/N\right)}{F\left(\pi p/N\right)}\right)^2$$

and

$$\left(\frac{F_{N,m}(\pi((k+1)N-p))}{F_{N,m}(\pi p)}\right)^2 = \left(\frac{p}{(k+1)N-p}\right)^{2(m+1)}$$
$$\times \left(\frac{F\left(\pi((k+1)N-p)/N\right)}{F\left(\pi p/N\right)}\right)^2.$$

It follows from the properties of the function f that

$$|F(t)| = 2 \cdot \left|\int_0^1 \cos(tx) f(x)dx\right| \le 2\int_0^1 |f(x)|dx$$

$$= \int_{-1}^1 f(x)dx = \int_{-\infty}^{\infty} f(x)dx = 1.$$

Moreover, by the above $F(\pi p/N) \ge f(\pi/2) > 0$ for $p = 1, 2, \ldots, N/2$.
Therefore

$$\left(\frac{F_{N,m}(\pi(p+kN))}{F_{N,m}(\pi p)}\right)^2 \le \frac{1}{F^2(\pi/2)} \cdot \left(\frac{p}{p+kN}\right)^{2(m+1)}$$

and

$$\left(\frac{F_{N,m}(\pi((k+1)N-p))}{F_{N,m}(\pi p)}\right)^2 \le \frac{1}{F^2(\pi/2)} \cdot \left(\frac{p}{(k+1)N-p}\right)^{2(m+1)}.$$

If we combine this with (56)–(59), we obtain

$$\sum_{k=1}^{\infty}\left(r^2_{N,m,p,k} + q^2_{N,m,p,k}\right) = \sum_{k=1}^{\infty}\left(s^2_{N,m,p,k} + t^2_{N,m,p,k}\right)$$

$$= \sum_{k=1}^{\infty}\left(\left(\frac{F_{N,m}(\pi(p+kN))}{F_{N,m}(\pi p)}\right)^2 + \left(\frac{F_{N,m}(\pi((k+1)N-p))}{F_{N,m}(\pi p)}\right)^2\right)$$

$$\leq \frac{1}{F^2(\pi/2)} \cdot \sum_{k=1}^{\infty} \left(\left(\frac{p}{p+kN} \right)^{2(m+1)} + \left(\frac{p}{(k+1)N-p} \right)^{2(m+1)} \right)$$

$$= \frac{1}{F^2(\pi/2)} \cdot \left(\frac{p}{N}\right)^{2(m+1)} \cdot \sum_{k=1}^{\infty} \left(\frac{1}{(k+p/N)^{2(m+1)}} + \frac{1}{(k+1-p/N)^{2(m+1)}} \right).$$

Since $1 \geq p \geq N/2$, we get

$$\sum_{k=1}^{\infty} \left(r_{N,m,p,k}^2 + q_{N,m,p,k}^2\right) = \sum_{k=1}^{\infty} \left(s_{N,m,p,k}^2 + t_{N,m,p,k}^2\right) \leq \left(\frac{p}{N}\right)^{2(m+1)} \cdot M,$$

where

$$M = \frac{1}{F^2(\pi/2)} \cdot \sum_{k=1}^{\infty} \left(\frac{1}{\left(k+\frac{1}{N}\right)^{2(m+1)}} + \frac{1}{\left(k+\frac{1}{N}+\frac{1}{2}\right)^{2(m+1)}} \right). \tag{60}$$

We see that all conditions of Theorem 7 are satisfied and inequality (40) holds. This completes the proof.

Remark. Actually, it was shown in the proof of Theorem 8 that the dimension of the space $L_{N,m}$ is equal to N. Therefore the system $\{f_{N,m,k}\}_{k=1}^{N}$ is linearly independent. Moreover, by definition of the Kolmogorov width, we see that

$$d_N\left(\widetilde{W}_2^r, L_2[-\pi,\pi]\right) \leq E_{L_2[-\pi,\pi]}\left(\widetilde{W}_2^r, L_{N,m}\right). \tag{61}$$

If we combine this with (38) and (40), we get

$$d_N\left(\widetilde{W}_2^r, L_2[-\pi,\pi]\right) \leq E_{L_2[-\pi,\pi]}\left(\widetilde{W}_2^r, L_{N,m}\right)$$

$$\leq d_N\left(\widetilde{W}_2^r, L_2[-\pi,\pi]\right) \sqrt{1+\varepsilon(N,m,r)},$$

where

$$\varepsilon(N,m,r) = \frac{M^2}{2^{4m+3}} + \frac{M}{2^{2m+1}} + \frac{\sqrt{2M}}{2^{m+r+1}}.$$

