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Explicit solution of the time-optimal control problem for one nonlinear three-dimensional system

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The time-optimal control problem for the system $\dot{x}_1 = u$, $\dot{x}_2 = x_1$, $\dot{x}_3 = x_1^3$ is considered. Explicit formulas for finding optimal controls are given. The explicit solution of the optimal synthesis problem is obtained.
Keywords: nonlinear control systems, time optimality, explicit solution.

Ігнатович С. Ю. **Явний розв'язок задачі швидкодії для одної нелінійної тривимірної системи.** Розглядається задача швидкодії для системи $\dot{x}_1 = u$, $\dot{x}_2 = x_1$, $\dot{x}_3 = x_1^3$. Даються явні формули для знаходження оптимальних керувань. Отримано явний розв'язок задачі оптимального синтезу.
Ключові слова: нелінійні керовані системи, швидкодія, явний розв'язок.

Игнатович С. Ю. **Явное решение задачи быстродействия для одной нелинейной трёхмерной системы.** Рассматривается задача быстродействия для системы $\dot{x}_1 = u$, $\dot{x}_2 = x_1$, $\dot{x}_3 = x_1^3$. Даются явные формулы для нахождения оптимальных управлений. Получено явное решение задачи оптимального синтеза.
Ключевые слова: нелинейные управляемые системы, быстродействие, явное решение.

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Introduction

The time-optimal problem is one of the most investigated optimal control problems. Different approaches were developed which give a description of optimal controls. However, in the general case an answer hardly can be obtained in an explicit form. So, for the simplest linear time-optimal control problem

$$\dot{x}_1 = u, \quad \dot{x}_i = x_{i-1},\ i = 2, \dots, n, |u(t)| \le 1, \quad x(0) = x^0,\ x(\theta) = 0, \quad \theta \to \min,$$

in the case $n = 2$ the well-known explicit solution directly follows from the Pontryagin Maximum Principle [1]. However, for $n \ge 3$ the answer is much more complicated and entirely non-obvious. Specifically, the Pontryagin Maximum Principle says that any optimal control equals ± 1 and has no more than $n-1$ switchings, however, it does not give a direct way for finding the optimal time and switching moments. The analytical solution of this problem was obtained in [2]. It was shown that for an arbitrary initial point x^0 the optimal time is a root of one of two special polynomials of degree $\frac{1}{4}n(n+2)$ for even n and $\frac{1}{4}(n+1)^2$ for odd n with coefficients depending on x^0. So, for $n = 3$ equations of degree 4 should be solved. Moments of switching can be found also as roots of certain polynomials.

For nonlinear systems the solution can be much more complicated; in particular, singular controls may occur. In [3], as an example, the time-optimal control problem for the system $\dot{x}_1 = u$, $\dot{x}_2 = x_1$, $\dot{x}_3 = x_1^2$ was considered and the explicit solution was given. By arguments essentially using the concrete form of the system, it was shown that the optimal control (if exists) takes the values $+1$, -1, 0 and has no more than two switchings.

Generally, it is an interesting problem to find classes of systems for which time-optimal controls can be described more or less explicitly. In the paper [4] one of such classes was presented, namely, the class of *dual to linear systems*,

$$\dot{x}_1 = u, \quad \dot{x}_i = P_i(x_1),\ i = 2, \dots, n,$$

where $P_2(x_1), \dots, P_n(x_1)$ are linearly independent real analytic functions *of one variable* such that $P_2(0) = \dots = P_n(0) = 0$. We emphasize that such systems are non-controllable w.r.t. the first approximation for $n \ge 3$. It was shown that a time-optimal control is piecewise constant and takes the values $+1$, -1 and 0 only. Moreover, for any initial point $x^0 \ne 0$ and any optimal control $\widehat{u}(t)$, $x \in [0, \widehat{\theta}]$, steering x^0 to the origin (if exists) there exists a function

$$P(z) = -\psi_0 - \psi_2 P_2(z) - \dots - \psi_n P_n(z), \tag{1}$$

where $\psi_0 \ge 0, \psi_2, \dots, \psi_n$ are real parameters, $\psi_0^2 + \psi_2^2 + \dots + \psi_n^2 > 0$, such that the first component of the optimal trajectory $\widehat{x}_1(t)$ satisfies the following properties:

– $P(\widehat{x}_1(t)) \ge 0$ for $t \in [0, \widehat{\theta}]$; hence, $\widehat{x}_1(t)$ belongs to the connected component of the set $\{z : P(z) \ge 0\}$ containing the point $z = 0$;

– if $\bar{t}$ is a switching moment for $\widehat{u}(t)$, then $\widehat{x}_1(\bar{t})$ is a root of the function $P(z)$;

– if $\bar{t}$ is a switching moment for $\widehat{u}(t)$ such that $\widehat{u}(\bar{t}+0) = 0$ or $\widehat{u}(\bar{t}-0) = 0$, then $\widehat{x}_1(\bar{t})$ is a multiple root of the function $P(z)$;

– any value can be taken by the function $\widehat{x}_1(t)$ no more than twice when $t \in [0, \widehat{\theta}]$, except of the value 0 which can be taken for three times if $x_1^0 = 0$.

These properties essentially reduce the set of possible optimal controls. In particular, if $P_i(x_1)$ are polynomials, the number of switchings can be estimated. As an example, in [4] the following time-optimal control problem was considered,

$$\dot{x}_1 = u,\ \dot{x}_2 = x_1,\ \dot{x}_3 = x_1^3, \quad |u(t)| \le 1,\ x(0) = x^0,\ x(\theta) = 0,\ \theta \to \min, \tag{2}$$

and all controls satisfying the above-mentioned conditions were described. Nevertheless, the questions remained whether all these controls are realized as optimal ones and whether an optimal control can be non-unique for some points.

In the present paper we give the complete solution of the time-optimal control problem (2). It turns out that all possible types of controls mentioned above are realized as optimal ones. Unlike the linear case, optimal controls and the optimal time can be found by *explicit* formulas. For each of such controls we describe the domain where it is optimal. We give the solution of the optimal synthesis problem, i.e., describe the domains where the optimal control equals +1 and −1, and the surfaces where it equals 0. Also, we describe surfaces where the optimal control is non-unique. In Sections 1–3 we consider all possible optimal controls in the case $x_1^0 > 0$ only; for $x_1^0 < 0$ the solution can be obtained by symmetry arguments. In Section 4 we sum up the results and briefly consider the case $x_1^0 = 0$.

1. Optimal controls

First, let us discuss the results of [4] in connection with the particular problem (2). For a given x^0, denote by $\widehat{\theta}$, $\widehat{u}(t)$, $\widehat{x}(t)$ the optimal time, an optimal control, and the corresponding optimal trajectory. Let us introduce the Pontryagin-Hamilton function $H = \psi_1 u + \psi_2 x_1 + \psi_3 x_1^3$ and consider the dual system

$$\dot{\psi}_1 = -\psi_2 - 3\psi_3 x_1^2, \quad \dot{\psi}_2 = 0, \quad \dot{\psi}_3 = 0, \tag{3}$$

hence, ψ_2 and ψ_3 are constants. According to the Pontryagin Maximum Principle, there exist numbers $\psi_0 \le 0$, ψ_2, ψ_3 and a function $\psi_1(t)$ satisfying (3) such that $\psi_0^2 + \psi_2^2 + \psi_3^2 + (\psi_1(t))^2 \ne 0$ for $t \in [0, \widehat{\theta}]$ and

$$\begin{gathered} \widehat{u}(t) = \operatorname{sign}(\psi_1(t)) \text{ a.e. for all } t \in [0, \widehat{\theta}] \text{ such that } \psi_1(t) \ne 0, \\ \psi_0 + |\psi_1(t)| + \psi_2 \widehat{x}_1(t) + \psi_3 \widehat{x}_1^3(t) = 0 \ \text{ for all } t \in [0, \widehat{\theta}]. \end{gathered} \tag{4}$$

In particular, we get $\psi_0^2 + \psi_2^2 + \psi_3^2 \ne 0$. Now we introduce the function (1); for this example it equals a (nontrivial) polynomial

$$P(z) = -\psi_0 - \psi_2 z - \psi_3 z^3, \tag{5}$$

then (3), (4) imply

$$|\psi_1(t)| = P(\widehat{x}_1(t)), \quad \dot{\psi}_1(t) = P'(\widehat{x}_1(t)), \ t \in [0, \widehat{\theta}]. \tag{6}$$

In particular, it follows that $\widehat{x}_1(t)$ belongs to the connected component of the set $\{z : P(z) \ge 0\}$ containing the point $z = 0$.

If $\psi_1(t) = 0$ identically in some segment (τ_1, τ_2), then (6) implies that $\widehat{x}_1(t)$ equals a root of $P(z)$ for $t \in (\tau_1, \tau_2)$. However, $P(z)$ has no more that three real roots, hence, $\widehat{x}_1(t)$ equals one of them, $\widehat{x}_1(t) = const$, therefore, $\widehat{u}(t) = \dot{\widehat{x}}_1(t) = 0$ for all $t \in (\tau_1, \tau_2)$. (Moreover, due to (6), $\widehat{x}_1(t)$ should equal the multiple root of $P(z)$.) The question arises whether the set of roots of $\psi_1(t)$ may have more complicated structure (for example, include convergent sequences of isolated roots or some nowhere dense subsets of positive measure).

It was proved in [4] that the answer is "no". More specifically, for any $\bar{t} \in (0, \widehat{\theta})$ there exists $\varepsilon > 0$ such that $\psi_1(t)$ keeps its sign on the intervals $(\bar{t} - \varepsilon, \bar{t})$ and $(\bar{t}, \bar{t} + \varepsilon)$; for the points $\bar{t} = 0$ and $\bar{t} = \widehat{\theta}$ the same is true with the intervals $(0, \varepsilon)$ and $(\widehat{\theta} - \varepsilon, \widehat{\theta})$. (Here we assume $\text{sign}(0) = 0$.) Clearly, this implies that the optimal control $\widehat{u}(t)$ is piecewise constant and can take the values ± 1 and 0 only.

In our example, let us consider all possible functions $P(z)$ of the form (5) for all (nontrivial) sets of parameters $\psi_0 \leq 0$, ψ_2, ψ_3. Since the coefficient of z^2 vanishes, a relation between roots arises. Fig. 1–4 show all four possible types of $P(z)$ admitting optimal controls with at least two switchings (controls with no more that one switching can be regarded as partial cases, so, we do not consider them separately).

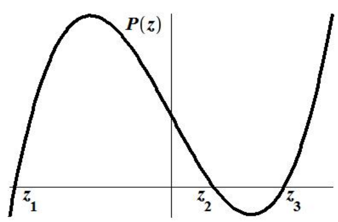

Fig. 1. Function $P(z)$ of type 1, $z_1 + z_2 + z_3 = 0$

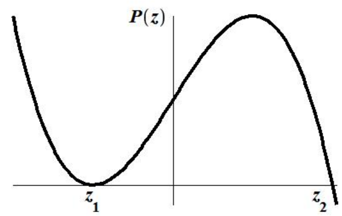

Fig. 2. Function $P(z)$ of type 2, $2z_1 + z_2 = 0$

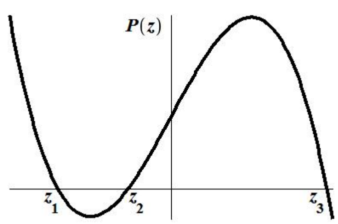

Fig. 3. Function $P(z)$ of type 3, $z_1 + z_2 + z_3 = 0$

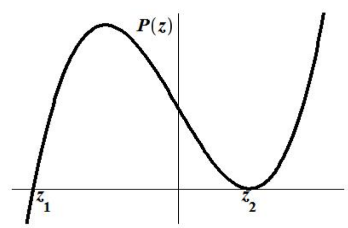

Fig. 4. Function $P(z)$ of type 4, $z_1 + 2z_2 = 0$

It was shown in [4] that any nonzero value can be taken by $\widehat{x}_1(t)$ no more than twice. Let us illustrate the reason for this by an example. It is convenient to draw $x_1(t)$ instead of $u(t)$. Suppose a control $u(t)$ taking values ± 1 steers some point x^0 to the origin in the time θ and assume that $x_1(t)$ has the form shown in Fig. 5 (a). Then $x_1(t)$ takes the value μ_1 for three times. Due to very special form of the system (2), $x_2(0)$ and $x_3(0)$ equal the area under the curves $-x_1(t)$ and $-x_1^3(t)$ respectively. Now, let us successively transform $x_1(t)$ as is shown in Fig. 5 (b) and (c); obviously, the mentioned areas are the same as in case (a), hence, the corresponding controls also steer x^0 to the origin in the same time θ. However, the control of case (c) cannot be optimal since four different vales μ_1, μ_2, μ_3, μ_4 cannot be roots of a function of the form (5).

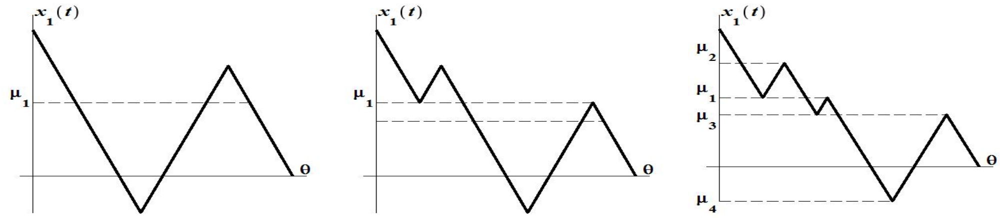

Fig. 5. Transformation of non-optimal trajectory; graphs of $x_1(t)$

2. Domains of solvability

Below we describe all possible controls compatible with the requirements mentioned above for the case $x_1^0 > 0$. For the sake of briefness, we omit the upper index of x^0, i.e., we write x_i instead of x_i^0. We use the notation

$$S_{11} = x_2 - \tfrac{1}{2}x_1^2, \quad S_{21} = x_3 - \tfrac{1}{4}x_1^4,$$
$$S_{12} = x_2 + \tfrac{1}{2}x_1^2, \quad S_{22} = x_3 + \tfrac{1}{4}x_1^4.$$

Case 1 corresponds to $P(z)$ of type 1 (Fig. 1), the control is of the form

$$u(t) = \begin{cases} 1 & \text{if } t \in [0, t_1), \\ -1 & \text{if } t \in [t_1, t_2), \\ 1 & \text{if } t \in [t_2, \theta]. \end{cases} \tag{7}$$

The graph of $x_1(t)$ is shown in Fig. 6.

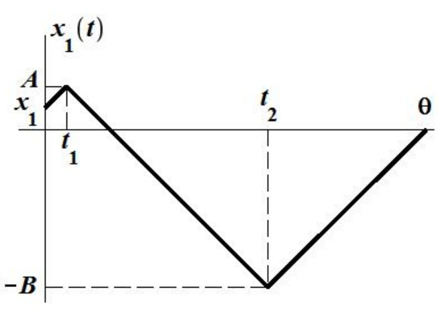

Fig. 6. Graph of $x_1(t)$, case 1

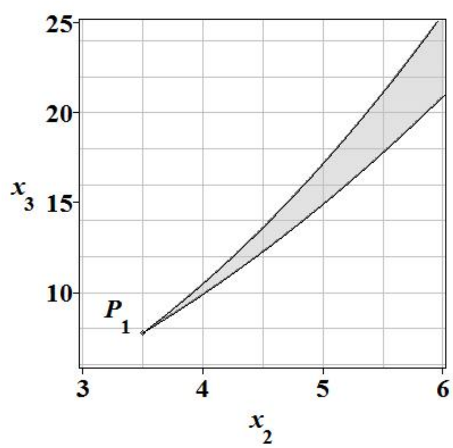

Fig. 7. Intersection of the domain D_1 and the plane $x_1 = 1$; $P_1 = (\frac{7}{2}, \frac{31}{4})$

Denote $x_1(t_1) = A$, $x_1(t_2) = -B$, then

$$A = x_1 + t_1 \geq x_1, \quad -B = x_1 + t_1 - (t_2 - t_1) = x_1 + 2t_1 - t_2 \leq 0.$$

Let $z_1 < 0 < z_2 \leq z_3$ be the roots of the function $P(z)$ (Fig. 1), then $A = z_2$, $B = -z_1$. Since $z_1 + z_2 + z_3 = 0$, we get $-z_1 = z_2 + z_3 \geq 2z_2$, therefore, $B \geq 2A$.

Integrating the equations $\dot{x}_2(t) = x_1(t)$ and $\dot{x}_3(t) = x_1^3(t)$ on the time interval $t \in [0, \theta]$ and taking into account the conditions $x_2(\theta) = x_3(\theta) = 0$ we get

$$-x_2 = -\tfrac{1}{2}x_1^2 + A^2 - B^2 \quad \text{and} \quad -x_3 = -\tfrac{1}{4}x_1^4 + \tfrac{1}{2}A^4 - \tfrac{1}{2}B^4.$$

Thus, in this case

$$\begin{cases} S_{11} = B^2 - A^2, \\ 2S_{21} = B^4 - A^4, \\ A \geq x_1, \ B \geq 2A, \end{cases} \Leftrightarrow \begin{cases} A^2 = \frac{S_{21}}{S_{11}} - \frac{1}{2}S_{11}, \\ B^2 = \frac{S_{21}}{S_{11}} + \frac{1}{2}S_{11}, \\ A \geq x_1, \ B \geq 2A. \end{cases}$$

Let us study the solvability of this system. If $S_{11} \leq 0$, then $B^2 \leq A^2$, which contradicts the requirement $B \geq 2A$. Hence, $S_{11} > 0$, therefore, the solvability conditions are

$$\begin{cases} S_{11} > 0, \\ \frac{S_{21}}{S_{11}} - \frac{1}{2}S_{11} \geq x_1^2, \\ \frac{S_{21}}{S_{11}} + \frac{1}{2}S_{11} \geq 4(\frac{S_{21}}{S_{11}} - \frac{1}{2}S_{11}), \end{cases} \Leftrightarrow \begin{cases} S_{11} > 0, \\ 2S_{21} - S_{11}^2 \geq 2x_1^2 S_{11}, \\ 6S_{21} - 5S_{11}^2 \leq 0. \end{cases} \tag{8}$$

This system implies $\frac{1}{2}(2x_1^2 S_{11} + S_{11}^2) \leq S_{21} \leq \frac{5}{6}S_{11}^2$, hence, $x_1^2 S_{11} \leq \frac{1}{3}S_{11}^2$. This gives $x_1^2 \leq \frac{1}{3}S_{11}$, which is equivalent to $x_2 \geq \frac{7}{2}x_1^2$. Substituting the expressions for S_{11} and S_{21} to (8), we get the solvability domain for case 1, i.e., the domain in which the control of case 1 exists:

$$D_1 = \left\{x : x_2 \geq \tfrac{7}{2}x_1^2, \ \tfrac{1}{2}x_2^2 + \tfrac{1}{2}x_1^2 x_2 - \tfrac{1}{8}x_1^4 \leq x_3 \leq \tfrac{5}{6}x_2^2 - \tfrac{5}{6}x_1^2 x_2 + \tfrac{11}{24}x_1^4\right\}.$$

For any point $x \in D_1$ the switching moments and the time of motion can be found explicitly by the formulas

$$t_1 = A - x_1, \quad t_2 = 2A + B - x_1, \quad \theta = 2A + 2B - x_1, \tag{9}$$

where

$$A = \sqrt{\frac{S_{21}}{S_{11}} - \tfrac{1}{2}S_{11}}, \quad B = \sqrt{\frac{S_{21}}{S_{11}} + \tfrac{1}{2}S_{11}}. \tag{10}$$

Case 2 corresponds to $P(z)$ of type 2 (Fig. 2), the control is of the form

$$u(t) = \begin{cases} -1 & \text{if } t \in [0, t_1), \\ 0 & \text{if } t \in [t_1, t_2), \\ 1 & \text{if } t \in [t_2, \theta]. \end{cases}$$

Denote $-A = x_1 - t_1 = z_1$ and $B = t_2 - t_1$, then $x_1 \leq z_2 = -2z_1 = 2A$. We have

$$-x_2 = \tfrac{1}{2}x_1^2 - A^2 - AB \quad \text{and} \quad -x_3 = \tfrac{1}{4}x_1^4 - \tfrac{1}{2}A^4 - A^3 B.$$

Then

$$\begin{cases} S_{12} = A^2 + AB, \\ S_{22} = \frac{1}{2}A^4 + A^3 B, \\ A \geq \frac{1}{2}x_1, \ B \geq 0, \end{cases} \Leftrightarrow \begin{cases} A^4 - 2S_{12}A^2 + 2S_{22} = 0, \\ B = \frac{S_{12}}{A} - A, \\ A \geq \frac{1}{2}x_1, \ B \geq 0. \end{cases}$$

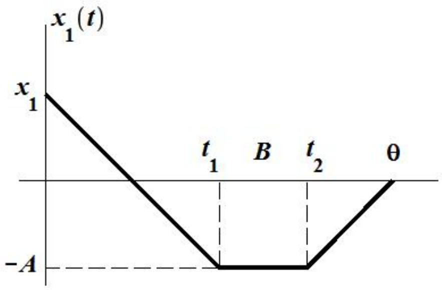

Fig. 8. Graph of $x_1(t)$, case 2

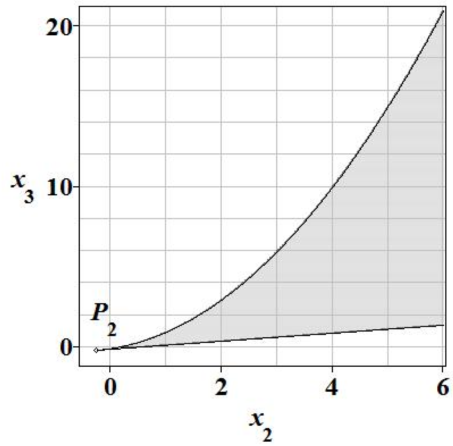

Fig. 9. Intersection of the domain D_2 and the plane $x_1 = 1$; $P_2 = (-\frac{1}{4}, -\frac{7}{32})$

The equation $A^4 - 2S_{12}A^2 + 2S_{22} = 0$ has real roots iff $d = S_{12}^2 - 2S_{22} \geq 0$, and then $A^2 = S_{12} \pm \sqrt{d}$. However, $B \geq 0$ iff $A^2 \leq S_{12}$. Hence, the minimal root should be chosen, $A^2 = S_{12} - \sqrt{d}$. The condition $A \geq \frac{1}{2}x_1$ can be rewritten as $A^2 = S_{12} - \sqrt{d} \geq \frac{1}{4}x_1^2$, which is equivalent to a pair of inequalities $S_{12} - \frac{1}{4}x_1^2 \geq 0$ and $(S_{12} - \frac{1}{4}x_1^2)^2 \geq d$. Substituting the expressions for S_{12} and S_{22}, we get the solvability domain for case 2:

$$D_2 = \left\{x : x_2 \geq -\tfrac{1}{4}x_1^2,\ \tfrac{1}{4}x_1^2 x_2 - \tfrac{5}{32}x_1^4 \leq x_3 \leq \tfrac{1}{2}x_2^2 + \tfrac{1}{2}x_1^2 x_2 - \tfrac{1}{8}x_1^4\right\}.$$