Furthermore, it follows from (60) and the equality

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi}{90},$$

that

$$M \leq \frac{2}{F^2(\pi/2)} \cdot \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{1}{F^2(\pi/2)} \cdot \frac{\pi^4}{45}.$$

Hence,

$$\lim_{N,m\to\infty} \frac{E_{L_2[-\pi,\pi]}\left(\widetilde{W}_2^r, L_{N,m}\right)}{d_N\left(\widetilde{W}_2^r, L_2[-\pi,\pi]\right)} = 1.$$

This means that $L_{N,m}$ is almost the best space for approximation of the class $\widetilde{W}_2^r$ in the norm of $L_2[-\pi,\pi]$. In other words, the generalized Fup-function $f_{N,m}$, which is a function with a local support, has good approximation properties.

Conclusion

In this paper we introduce a method of construction of function spaces that combine the following convenient properties:

1) an existence of the basis that constructed using shifts of one compactly supported function; besides, as it can be seen from (39) a support of this function can be made arbitrarily small;

2) smoothness of functions of these spaces; moreover, the degree of smoothness of these functions can be arbitrarily large; for example, if the mother function is $mup_s(x)$, we get infinitely differentiable generalized Fup-function;

3) the dimension of these spaces can be quite arbitrarily;

4) good approximation properties; the class $\widetilde{W}_2^r$ is a classic object of investigation in approximation theory, theorem 8 actually means that spaces of shifts of the generalized Fup-functions approximate $\widetilde{W}_2^r$ well; furthermore, these spaces are almost the best spaces for approximation of $\widetilde{W}_2^r$ in the norm of $L_2[-\pi,\pi]$;

The last property is the most important.

Notice that spaces of shifts of the generalized Fup-functions have good approximation properties because of existence of the almost-trigonometric basis.

We stress that the almost-trigonometric basis theorem is another important result. Actually, if it can be proved that some space of functions has an almost-trigonometric basis, then this space has good approximation properties.

In spite of all convenient properties, there are many unsolved problems relating generalized Fup-functions. The following open questions are of interest:

1) How can some generalize Fup-function be computed (generally, for this purpose the Fourier series can be used)?

2) Can convenient asymptotic expansions of generalized Fup-functions be obtained (we note that the first term of asymptotic expansions of these functions was already obtained in [26])?

3) Can the inequality (61) be replaced by equality

$$E_{L_2[-\pi,\pi]}\left(\widetilde{W}_2^r, L_{N,m}\right) = d_N\left(\widetilde{W}_2^r, L_2[-\pi,\pi]\right)$$

(notice that by theorem 5 this equality holds, if $up(x)$ is a mother-function of the generalized Fup-function)?

4) Can it be proved that

$$E_{C_{[-\pi,\pi]}}\left(\widetilde{W}^r_\infty, L_{N,m}\right) \le d_N\left((\widetilde{W}^r_\infty, C_{[-\pi,\pi]}\right)\cdot(1+\alpha(N,m)),$$

where $\alpha(N,m)\to\infty$ as $N,m\to\infty$ (see theorem 4 for the *up*-function case)?

These will be the object of another papers.

Acknowledgement. Research of V.A. Makarichev is supported in part by N.I. Akhiezer Foundation.

REFERENCES

1. Kolmogoroff A.N. Über die beste Annäherung von Funktionen einer gegebener Funktionenklasse. // Ann. of Math., 1936. – **37**. – P. 107–110.

2. Rvachev V.A. Compactly supported solutions of functional–differential equations and their applications. // Russian Math. Surveys, 1990. – **45**. – P. 87–120.

3. Rvachev V.L., Rvachev V.A. Non-classical methods of approximation theory in boundary value problem (in Russian). – K.: Naukova Dumka, 1979. – 196 p.

4. Makarichev V.A. Approximation of periodic functions by $mup_s(x)$. // Math. Notes, 2013. – **93**. – P. 858–880.

5. Rvachev V.L., Rvachev V.A. A certain finite function (in Ukrainian). // Proc. Ukr. SSR Acad. Sci., Ser. A., 1971. – **8**. – P. 705–707.

6. Rvachev V.A., Starets G.A. Some atomic functions and their apllications (in Ukrainian). // Proc. Ukr. SSR Acad. Sci., Ser. A., 1983. – **11**. – P. 22–24.