Then

$$t_1 = A + x_1, \quad t_2 = A + B + x_1, \quad \theta = 2A + B + x_1, \tag{11}$$

where

$$A = \sqrt{S_{12} - \sqrt{d}}, \quad d = S_{12}^2 - 2S_{22}, \quad B = \frac{S_{12}}{A} - A. \tag{12}$$

Case 3 also corresponds to $P(z)$ of type 2 (Fig. 2), the control is of the form

$$u(t) = \begin{cases} 1 & \text{if } t \in [0, t_1), \\ -1 & \text{if } t \in [t_1, t_2), \\ 0 & \text{if } t \in [t_2, t_3), \\ 1 & \text{if } t \in [t_3, \theta]. \end{cases}$$

Denote $A = x_1 + t_1 = z_2$ and $B = t_3 - t_2$, then

$$-x_2 = -\tfrac{1}{2}x_1^2 + \tfrac{3}{4}A^2 - \tfrac{1}{2}AB \quad \text{and} \quad -x_3 = -\tfrac{1}{4}x_1^4 + \tfrac{15}{32}A^4 - \tfrac{1}{8}A^3B.$$

The solvability conditions are

$$\begin{cases} S_{11} = \frac{1}{2}AB - \frac{3}{4}A^2, \\ S_{21} = \frac{1}{8}A^3B - \frac{15}{32}A^4, \\ A \geq x_1,\ B \geq 0, \end{cases} \quad \Leftrightarrow \quad \begin{cases} 9A^4 - 8S_{11}A^2 + 32S_{21} = 0, \\ B = \frac{2S_{11}}{A} + \frac{3}{2}A, \\ A \geq x_1,\ B \geq 0. \end{cases}$$

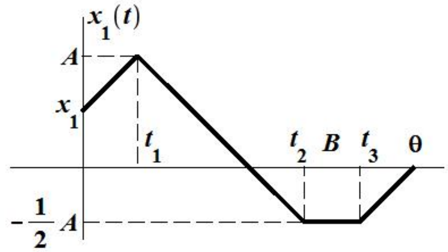

Fig. 10. Graph of $x_1(t)$, case 3

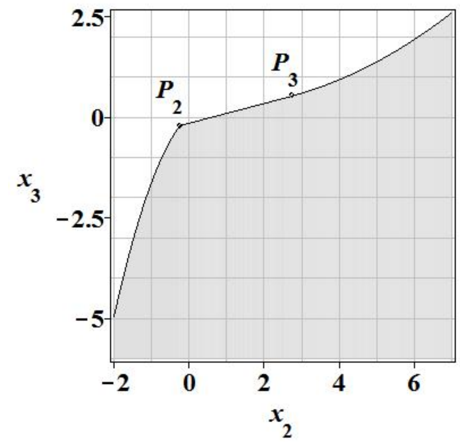

Fig. 11. Intersection of the domain D_3 and the plane $x_1 = 1$; $P_3 = (\frac{11}{4}, \frac{17}{32})$

To analyze the biquadratic equation $9A^4 - 8S_{11}A^2 + 32S_{21} = 0$, let us introduce the function $f(z) = 9z^2 - 8S_{11}z + 32S_{21}$; then A^2 is a (positive) root of $f(z)$.

(a) If $S_{21} \le 0$, then the function $f(z)$ has one non-negative root. Hence, the biquadratic equation has one non-negative root (the maximal one). The condition $A \ge x_1$, which can be expressed as $A^2 \ge x_1^2$, is equivalent to

$$f(x_1^2) \le 0 \quad \Leftrightarrow \quad 9x_1^4 - 8S_{11}x_1^2 + 32S_{21} \le 0. \tag{13}$$

If $S_{11} \ge 0$, then the condition $B \ge 0$ is obviously satisfied. If $S_{11} \le 0$, then this condition can be expressed as $A^2 \ge -\frac{4}{3}S_{11}$ and is equivalent to

$$f(-\tfrac{4}{3}S_{11}) \le 0 \Leftrightarrow 9(-\tfrac{4}{3}S_{11})^2 - 8S_{11}(-\tfrac{4}{3}S_{11}) + 32S_{21} \le 0 \;\Leftrightarrow\; 5S_{11}^2 + 6S_{21} \le 0. \tag{14}$$

We note that condition (13) implies (14) if $x_1^2 \ge -\frac{4}{3}S_{11}$, and (14) implies (13) otherwise; recall that if $S_{11} \ge 0$, then only condition (13) should be required. Hence, the solvability domain in case (a) is

$$\big\{x:\ x_3 \le \tfrac{1}{4}x_1^4,\quad x_3 \le -\tfrac{5}{6}x_2^2 + \tfrac{5}{6}x_1^2x_2 + \tfrac{1}{24}x_1^4 \ \text{ if } x_2 \le -\tfrac{1}{4}x_1^2,$$
$$x_3 \le \tfrac{1}{4}x_1^2x_2 - \tfrac{5}{32}x_1^4 \ \text{ if } x_2 \ge -\tfrac{1}{4}x_1^2 \ \big\}$$

and

$$t_1 = A - x_1, \quad t_2 = \tfrac{5}{2}A - x_1, \quad t_3 = \tfrac{5}{2}A + B - x_1, \quad \theta = 3A + B - x_1, \tag{15}$$

where

$$A = \tfrac{2}{3}\sqrt{S_{11} + \sqrt{d}}, \quad d = S_{11}^2 - 18S_{21}, \quad B = \frac{2S_{11}}{A} + \tfrac{3}{2}A. \tag{16}$$

(b) Let $S_{21} > 0$. If $S_{11} < 0$, then the function $f(z)$ has no nonnegative roots, therefore, the biquadratic equation has no real roots. If $S_{11} \ge 0$, then $f(z)$ has nonnegative roots iff $d = S_{11}^2 - 18S_{21} \geq 0$. The condition $B \geq 0$ is obviously satisfied. The condition $A \geq x_1$ will be considered later.

Now, suppose the equation $9A^4 - 8S_{11}A^2 + 32S_{21} = 0$ has two different positive roots $A_{max} > A_{min} \geq x_1$. Let us compare the corresponding times of motion θ_{max} and θ_{min}. For both values (15) holds, hence,

$$\theta_{min} = \tfrac{9}{2}A_{min} + \frac{2S_{11}}{A_{min}} - x_1, \quad \theta_{max} = \tfrac{9}{2}A_{max} + \frac{2S_{11}}{A_{max}} - x_1.$$

Since A_{min}^2 and A_{max}^2 are different roots of $f(z)$, we have $\frac{8}{9}S_{11} = A_{max}^2 + A_{min}^2$. Therefore, $\theta_{min} \geq \theta_{max}$ iff

$$\frac{\frac{9}{2}A_{min}^2 + 2S_{11}}{A_{min}} \geq \frac{\frac{9}{2}A_{max}^2 + 2S_{11}}{A_{max}} \quad \Leftrightarrow \quad \frac{3A_{min}^2 + A_{max}^2}{A_{min}} \geq \frac{3A_{max}^2 + A_{min}^2}{A_{max}},$$

which is equivalent to the obvious inequality $(A_{max} - A_{min})^3 \geq 0$. Thus, θ_{min} cannot be the optimal time. This means that the maximal root of the biquadratic equation should be taken, $A = A_{max}$, therefore, in this case (15), (16) hold as well. The condition $A^2 = \frac{4}{9}(S_{11} + \sqrt{d}) \geq x_1^2$ implies $S_{11} \geq 0$ and is equivalent to

$$\tfrac{9}{4}x_1^2 - S_{11} \leq 0 \quad \text{or} \quad d \geq (\tfrac{9}{4}x_1^2 - S_{11})^2 \;\Leftrightarrow\; 9x_1^4 - 8S_{11}x_1^2 + 32S_{21} \leq 0.$$

We note that $d \geq (\frac{9}{4}x_1^2 - S_{11})^2$ implies $d \geq 0$. Thus, the solvability domain in case (b) is

$$\big\{x: \; x_3 \geq \tfrac{1}{4}x_1^4, \;\; x_3 \leq \tfrac{1}{4}x_1^2x_2 - \tfrac{5}{32}x_1^4 \text{ if } x_2 \leq \tfrac{11}{4}x_1^2,$$
$$x_3 \leq \tfrac{1}{18}x_2^2 - \tfrac{1}{18}x_1^2x_2 + \tfrac{19}{72}x_1^4 \text{ if } x_2 \geq \tfrac{11}{4}x_1^2\big\}.$$

Combining the obtained results, we get the solvability domain in case 3

$$D_3 = \big\{x: \quad x_3 \leq -\tfrac{5}{6}x_2^2 + \tfrac{5}{6}x_1^2x_2 + \tfrac{1}{24}x_1^4 \text{ if } x_2 \leq -\tfrac{1}{4}x_1^2,$$
$$x_3 \leq \tfrac{1}{4}x_1^2x_2 - \tfrac{5}{32}x_1^4 \text{ if } -\tfrac{1}{4}x_1^2 \leq x_2 \leq \tfrac{11}{4}x_1^2,$$
$$x_3 \leq \tfrac{1}{18}x_2^2 - \tfrac{1}{18}x_1^2x_2 + \tfrac{19}{72}x_1^4 \text{ if } x_2 \geq \tfrac{11}{4}x_1^2\big\}.$$

The time of motion and switching moments are found by formulas (15), (16).

Case 4 corresponds to $P(z)$ of type 3 (Fig. 3), the control is of the form (7).

Using the notation of case 1, we have

$$\begin{cases} S_{11} = B^2 - A^2, \\ 2S_{21} = B^4 - A^4, \\ A \geq x_1, \; A \geq 2B \geq 0, \end{cases} \quad \Leftrightarrow \quad \begin{cases} A^2 = \frac{S_{21}}{S_{11}} - \frac{1}{2}S_{11}, \\ B^2 = \frac{S_{21}}{S_{11}} + \frac{1}{2}S_{11}, \\ A \geq x_1, \; A \geq 2B \geq 0. \end{cases}$$

If $S_{11} \geq 0$, then $B^2 \geq A^2$, which contradicts the requirement $A \geq 2B$. If $S_{11} < 0$, then the solvability conditions are

$$\begin{cases} S_{11} < 0, \; 2S_{21} + S_{11}^2 \leq 0, \\ \frac{S_{21}}{S_{11}} - \frac{1}{2}S_{11} \geq x_1^2, \\ \frac{S_{21}}{S_{11}} - \frac{1}{2}S_{11} \geq 4(\frac{S_{21}}{S_{11}} + \frac{1}{2}S_{11}), \end{cases} \quad \Leftrightarrow \quad \begin{cases} S_{11} < 0, \; 2S_{21} \leq -S_{11}^2, \\ 2S_{21} \leq S_{11}^2 + 2x_1^2S_{11}, \\ 5S_{11}^2 + 6S_{21} \geq 0. \end{cases}$$

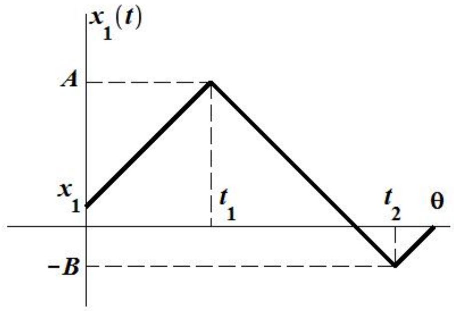

Fig. 12. Graph of $x_1(t)$, case 4

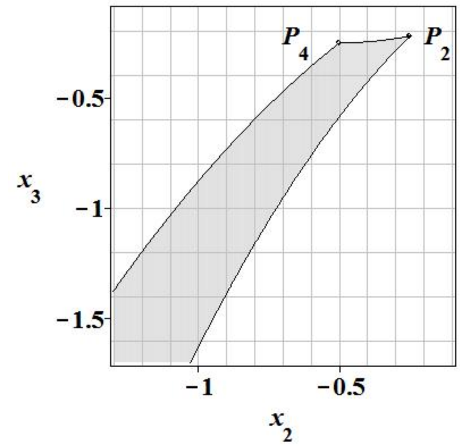

Fig. 13. Intersection of the domain D_4 and the plane $x_1 = 1$; $P_4 = (-\frac{1}{2}, -\frac{1}{4})$

Notice that these conditions imply $S_{11} \le -\frac{3}{4}x_1^2$. Notice also that in this case $-S_{11}^2 \le S_{11}^2 + 2x_1^2 S_{11}$ iff $S_{11} \le -x_1^2$. Substituting the expressions for S_{11} and S_{21}, we get the solvability domain for case 4:

$$D_4 = \big\{x: \ x_2 \le -\tfrac{1}{4}x_1^2, \quad x_3 \ge -\tfrac{5}{6}x_2^2 + \tfrac{5}{6}x_1^2 x_2 + \tfrac{1}{24}x_1^4,$$
$$x_3 \le -\tfrac{1}{2}x_2^2 + \tfrac{1}{2}x_1^2 x_2 + \tfrac{1}{8}x_1^4 \ \text{ if } \ x_2 \le -\tfrac{1}{2}x_1^2,$$
$$x_3 \le \tfrac{1}{2}x_2^2 + \tfrac{1}{2}x_1^2 x_2 - \tfrac{1}{8}x_1^4 \ \text{ if } \ x_2 \ge -\tfrac{1}{2}x_1^2\big\}$$

and the time of motion and switching moments are found by (9), (10).

Case 5 corresponds to $P(z)$ of type 4 (Fig. 4) with the control of the form

$$u(t) = \begin{cases} 1 & \text{if } t \in [0, t_1), \\ 0 & \text{if } t \in [t_1, t_2), \\ -1 & \text{if } t \in [t_2, \theta]. \end{cases}$$

Denote $A = x_1 + t_1 = z_2$ and $B = t_2 - t_1$, then

$$-x_2 = -\tfrac{1}{2}x_1^2 + A^2 + AB \quad \text{and} \quad -x_3 = -\tfrac{1}{4}x_1^4 + \tfrac{1}{2}A^4 + A^3 B.$$

Hence,

$$\begin{cases} S_{11} = -A^2 - AB, \\ S_{21} = -\frac{1}{2}A^4 - A^3 B, \\ A \ge x_1, \ B \ge 0, \end{cases} \quad \Leftrightarrow \quad \begin{cases} A^4 + 2S_{11}A^2 - 2S_{21} = 0, \\ B = -\frac{S_{11}}{A} - A, \\ A \ge x_1, \ B \ge 0. \end{cases}$$

The biquadratic equation $A^4 + 2S_{11}A^2 - 2S_{21} = 0$ has real roots iff $d = S_{11}^2 + 2S_{21} \ge 0$, and then $A^2 = -S_{11} \pm \sqrt{d}$. However, $B \ge 0$ iff $A^2 \le -S_{11}$. Hence, the minimal root should be chosen, $A^2 = -S_{11} - \sqrt{d}$. The condition $A \ge x_1$ can be written as $A^2 = -S_{11} - \sqrt{d} \ge x_1^2$ and is equivalent to $S_{11} + x_1^2 \le 0$ and $d \le (S_{11} + x_1^2)^2$. Therefore, the solvability domain is

$$D_5 = \big\{x: \ x_2 \le -\tfrac{1}{2}x_1^2, \ -\tfrac{1}{2}x_2^2 + \tfrac{1}{2}x_1^2 x_2 + \tfrac{1}{8}x_1^4 \le x_3 \le x_1^2 x_2 + \tfrac{1}{4}x_1^4\big\}.$$

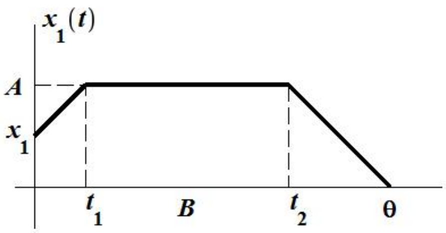

Fig. 14. Graph of $x_1(t)$, case 5

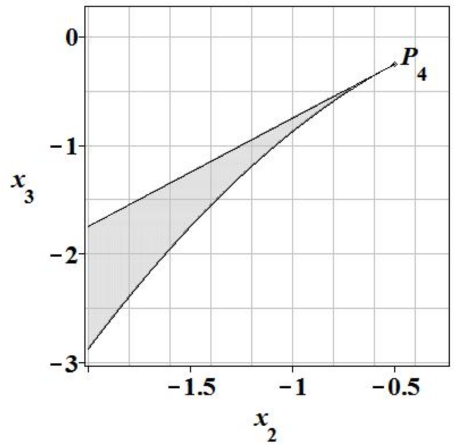

Fig. 15. Intersection of the domain D_5 and the plane $x_1 = 1$

In this case

$$t_1 = A - x_1, \quad t_2 = A + B - x_1, \quad \theta = 2A + B - x_1, \tag{17}$$

where

$$A = \sqrt{-S_{11} - \sqrt{d}}, \quad d = S_{11}^2 + 2S_{21}, \quad B = -\frac{S_{11}}{A} - A. \tag{18}$$

Case 6 corresponds to $P(z)$ of type 4 (Fig. 4) with the control of the form

$$u(t) = \begin{cases} -1 & \text{if } t \in [0, t_1), \\ 0 & \text{if } t \in [t_1, t_2), \\ -1 & \text{if } t \in [t_2, \theta]. \end{cases}$$

Denote $A = x_1 - t_1 = z_2$ and $B = t_2 - t_1$, then

$$-x_2 = \tfrac{1}{2}x_1^2 + AB \quad \text{and} \quad -x_3 = \tfrac{1}{4}x_1^4 + A^3B.$$

If $A = 0$ or $B = 0$, then $x_2 = -\frac{1}{2}x_1^2$ and $x_3 = -\frac{1}{4}x_1^4$; obviously, for this point the optimal control has no switchings and equals -1. Below we assume $A > 0$ and $B > 0$. Then

$$\begin{cases} S_{12} = -AB, \\ S_{22} = -A^3B, \\ 0 < A \le x_1,\ B > 0, \end{cases} \quad \Leftrightarrow \quad \begin{cases} A^2 = \frac{S_{22}}{S_{12}}, \\ B = -\frac{S_{12}}{A}, \\ 0 < A \le x_1,\ B > 0. \end{cases}$$

The solvability domain equals

$$D_6 = \left\{x:\ x_2 < -\tfrac{1}{2}x_1^2,\ x_1^2x_2 + \tfrac{1}{4}x_1^4 \le x_3 < -\tfrac{1}{4}x_1^4\right\},$$

and in this case

$$t_1 = x_1 - A, \quad t_2 = x_1 - A + B, \quad \theta = x_1 + B, \tag{19}$$

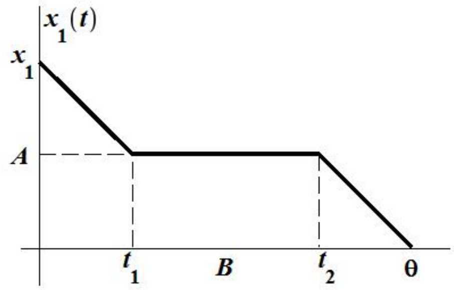

Fig. 16. Graph of $x_1(t)$, case 6

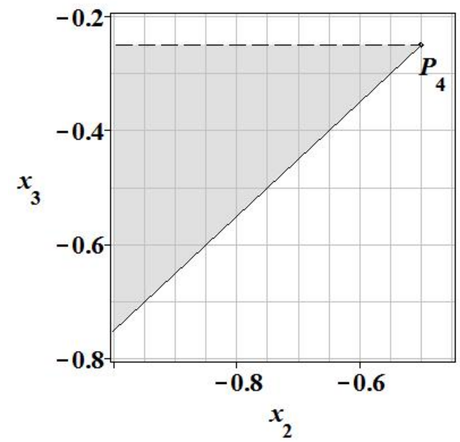

Fig. 17. Intersection of the domain D_6 and the plane $x_1 = 1$

where

$$A = \sqrt{\frac{S_{22}}{S_{12}}}, \quad B = -\frac{S_{12}}{A}. \tag{20}$$

Case 7 corresponds to $P(z)$ of type 4 (Fig. 4) with the control of the form

$$u(t) = \begin{cases} 1 & \text{if } t \in [0, t_1), \\ 0 & \text{if } t \in [t_1, t_2), \\ -1 & \text{if } t \in [t_2, t_3), \\ 1 & \text{if } t \in [t_3, \theta]. \end{cases}$$

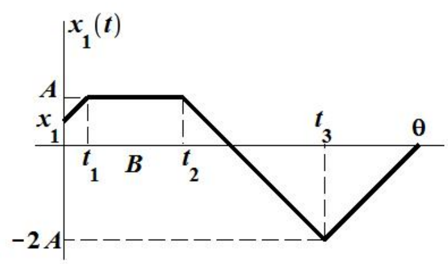

Fig. 18. Graph of $x_1(t)$, case 7

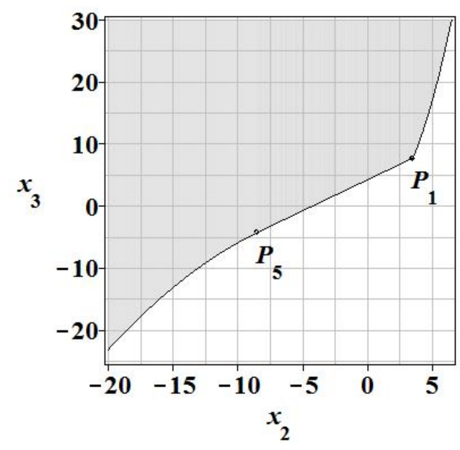

Fig. 19. Intersection of the domain D_7 and the plane $x_1 = 1$; $P_5 = (-\frac{17}{2}, -\frac{17}{4})$

Denote $A = x_1 + t_1 = z_2$ and $B = t_2 - t_1$, then

$$-x_2 = -\tfrac{1}{2}x_1^2 - 3A^2 + AB \quad \text{and} \quad -x_3 = -\tfrac{1}{4}x_1^4 - \tfrac{15}{2}A^4 + A^3B.$$

Hence,

$$\begin{cases} S_{11} = 3A^2 - AB, \\ S_{21} = \frac{15}{2}A^4 - A^3B, \\ A \geq x_1,\ B \geq 0, \end{cases} \quad \Leftrightarrow \quad \begin{cases} 9A^4 + 2S_{11}A^2 - 2S_{21} = 0, \\ B = -\frac{S_{11}}{A} + 3A, \\ A \geq x_1,\ B \geq 0. \end{cases}$$

Analogously to case 3, we introduce the function $f(z) = 9z^2 + 2S_{11}z - 2S_{21}$.