7. Makarichev V.A. Asymptotics of the basis functions of generalized Taylor series for the class $H_{\rho,2}$. // Math. Notes, 2011. – **89**. – P. 689–705.

8. Dyn N., Ron A. Multiresolution analysis by infinitely differentiable compactly supported functions. // Appl. Comput. Harmon. Anal., 1995. – **2**. – P. 15–20.

9. Cooklev T., Berbecel G.I., Venetsanopoulos A.N. Wavelets and differential-dilatation equations. // IEEE Transactions on signal processing, 2000. – **48**. – P. 670–681.

10. Charina M., Stockler J. Tight wavelet frames for irregular multiresolution analysis. // Appl. Comput. Harmon. Anal., 2008. – **25**. – P. 98–113.

11. Makarichev V.A. Applications of the function $mup_s(x)$. // Progress in analysis. Proceedings of the 8th congress of the International Society for Analysis, its Applications, and Computation (ISAAC), 2012. – **2**. – P. 297–304.

12. Makarichev V.A. The function $mup_s(x)$ and its applications to the theory of generalized Taylor series, approximation theory and wavelet theory. // In book: Contemporary problems of mathematics, mechanics and computing sciences. – Kharkiv: Apostrophe, 2011. – P. 279–287.

13. Brysina I.V., Makarichev V.A. Atomic wavelets. // Radioelectronic and Computer Systems, 2012. – **53**. – P. 37–45.

14. Rvachova T.V. On a nonstationary system of infinitely differentiable wavelets with compact support (in Russian). // Visn. Hark. nac. univ. im. V.N. Karazina, Ser.: Mat. prikl. mat. meh., 2011. – **967**. – P. 63–80.

15. Lazorenko O.V. The use of atomic functions in the Choi–Williams analysis of ultrawideband signals. // Radioelectronics and Communications Systems, 2009. **52**. – P. 397–404.

16. Ulises Moya-Sanchez E., Bayro-Corrochano E. Quaternionic analytic signal using atomic functions. // Progress in Pattern Recognition, Image Analysis, Computer Vision, and Applications, Lecture Notes in Computer Science, 2012. – **7441**. – P. 699–706.

17. Gotovac H., Andricevic R., Gotovac B. Multi-resolution adaptive modeling of groundwater flow and transport problems. // Adv. Water Resour., 2007. – **30**. – P. 1105–1126.

18. Gotovac H., Cvetkovic V., Andricevic R. Adaptive Fup multi-resolution approach to flow and advective transport in heterogeneous porous media. // Adv. Water Resour., 2009. – **32**. – P. 885–905.

19. Gotovac H., Gotovac B. Maximum entropy algorithm with inexact upper entropy bound based on Fup basis functions with compact support. // J. Comput. Phys., 2009. – **228**. – P. 9079–9091.

20. Basarab M.A. Periodic atomic quasiinterpolation. // Ukrainian Math. J., – 2001. – **53**. – P. 1728 – 1734.

21. Stoyan Y.G., Protsenko V.S., Man'ko G.P., Goncharyuk I.V., Kurpa L.V., Rvachev V.A., Sinekop N.S., Sirodzha I.B., Shevchenko A.N., Sheiko T.I. The theory of R-functions and current problems of applied mathematics (in Russian). – Kiev: Naukova Dumka, 1986. – 264 p.

22. RvachovaT.V. On a relation between the coefficients and sum of the generalized Taylor series. // Mathematical physics, analysis and geometry, 2003. – **10**. – P. 262–268.

23. Rvachova T.V. On the asymptotics of the basic functions of a generalized Taylor series (in Russian). // Visn. Hark. nac. univ. im. V.N. Karazina, Ser.: Mat. prikl. mat. meh., 2003. – **602**. – P. 94–104.

24. Rvachova T.V. On the rate of approximation of the infinitely differentiable functions by the partial sums of the generalized Taylor series (in Russian). // Visn. Hark. nac. univ. im. V.N. Karazina, Ser.: Mat. prikl. mat. meh., 2010. – **931**. – P. 93–98.

25. Makarichev V.A. On the asymptotics of the basic functions of a generalized Taylor series for some classes of infinitely differentiable functions (in Russian). // Del'nevostochniy mathematicheskiy zhurnal, 2011. – **11**. – P. 56–75.

26. Brysina I.V., Makarichev V.A. On the asymptotics of the generalized Fup-functions. // Adv. Pure Appl. Math., 2014. – **5**. – P. 131–138.

Article history: Received: 29 December 2015; Accepted: 12 December 2016.