(a) If $S_{21} \geq 0$ then $f(z)$ has one non-negative root. The condition $A \geq x_1$, which can be expressed as $A^2 \geq x_1^2$, is equivalent to

$$f(x_1^2) \leq 0 \quad \Leftrightarrow \quad 9x_1^4 + 2S_{11}x_1^2 - 2S_{21} \leq 0. \tag{21}$$

If $S_{11} \leq 0$, then the condition $B \geq 0$ is obviously satisfied. If $S_{11} \geq 0$, then the condition $B \geq 0$, which can be expressed as $A^2 \geq \frac{1}{3}S_{11}$, is equivalent to

$$f(\tfrac{1}{3}S_{11}) \leq 0 \ \Leftrightarrow\ 9(\tfrac{1}{3}S_{11})^2 + 2S_{11}(\tfrac{1}{3}S_{11}) - 2S_{21} \leq 0 \ \Leftrightarrow\ 5S_{11}^2 - 6S_{21} \leq 0. \tag{22}$$

Condition (21) implies (22) if $x_1^2 \geq \frac{1}{3}S_{11}$, and (22) implies (21) otherwise; if $S_{11} \leq 0$ then only (21) should be required. Thus, the solvability domain in case (a) is

$$\big\{x:\ x_3 \geq \tfrac{1}{4}x_1^4, \quad x_3 \geq x_1^2x_2 + \tfrac{17}{4}x_1^4 \ \text{ if } \ x_2 \leq \tfrac{7}{2}x_1^2, \\ x_3 \geq \tfrac{5}{6}x_2^2 - \tfrac{5}{6}x_1^2x_2 + \tfrac{11}{24}x_1^4 \ \text{ if } \ x_2 \geq \tfrac{7}{2}x_1^2\big\},$$

and the formulas for switching moments and the optimal time are

$$t_1 = A - x_1, \quad t_2 = A + B - x_1, \quad t_3 = 4A + B - x_1, \quad \theta = 6A + B - x_1, \tag{23}$$

where

$$A = \tfrac{1}{3}\sqrt{-S_{11} + \sqrt{d}}, \quad d = S_{11}^2 + 18S_{21}, \quad B = -\frac{S_{11}}{A} + 3A. \tag{24}$$

(b) Let $S_{21} < 0$. If $S_{11} > 0$, then the function $f(z)$ has no nonnegative roots. If $S_{11} \leq 0$, then $f(z)$ has nonnegative roots iff $d = S_{11}^2 + 18S_{21} \geq 0$. The condition $B \geq 0$ is obviously satisfied.

Suppose the equation $9A^4 + 2S_{11}A^2 - 2S_{21} = 0$ has two different positive roots $A_{max} > A_{min} \geq x_1$. Let us compare the corresponding times of motion θ_{min} and θ_{max}. For both values (23) holds, then

$$\theta_{min} = \frac{9A_{min}^2 - S_{11}}{A_{min}} - x_1, \quad \theta_{max} = \frac{9A_{max}^2 - S_{11}}{A_{max}} - x_1.$$

Since A_{min}^2 and A_{max}^2 are different roots of $f(z)$, we have $-\frac{2}{9}S_{11} = A_{min}^2 + A_{max}^2$. Then $\theta_{min} \geq \theta_{max}$ iff

$$\frac{9A_{min}^2 - S_{11}}{A_{min}} \geq \frac{9A_{max}^2 - S_{11}}{A_{max}} \quad \Leftrightarrow \quad \frac{3A_{min}^2 + A_{max}^2}{A_{min}} \geq \frac{3A_{max}^2 + A_{min}^2}{A_{max}},$$

which is equivalent to $(A_{max} - A_{min})^3 \geq 0$. Hence, θ_{min} cannot be the optimal time and the maximal root of the biquadratic equation should be taken, $A = A_{max}$. The condition $A^2 = \frac{1}{9}(-S_{11} + \sqrt{d}) \geq x_1^2$ implies $S_{11} \leq 0$ and is equivalent to

$$9x_1^2 + S_{11} \leq 0 \quad \text{or} \quad d \geq (9x_1^2 + S_{11})^2 \ \Leftrightarrow\ 9x_1^4 + 2S_{11}x_1^2 - 2S_{21} \leq 0.$$

The condition $d \geq (9x_1^2 + S_{11})^2$ implies $d \geq 0$. Therefore, the solvability domain in case (b) is

$$\big\{x:\ x_3 \leq \tfrac{1}{4}x_1^4,\quad x_3 \geq -\tfrac{1}{18}x_2^2 + \tfrac{1}{18}x_1^2x_2 + \tfrac{17}{72}x_1^4 \ \text{ if }\ x_2 \leq -\tfrac{17}{2}x_1^2,\\ x_3 \geq x_1^2x_2 + \tfrac{17}{4}x_1^4 \ \text{ if }\ x_2 \geq -\tfrac{17}{2}x_1^2\big\}.$$

Combining the obtained results, we get the solvability domain in case 7

$$D_7 = \big\{x:\ \ x_3 \geq -\tfrac{1}{18}x_2^2 + \tfrac{1}{18}x_1^2x_2 + \tfrac{17}{72}x_1^4 \ \text{ if } x_2 \leq -\tfrac{17}{2}x_1^2,\\ x_3 \geq x_1^2x_2 + \tfrac{17}{4}x_1^4 \ \text{ if } \ -\tfrac{17}{2}x_1^2 \leq x_2 \leq \tfrac{7}{2}x_1^2,\\ x_3 \geq \tfrac{5}{6}x_2^2 - \tfrac{5}{6}x_1^2x_2 + \tfrac{11}{24}x_1^4 \ \text{ if } x_2 \geq \tfrac{7}{2}x_1^2\big\}. \tag{25}$$

The time of motion and switching moments are found by (23), (24).

Case 8 corresponds to $P(z)$ of type 4 (Fig. 4) with the control of the form

$$u(t) = \begin{cases} -1 & \text{if } t \in [0, t_1), \\ \ \ 0 & \text{if } t \in [t_1, t_2), \\ -1 & \text{if } t \in [t_2, t_3), \\ \ \ 1 & \text{if } t \in [t_3, \theta]. \end{cases}$$

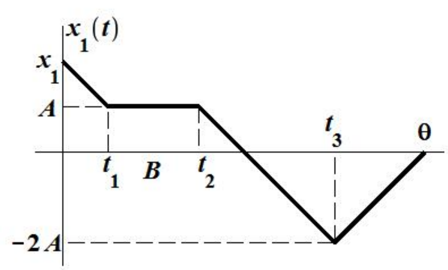

Fig. 20. Graph of $x_1(t)$, case 8

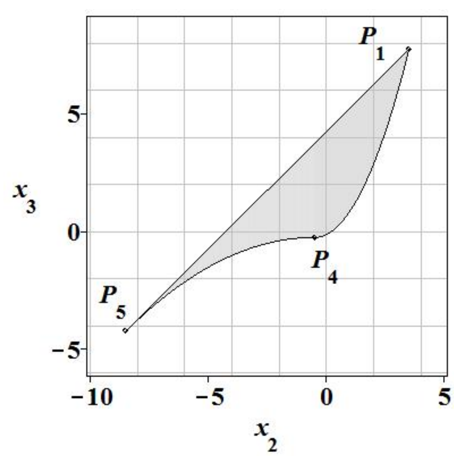

Fig. 21. Intersection of the domain of solvability D_8 and the plane $x_1 = 1$

Denote $A = x_1 - t_1 = z_2$ and $B = t_2 - t_1$, then

$$-x_2 = \tfrac{1}{2}x_1^2 - 4A^2 + AB \quad \text{ and } \quad -x_3 = \tfrac{1}{4}x_1^4 - 8A^4 + A^3B.$$

If $A = 0$, then $B = 0$ and, therefore, $x_2 = -\frac{1}{2}x_1^2$ and $x_3 = -\frac{1}{4}x_1^4$; for this point the optimal control equals -1. Below we require $A > 0$. Then

$$\begin{cases} S_{12} = 4A^2 - AB, \\ S_{22} = 8A^4 - A^3B, \\ 0 < A \leq x_1,\ B \geq 0, \end{cases} \quad \Leftrightarrow \quad \begin{cases} 4A^4 + S_{12}A^2 - S_{22} = 0, \\ B = -\frac{S_{12}}{A} + 4A, \\ 0 < A \leq x_1,\ B \geq 0. \end{cases}$$

Analogously to the cases 3 and 7, we introduce $f(z) = 4z^2 + S_{12}z - S_{22}$.

(a) If $S_{22} \geq 0$, then $f(z)$ has one non-negative root. The condition $A \leq x_1$ is equivalent to

$$f(x_1^2) \geq 0 \quad \Leftrightarrow \quad 4x_1^4 + S_{12}x_1^2 - S_{22} \geq 0. \tag{26}$$

If $S_{12} \leq 0$, then the condition $B \geq 0$ is satisfied. If $S_{12} \geq 0$, then $B \geq 0$ iff

$$f(\tfrac{1}{4}S_{12}) \leq 0 \;\Leftrightarrow\; 4(\tfrac{1}{4}S_{12})^2 + S_{12}(\tfrac{1}{4}S_{12}) - S_{22} \leq 0 \;\Leftrightarrow\; S_{12}^2 - 2S_{22} \leq 0. \tag{27}$$

Conditions (26) and (27) imply $S_{12} \leq 4x_1^2$. Hence, the solvability domain in case (a) is

$$\left\{x:\; -\tfrac{1}{4}x_1^4 \leq x_3 \leq x_1^2x_2 + \tfrac{17}{4}x_1^4,\; x_3 \geq \tfrac{1}{2}x_2^2 + \tfrac{1}{2}x_1^2x_2 - \tfrac{1}{8}x_1^4 \text{ if } -\tfrac{1}{2}x_1^2 \leq x_2 \leq \tfrac{7}{2}x_1^2\right\}$$

and

$$t_1 = x_1 - A, \quad t_2 = x_1 - A + B, \quad t_3 = x_1 + 2A + B, \quad \theta = x_1 + 4A + B, \tag{28}$$

where

$$A = \sqrt{\tfrac{1}{8}(-S_{12} + \sqrt{d})}, \quad d = S_{12}^2 + 16S_{22}, \quad B = -\frac{S_{12}}{A} + 4A. \tag{29}$$

(b) Let $S_{22} < 0$. If $S_{12} > 0$, then the function $f(z)$ has no nonnegative roots. If $S_{12} \leq 0$, then $f(z)$ has nonnegative roots iff $d = S_{12}^2 + 16S_{22} \geq 0$. The condition $B \geq 0$ is satisfied. Now we consider the condition $A \leq x_1$. Suppose the roots of the equation $4A^4 + S_{12}A^2 - S_{22} = 0$ are $A_{min} \leq A_{max}$.

(b1) First, let us consider the case when $A_{min}^2 \leq x_1^2 \leq A_{max}^2$, which is equivalent to $f(x_1^2) \leq 0$; this inequality implies $S_{12} \leq 0$. Then we get the condition

$$S_{22} < 0 \text{ and } 4x_1^4 + S_{12}x_1^2 - S_{22} \leq 0 \quad \Leftrightarrow \quad x_1^2x_2 + \tfrac{17}{4}x_1^4 \leq x_3 < -\tfrac{1}{4}x_1^4,$$

which implies $x_2 < -\frac{9}{2}x_1^2$. Analogously to (28), the time of motion θ_{8min} corresponding to A_{min} equals $\theta_{8min} = 8A_{min} - \frac{S_{12}}{A_{min}} + x_1$. It is easy to see that in this domain the control corresponding to case 6 exists; the time of motion θ_6 can be found by (19), (20). Let us show that $\theta_{8min} > \theta_6$. Since $A_{min}^2 + A_{max}^2 = -\frac{1}{4}S_{12}$, $A_{min}^2A_{max}^2 = -\frac{1}{4}S_{22}$, we get

$$\theta_{8min} = \frac{8A_{min}^2 + 4(A_{min}^2 + A_{max}^2)}{A_{min}} + x_1 = 4\frac{3A_{min}^2 + A_{max}^2}{A_{min}} + x_1,$$

$$\theta_6 = \sqrt{\frac{16(A_{min}^2 + A_{max}^2)^3}{A_{min}^2A_{max}^2}} + x_1 = 4\frac{\sqrt{(A_{min}^2 + A_{max}^2)^3}}{A_{min}A_{max}} + x_1.$$

Hence, $\theta_{8min} > \theta_6$ iff

$$\frac{3A_{min}^2 + A_{max}^2}{A_{min}} > \frac{\sqrt{(A_{min}^2 + A_{max}^2)^3}}{A_{min}A_{max}} \Leftrightarrow (3A_{min}^2 + A_{max}^2)^2A_{max}^2 > (A_{min}^2 + A_{max}^2)^3.$$

This is equivalent to the obvious inequality $A_{min}^2(6A_{min}^2A_{max}^2+3A_{max}^4-A_{min}^4)>0$. Thus, the control in case (b1) cannot be optimal. In Fig. 21 and in formula (30) we do not indicate points satisfying case (b1).

(b2) Now let us consider the case when $A_{max}^2=\frac{1}{8}(-S_{12}+\sqrt{d})\le x_1^2$, which is equivalent to a pair of conditions $8x_1^2+S_{12}\ge 0$ and $d\le(8x_1^2+S_{12})^2$. Let θ_{8max} be the time of motion corresponding to A_{max}. As above, we have

$$\theta_{8min}=4\frac{3A_{min}^2+A_{max}^2}{A_{min}}+x_1,\quad \theta_{8max}=4\frac{3A_{max}^2+A_{min}^2}{A_{max}}+x_1,$$

so, $\theta_{8min}\ge\theta_{8max}$ is equivalent to $(A_{max}-A_{min})^3\ge 0$. Thus, the maximal root $A=A_{max}$ should be chosen. The solvability domain in case (b2) is

$$\Big\{x:\ -\tfrac{17}{2}x_1^2\le x_2\le-\tfrac{1}{2}x_1^2,\quad x_3<-\tfrac{1}{4}x_1^4,\ x_3\le x_2x_1^2+\tfrac{17}{4}x_1^4,\\ x_3\ge-\tfrac{1}{16}x_2^2-\tfrac{1}{16}x_1^2x_2-\tfrac{17}{64}x_1^4\Big\}.$$

Combining the obtained results, we get the solvability domain in case 8 (recall that we do not include points corresponding to the case (b1))

$$D_8=\Big\{x:\ -\tfrac{17}{2}x_1^2\le x_2\le\tfrac{7}{2}x_1^2,\quad x_3\le x_1^2x_2+\tfrac{17}{4}x_1^4,\\ x_3\ge-\tfrac{1}{16}x_2^2-\tfrac{1}{16}x_1^2x_2-\tfrac{17}{64}x_1^4\ \text{ if }\ x_2\le-\tfrac{1}{2}x_1^2,\\ x_3\ge\tfrac{1}{2}x_2^2+\tfrac{1}{2}x_1^2x_2-\tfrac{1}{8}x_1^4\ \text{ if }\ x_2\ge-\tfrac{1}{2}x_1^2\Big\}. \qquad (30)$$

The time of motion and switching moments are found by (28), (29).

3. Overlapping solvability domains

In this section we analyze the solvability domains which overlap.

Cases 2 and 3. The domain where both controls exist is

$$D_{2,3}=\Big\{x:\ x_2\ge\tfrac{11}{4}x_1^2,\ \tfrac{1}{4}x_1^2x_2-\tfrac{5}{32}x_1^4\le x_3\le\tfrac{1}{18}x_2^2-\tfrac{1}{18}x_1^2x_2+\tfrac{19}{72}x_1^4\Big\}$$

(see Fig. 22). The times of motion θ_2 and θ_3 for cases 2 and 3 can be found by (11), (12) and (15), (16). Let us introduce the function $F=\theta_3-\theta_2$, i.e.,

$$F(x)=\frac{6S_{11}+3\sqrt{S_{11}^2-18S_{21}}}{\sqrt{S_{11}+\sqrt{S_{11}^2-18S_{21}}}}-\frac{2S_{12}-\sqrt{S_{12}^2-2S_{22}}}{\sqrt{S_{12}-\sqrt{S_{12}^2-2S_{22}}}}-2x_1. \qquad (31)$$

Then $\theta_2=\theta_3$ iff x belongs to the surface

$$M_{2,3}=\{x:\ x_2\ge\tfrac{11}{4}x_1^2,\ \tfrac{1}{4}x_1^2x_2-\tfrac{5}{32}x_1^4\le x_3\le\tfrac{1}{18}x_2^2-\tfrac{1}{18}x_1^2x_2+\tfrac{19}{72}x_1^4,\ F(x)=0\}$$

and for any point $x\in D_{2,3}$ one has $\theta_2<\theta_3$ iff $F(x)>0$.

Our nearest goal is to show that the surface $M_{2,3}$ has a unique point of intersection with any vertical line with fixed $x_1>0$ and $x_2\ge\frac{11}{4}x_1^2$. To this end, let us fix any $x_1>0$ and $x_2>\frac{11}{4}x_1^2$ and suppose x_3 runs through the segment $[x_{3min},x_{3max}]=[\frac{1}{4}x_1^2x_2-\frac{5}{32}x_1^4,\frac{1}{18}x_2^2-\frac{1}{18}x_1^2x_2+\frac{19}{72}x_1^4]$. Then

$$\theta_2=\theta_2(x_3)=A_2+\frac{S_{12}}{A_2}+x_1,\qquad \theta_3=\theta_3(x_3)=\tfrac{9}{2}A_3+\frac{2S_{11}}{A_3}-x_1,$$

where

$$A_2 = A_2(x_3) = \sqrt{S_{12} - \sqrt{S_{12}^2 - 2S_{22}}}, \quad A_3 = A_3(x_3) = \tfrac{2}{3}\sqrt{S_{11} + \sqrt{S_{11}^2 - 18S_{21}}}.$$

By $\widehat{\theta}(x)$ we denote the optimal time for the point x; it is continuous as a function of x, what follows from [5].

First, consider the lower bound, i.e., $x_3 = x_{3min}$. Let us notice that for points $x_\delta = (x_1, x_2, x_{3,\delta})$, where $x_{3,\delta} = x_{3min} - \delta$ with $\delta > 0$, the control of case 2 does not exist and the control of case 3 is optimal. Then $\widehat{\theta}(x_\delta) = \theta_3(x_{3,\delta})$. We notice that the function $\theta_3(x_3)$ is continuous. Hence,

$$\theta_3(x_{3min}) = \lim_{\delta\to 0} \theta_3(x_{3,\delta}) = \lim_{\delta\to 0} \widehat{\theta}(x_\delta) = \widehat{\theta}(x_0), \quad \text{where} \ \ x_0 = (x_1, x_2, x_{3min}),$$

which implies $\theta_3(x_{3min}) \le \theta_2(x_{3min})$. Analogously, for the upper bound we get $\theta_2(x_{3max}) \le \theta_3(x_{3max})$.

Notice that S_{11} and S_{12} are constants while S_{21} and S_{22} are increasing functions of x_3. Hence, A_2 increases and A_3 decreases (as functions of x_3). Since $A_2^2 \le S_{12}$, we see that θ_2 decreases as function of A_2. Analogously, $A_3^2 \ge \frac{4}{9}S_{11}$ implies that θ_3 increases as function of A_3. As a result, both functions θ_2 and θ_3 decrease as functions of x_3.

Let us introduce the functions

$$h_2(x_3) = \theta_2(x_3) + \frac{27x_3}{2\sqrt{S_{11}^3}}, \qquad h_3(x_3) = \theta_3(x_3) + \frac{27x_3}{2\sqrt{S_{11}^3}},$$

and show that $h_2(x_3)$ decreases and $h_3(x_3)$ increases. To this end, we find their derivatives. Since $\sqrt{S_{12}^2 - 2S_{22}} = -(A_2^2 - S_{12})$, we get

$$\frac{\partial\theta_2}{\partial x_3} = \frac{\partial\theta_2}{\partial A_2}\cdot\frac{\partial A_2}{\partial x_3} = \left(1 - \frac{S_{12}}{A_2^2}\right)\frac{-2}{4A_2(A_2^2 - S_{12})} = -\frac{1}{2A_2^3}$$

and analogously

$$\frac{\partial\theta_3}{\partial x_3} = \frac{\partial\theta_3}{\partial A_3}\cdot\frac{\partial A_3}{\partial x_3} = \left(\frac{9}{2} - \frac{2S_{11}}{A_3^2}\right)\frac{-\frac{2}{3}\cdot 18}{6A_3(\frac{9}{4}A_3^2 - S_{11})} = -\frac{4}{A_3^3}.$$

Hence,

$$\frac{\partial h_2(x_3)}{\partial x_3} = -\frac{1}{2A_2^3(x_3)} + \frac{27}{2\sqrt{S_{11}^3}}, \quad \frac{\partial h_3(x_3)}{\partial x_3} = -\frac{4}{A_3^3(x_3)} + \frac{27}{2\sqrt{S_{11}^3}}.$$

Then

$$\frac{\partial h_2(x_3)}{\partial x_3} \le 0 \quad\Leftrightarrow\quad 9A_2^2(x_3) \le S_{11} \quad\Leftrightarrow\quad x_3 \le \tfrac{1}{648}(68x_2^2 + 4x_1^2x_2 - 181x_1^4).$$

However, $x_3 \le \frac{1}{18}x_2^2 - \frac{1}{18}x_1^2x_2 + \frac{19}{72}x_1^4$ for $x \in D_{2,3}$ and

$$\tfrac{1}{18}x_2^2 - \tfrac{1}{18}x_1^2x_2 + \tfrac{19}{72}x_1^4 \le \tfrac{1}{648}(68x_2^2 + 4x_1^2x_2 - 181x_1^4) \ \Leftrightarrow\ (x_2 + 4x_1^2)(x_2 - \tfrac{11}{4}x_1^2) \ge 0,$$

which is true for $x \in D_{2,3}$. Hence, $\frac{\partial h_2(x_3)}{\partial x_3} \le 0$, i.e., $h_2(x_3)$ decreases. For h_3 we have

$$\frac{\partial h_3(x_3)}{\partial x_3} \ge 0 \quad \Leftrightarrow \quad 9A_3^2(x_3) \ge 4S_{11} \quad \Leftrightarrow \quad 4\sqrt{S_{11}^2 - 18S_{21}} \ge 0,$$

which is obvious. Hence, $\frac{\partial h_3(x_3)}{\partial x_3} \ge 0$. i.e., $h_3(x_3)$ increases. As was shown above, $\theta_2(x_{3min}) \ge \theta_3(x_{3min})$ and $\theta_3(x_{3max}) \ge \theta_2(x_{3max})$, hence,

$$h_2(x_{3min}) \ge h_3(x_{3min}) \ \text{ and } \ h_3(x_{3max}) \ge h_2(x_{3max}).$$

Thus, there exists a unique point $\widetilde{x}_3 \in [x_{3min}, x_{3max}]$ such that $h_2(\widetilde{x}_3) = h_3(\widetilde{x}_3)$ or, equivalently, $\theta_2(\widetilde{x}_3) = \theta_3(\widetilde{x}_3)$ for any fixed $x_1 > 0$ and $x_2 \ge \frac{11}{4}x_1^2$.

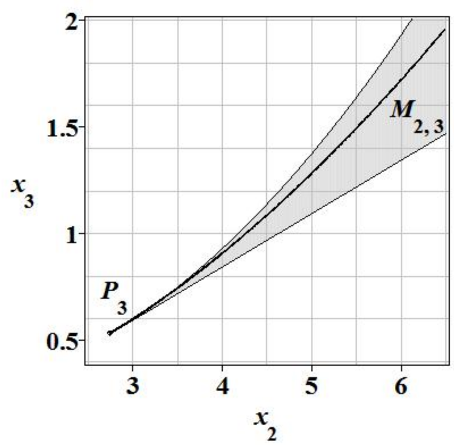

Fig. 22. Intersection of the domain $D_{2,3}$ and the surface $M_{2,3}$ with the plane $x_1 = 1$

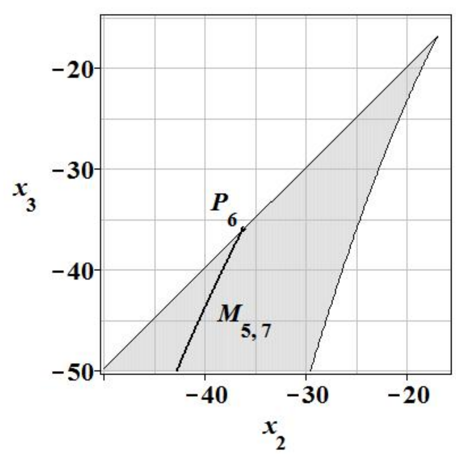

Fig. 23. Intersection of the domain $D_{5,7}$ and the surface $M_{5,7}$ with the plane $x_1 = 1$; $P_6 = (c_2, c_2 + \frac{1}{4}) \approx (-36.175, -35.925)$

Cases 5 and 7. The domain where both controls exist is

$$D_{5,7} = \big\{x: \ x_2 \le -\tfrac{17}{2}x_1^2, \ -\tfrac{1}{18}x_2^2 + \tfrac{1}{18}x_1^2x_2 + \tfrac{17}{72}x_1^4 \le x_3 \le x_1^2x_2 + \tfrac{1}{4}x_1^4\big\}.$$

These conditions imply $x_2 \le rx_1^2$, where $r = (-\frac{17}{2} - 6\sqrt{2}) \approx -16.98528$. Denote the corresponding times of motion by θ_5 and θ_7. Formulas (17), (18) and (23), (24) imply

$$\theta_5 = \frac{-2S_{11} - \sqrt{S_{11}^2 + 2S_{21}}}{\sqrt{-S_{11} - \sqrt{S_{11}^2 + 2S_{21}}}} - x_1, \quad \theta_7 = \frac{-6S_{11} + 3\sqrt{S_{11}^2 + 18S_{21}}}{\sqrt{-S_{11} + \sqrt{S_{11}^2 + 18S_{21}}}} - x_1.$$

Hence, $\theta_5 \ge \theta_7$ iff

$$\frac{(-2S_{11} - \sqrt{S_{11}^2 + 2S_{21}})^2}{-S_{11} - \sqrt{S_{11}^2 + 2S_{21}}} \ge \frac{(-6S_{11} + 3\sqrt{S_{11}^2 + 18S_{21}})^2}{-S_{11} + \sqrt{S_{11}^2 + 18S_{21}}}. \tag{32}$$

Let us write down this relation in an explicit form w.r.t. x_3. Taking into account that in $D_{5,7}$ the inequalities $S_{11} < 0$ and $S_{21} < 0$ hold, we denote $v = \sqrt{1 + 18\frac{S_{21}}{S_{11}^2}} < 1$ and $w = \sqrt{1 + 2\frac{S_{21}}{S_{11}^2}} = \frac{1}{3}\sqrt{v^2+8} < 1$. Then (32) reads

$$\frac{2-w}{\sqrt{1-w}} \geq \frac{6+3v}{\sqrt{1+v}} \;\Leftrightarrow\; (2-w)^2(1+v) \geq (6+3v)^2(1-w).$$

Substituting $w^2 = \frac{1}{9}v^2 + \frac{8}{9}$, we get the equivalent inequality

$$9w(9v^2+32v+32) \geq -v^3 + 80v^2 + 280v + 280;$$

its both sides are positive for $0 \leq v < 1$. Hence, we get

$$9(v^2+8)(9v^2+32v+32)^2 \geq (-v^3+80v^2+280v+280)^2,$$

which is equivalent to

$$(91v^4 + 486v^3 + 736v^2 - 584)(1+v)^2 \geq 0.$$

The function $91v^4 + 486v^3 + 736v^2 - 584$ increases as $v \geq 0$ and its unique positive root equals $v_1 \approx 0.71826$. Hence, (32) holds iff $v \geq v_1$. Substituting the expression of v we get that (32) holds iff $S_{21} \geq c_1 S_{11}^2$, i.e., $x_3 \geq \frac{1}{4}x_1^4 + c_1(x_2 - \frac{1}{2}x_1^2)^2$, where $c_1 = \frac{1}{18}(v_1^2 - 1) \approx -0.026895$. Due to the definition of the domain $D_{5,7}$, this condition implies $c_1(x_2 - \frac{1}{2}x_1^2)^2 \leq x_1^2 x_2$ or, equivalently, $x_2 \leq c_2 x_1^2$, where $c_2 = \frac{1+c_1+\sqrt{1+2c_1}}{2c_1} \approx -36.17491$.

Thus, $\theta_5 = \theta_7$ iff x belongs to the surface

$$M_{5,7} = \{x : \; x_2 \leq c_2 x_1^2, \; x_3 = \tfrac{1}{4}x_1^4 + c_1(x_2 - \tfrac{1}{2}x_1^2)^2\}$$

and for any point $x \in D_{5,7}$ one has $\theta_7 < \theta_5$ iff $x_3 > \frac{1}{4}x_1^4 + c_1(x_2 - \frac{1}{2}x_1^2)^2$.

Cases 6 and 8. The domain where both controls exist is

$$D_{6,8} = \big\{x : \; -\tfrac{17}{2}x_1^2 \leq x_2 \leq -\tfrac{1}{2}x_1^2, \quad x_3 < -\tfrac{1}{4}x_1^4, \; x_3 \leq x_1^2 x_2 + \tfrac{17}{4}x_1^4, \\ x_3 \geq -\tfrac{1}{16}x_2^2 - \tfrac{1}{16}x_1^2 x_2 - \tfrac{17}{64}x_1^4\big\}.$$

Let us compare $\theta_8 = \theta_{8max}$ and θ_6. We use the arguments and notation of case 8 (b1). Namely, let $0 < A_{min}^2 \leq A_{max}^2$ be the roots of the equation $f(z) = 4z^2 + S_{12}z - S_{22} = 0$. Then $\theta_8 \leq \theta_6$ iff

$$\frac{3A_{max}^2 + A_{min}^2}{A_{max}} \leq \frac{\sqrt{(A_{min}^2 + A_{max}^2)^3}}{A_{min}A_{max}} \Leftrightarrow (3A_{max}^2 + A_{min}^2)^2 A_{min}^2 \leq (A_{min}^2 + A_{max}^2)^3,$$

which is equivalent to the inequality $6A_{min}^2 A_{max}^2 + 3A_{min}^4 - A_{max}^4 \leq 0$. Substituting expressions for A_{min}^2 and A_{max}^2 and taking into account that $S_{12} \leq 0$ and $S_{22} \leq 0$, we get that $\theta_8 \leq \theta_6$ iff

$$S_{12}^2 - 16S_{22} \leq -2S_{12}\sqrt{S_{12}^2 + 16S_{22}} \;\Leftrightarrow\; 256S_{22}^2 - 96S_{12}^2 S_{22} - 3S_{12}^4 \leq 0.$$

This condition is equivalent to the inequality $S_{22} \geq k_1 S_{12}^2$ or, what is the same, $x_3 \geq -\frac{1}{4}x_1^4 + k_1(x_2 + \frac{1}{2}x_1^2)^2$, where $k_1 = \frac{1}{16}(3 - 2\sqrt{3}) \approx -0.0290064$. Due to the definition of the domain $D_{6,8}$, this condition implies $k_1(x_2 + \frac{1}{2}x_1^2)^2 \leq x_2 x_1^2 + \frac{9}{2}x_1^4$ or, equivalently, $x_2 \geq k_2 x_1^2$, where $k_2 = \frac{1-k_1+\sqrt{1+16k_1}}{2k_1} = -\frac{1}{2} - \frac{8}{\sqrt{3}} \approx -5.118802$.

Thus, $\theta_8 = \theta_6$ iff x belongs to the surface

$$M_{6,8} = \{x:\ k_2 x_1^2 \leq x_2 < -\tfrac{1}{2}x_1^2,\ x_3 = -\tfrac{1}{4}x_1^4 + k_1(x_2 + \tfrac{1}{2}x_1^2)^2\}$$

and for any point $x \in D_{6,8}$ one has $\theta_8 < \theta_6$ iff $x_3 > -\frac{1}{4}x_1^4 + k_1(x_2 + \frac{1}{2}x_1^2)^2$.

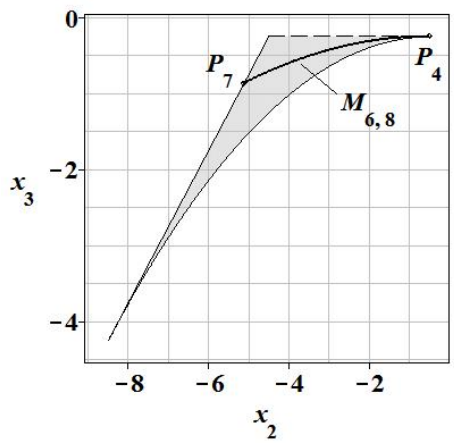

Fig. 24. Intersection of the domain $D_{6,8}$ and the surface $M_{6,8}$ with the plane $x_1 = 1$; $P_7 = (k_2, k_2 + \frac{17}{4}) \approx (-5.119, -0.869)$

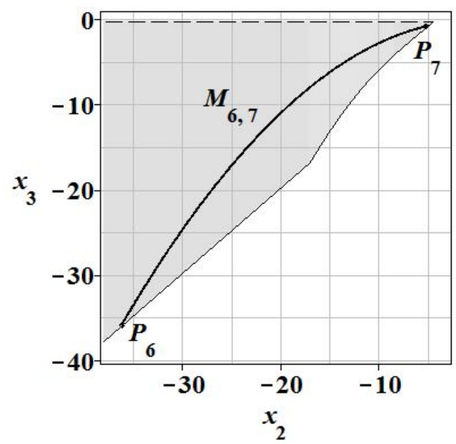

Fig. 25. Intersection of the domain $D_{6,7}$ and the surface $M_{6,7}$ with the plane $x_1 = 1$

Cases 6 and 7. The domain where both controls exist is

$$\begin{aligned} D_{6,7} = \big\{x : x_3 < -\tfrac{1}{4}x_1^4,\quad & x_3 \geq x_1^2 x_2 + \tfrac{1}{4}x_1^4 \ \text{ if }\ x_2 \leq r x_1^2, \\ & x_3 \geq -\tfrac{1}{18}x_2^2 + \tfrac{1}{18}x_1^2 x_2 + \tfrac{17}{72}x_1^4 \ \text{ if } r x_1^2 \leq x_2 \leq -\tfrac{17}{2}x_1^2, \\ & x_3 \geq x_1^2 x_2 + \tfrac{17}{4}x_1^4 \ \text{ if }\ -\tfrac{17}{2}x_1^2 \leq x_2 < -\tfrac{9}{2}x_1^2\big\}, \end{aligned}$$

where $r = -\frac{17}{2} - 6\sqrt{2} \approx -16.98528$ was introduced above. The times of motion θ_6 and θ_7 for cases 6 and 7 can be found by (19), (20) and (23), (24). Let us introduce the function $G = \theta_6 - \theta_7$, i.e.,

$$G(x) = \sqrt{\frac{S_{12}^3}{S_{22}}} - \frac{-6S_{11} + 3\sqrt{S_{11}^2 + 18S_{21}}}{\sqrt{-S_{11} + \sqrt{S_{11}^2 + 18S_{21}}}} + 2x_1, \tag{33}$$

then $\theta_6 = \theta_7$ iff x belongs to the surface

$$\begin{aligned} M_{6,7} = \big\{x :\ & x_1^2 x_2 + \tfrac{1}{4}x_1^4 \leq x_3 < -\tfrac{1}{4}x_1^4 \text{ if }\ c_2 x_1^2 \leq x_2 \leq r x_1^2, \\ & -\tfrac{1}{18}x_2^2 + \tfrac{1}{18}x_1^2 x_2 + \tfrac{17}{72}x_1^4 \leq x_3 < -\tfrac{1}{4}x_1^4 \ \text{ if } r x_1^2 \leq x_2 \leq -\tfrac{17}{2}x_1^2, \\ & x_1^2 x_2 + \tfrac{17}{4}x_1^4 \leq x_3 < -\tfrac{1}{4}x_1^4 \ \text{ if }\ -\tfrac{17}{2}x_1^2 \leq x_2 \leq k_2 x_1^2, \\ & G(x) = 0\big\} \end{aligned}$$

and for any point $x \in D_{6,7}$ one has $\theta_7 < \theta_6$ iff $G(x) > 0$.

Now we study this surface in detail. Let us fix any $x_1 > 0$ and $x_2 < -\frac{9}{2}x_1^2$ and suppose x_3 runs through the segment $[x_{3min}, -\frac{1}{4}x_1^4)$, where x_{3min} is given by the description of the domain $D_{6,7}$. First let us consider the lower bound, $x_3 = x_{3min}$.

(a) If $x_2 < rx_1^2$, then $x_{3min} = x_1^2 x_2 + \frac{1}{4}x_1^4$. For these points using (17), (18) one easily finds $S_{11}^2 + 2S_{21} = (x_2 + \frac{1}{2}x_1^2)^2$, hence, $\theta_5(x_{3min}) = -\frac{x_2}{x_1} + \frac{1}{2}x_1$. On the other hand, $S_{22} = S_{12}x_1^2$, hence, by (19), (20) we get $\theta_6(x_{3min}) = -\frac{x_2}{x_1} + \frac{1}{2}x_1$. Thus, $\theta_5(x_{3min}) = \theta_6(x_{3min})$. Using the results obtained above for the domain $D_{5,7}$, we get

– if $x_2 < c_2x_1^2$, then $\theta_6(x_{3min}) = \theta_5(x_{3min}) > \theta_7(x_{3min})$;

– if $c_2x_1^2 \le x_2 \le rx_1^2$, then $\theta_6(x_{3min}) = \theta_5(x_{3min}) \le \theta_7(x_{3min})$.

(b) If $rx_1^2 < x_2 < -\frac{17}{2}x_1^2$, then $x_{3min} = -\frac{1}{18}x_2^2 + \frac{1}{18}x_1^2x_2 + \frac{17}{72}x_1^4$. As above, we consider points $x_\delta = (x_1, x_2, x_{3,\delta})$, where $x_{3,\delta} = x_{3min} - \delta$ with small $\delta > 0$. For points x_δ the control of case 7 does not exist and the control of case 6 is optimal, i.e., $\widehat{\theta}(x_\delta) = \theta_6(x_{3,\delta})$. Due to continuity of $\widehat{\theta}$ and θ_6, we have

$$\theta_6(x_{3min}) = \lim_{\delta\to 0}\theta_6(x_{3,\delta}) = \lim_{\delta\to 0}\widehat{\theta}(x_\delta) = \widehat{\theta}(x_0), \quad \text{where} \quad x_0 = (x_1, x_2, x_{3min}),$$

therefore, $\theta_6(x_{3min}) \le \theta_7(x_{3min})$.

(c) If $-\frac{17}{2}x_1^2 < x_2 < -\frac{9}{2}x_1^2$, then $x_{3min} = x_1^2x_2 + \frac{17}{4}x_1^4$. For these points $S_{21} = S_{11}x_1^2 + \frac{9}{2}x_1^4$, hence, $S_{11}^2 + 18S_{21} = (S_{11} + 9x_1^2)^2$. Since $S_{11} + 9x_1^2 \ge 0$, using (23), (24) we get $\theta_7(x_{3min}) = -\frac{x_2}{x_1} + \frac{17}{2}x_1$. On the other hand, $S_{22} = S_{12}x_1^2 + 4x_1^4$, therefore, $S_{12}^2 + 16S_{22} = (S_{12} + 8x_1^2)^2$. Since $S_{12} + 8x_1^2 \ge 0$, using (28), (29) we get $\theta_8(x_{3min}) = -\frac{x_2}{x_1} + \frac{17}{2}x_1$. Thus, $\theta_7(x_{3min}) = \theta_8(x_{3min})$. Using the results obtained for the domain $D_{6,8}$, we get

– if $k_2x_1^2 < x_2 < -\frac{9}{2}x_1^2$, then $\theta_7(x_{3min}) = \theta_8(x_{3min}) < \theta_6(x_{3min})$;

– if $x_2 \le k_2x_1^2$, then $\theta_7(x_{3min}) = \theta_8(x_{3min}) \ge \theta_6(x_{3min})$.

Thus, we get the following relations.

$$\begin{aligned} &\text{– If } c_2x_1^2 \le x_2 \le k_2x_1^2, \quad \text{then} \quad \theta_6(x_{3min}) \le \theta_7(x_{3min}). \\ &\text{– If } x_2 < c_2x_1^2 \text{ or } k_2x_1^2 < x_2 < -\tfrac{9}{2}x_1^2, \quad \text{then} \quad \theta_7(x_{3min}) < \theta_6(x_{3min}). \end{aligned} \tag{34}$$

Now let us study $\theta_6(x_3)$ and $\theta_7(x_3)$ as functions of $x_3 \in [x_{3min}, -\frac{1}{4}x_1^4)$. By (19), (20) and (23), (24),

$$\theta_6 = \theta_6(x_3) = \sqrt{\frac{S_{12}^3}{S_{22}}} + x_1, \quad \theta_7 = \theta_7(x_3) = 9A_7 - \frac{S_{11}}{A_7} - x_1,$$

and

$$A_7 = A_7(x_3) = \tfrac{1}{3}\sqrt{-S_{11} + \sqrt{S_{11}^2 + 18S_{21}}}.$$

Since $S_{11} < 0$ and $S_{12} < 0$ are constants while $S_{21} < 0$ and $S_{22} < 0$ are increasing functions of x_3, we see that $\theta_6(x_3)$ and $A_7(x_3)$ increase. However, $9A_7^2 > -S_{11}$, hence, $\theta_7(x_3)$ also increases.

Let us introduce the functions

$$h_6(x_3) = \theta_6(x_3) - \frac{27x_3}{2\sqrt{-S_{11}^3}}, \quad h_3(x_3) = \theta_7(x_3) - \frac{27x_3}{2\sqrt{-S_{11}^3}},$$

and show that $h_6(x_3)$ increases and $h_7(x_3)$ decreases. We have

$$\frac{\partial h_6(x_3)}{\partial x_3} = \frac{\sqrt{-S_{12}^3}}{2\sqrt{-S_{22}^3}} - \frac{27}{2\sqrt{-S_{11}^3}}, \quad \frac{\partial h_7(x_3)}{\partial x_3} = \frac{1}{2A_7^3(x_3)} - \frac{27}{2\sqrt{-S_{11}^3}}.$$

Hence, $\frac{\partial h_7(x_3)}{\partial x_3} \le 0$ iff $A_7 \ge \frac{1}{3}\sqrt{-S_{11}}$ which is obvious. Thus, $h_7(x_3)$ decreases.

For $h_6(x_3)$ we have $\frac{\partial h_6(x_3)}{\partial x_3} \ge 0$ iff $9S_{22} + S_{11}S_{12} \ge 0$ or, what is the same, $x_3 \ge -\frac{1}{9}(x_2^2 + 2x_1^4)$. If $x \in D_{6,7}$, then the inequality $x_3 \ge -\frac{1}{18}x_2^2 + \frac{1}{18}x_1^2x_2 + \frac{17}{72}x_1^4$ holds. Moreover, $-\frac{1}{18}x_2^2 + \frac{1}{18}x_1^2x_2 + \frac{17}{72}x_1^4 \ge -\frac{1}{9}(x_2^2 + 2x_1^4)$ for any x_1, x_2. Therefore, $x_3 \ge -\frac{1}{9}(x_2^2 + 2x_1^4)$ in $D_{6,7}$, hence, $\frac{\partial h_6(x_3)}{\partial x_3} \ge 0$.

Thus, $h_6(x_3)$ increases and $h_7(x_3)$ decreases and, besides, relations (34) imply that

– if $c_2x_1^2 \le x_2 \le k_2x_1^2$, then $h_6(x_{3min}) \le h_7(x_{3min})$,
– if $x_2 < c_2x_1^2$ or $k_2x_1^2 < x_2 < -\frac{9}{2}x_1^2$, then $h_7(x_{3min}) < h_6(x_{3min})$.

Concerning the upper bound, we have $h_6(x_3) \to +\infty$ as $x_3 \to -\frac{1}{4}x_1^4$ while $h_7(-\frac{1}{4}x_1^4) < +\infty$. Therefore, we obtain the following result.

– If $x_2 < c_2x_1^2$ or $k_2x_1^2 < x_2 < -\frac{9}{2}x_1^2$, then $h_7(x_3) < h_6(x_3)$, and therefore, $\theta_7(x_3) < \theta_6(x_3)$ for all $x_3 \in [x_{3min}, -\frac{1}{4}x_1^4)$.

– If $c_2x_1^2 \le x_2 \le k_2x_1^2$, then there exists a unique point $\widetilde{x}_3 \in [x_{3min}, -\frac{1}{4}x_1^4)$ such that $h_6(\widetilde{x}_3) = h_7(\widetilde{x}_3)$ or, equivalently, $\theta_6(\widetilde{x}_3) = \theta_7(\widetilde{x}_3)$.

In other words, if $x \in M_{6,7}$, then $c_2x_1^2 \le x_2 \le k_2x_1^2$. Moreover, the surface $M_{6,7}$ has a unique point of intersection with any vertical line with fixed $x_1 > 0$ and $c_2x_1^2 \le x_2 \le k_2x_1^2$.

4. Time-optimal controls

Combining the results obtained above we formulate the explicit solution of the time-optimal control problem (2). Suppose a point x with $x_1 > 0$ is given. In order to set the point to a certain case, one has to check *all the conditions* from the list corresponding to this case; they are collected in Table 1. The optimal time and the optimal control are found by explicit formulas depending on the case.

Recall that $c_1 = \frac{1}{18}(v_1^2 - 1) \approx -0.026895$, where v_1 is the unique positive root of the equation $91v^4 + 486v^3 + 736v^2 - 584 = 0$, $c_2 = \frac{1+c_1+\sqrt{1+2c_1}}{2c_1} \approx -36.17491$, $k_1 = \frac{1}{16}(3 - 2\sqrt{3}) \approx -0.0290064$, $k_2 = \frac{1-k_1+\sqrt{1+16k_1}}{2k_1} = -\frac{1}{2} - \frac{8}{\sqrt{3}} \approx -5.118802$, and $r = (-\frac{17}{2} - 6\sqrt{2}) \approx -16.98528$; the functions $F(x)$ and $G(x)$ are given by formulas (31) and (33). Fig. 26 shows the intersection of the plane $x_1 = 1$ with domains where controls corresponding to cases 1–8 are optimal.

Case 1: $(1,-1,1)$	$x_2 \geq \frac{7}{2}x_1^2$ and $\frac{1}{2}x_2^2 + \frac{1}{2}x_1^2x_2 - \frac{1}{8}x_1^4 \leq x_3 \leq \frac{5}{6}x_2^2 - \frac{5}{6}x_1^2x_2 + \frac{11}{24}x_1^4$.
Case 2: $(-1,0,1)$	$x_2 \geq -\frac{1}{4}x_1^2$ and $\frac{1}{4}x_1^2x_2 - \frac{5}{32}x_1^4 \leq x_3 \leq \frac{1}{2}x_2^2 + \frac{1}{2}x_1^2x_2 - \frac{1}{8}x_1^4$, if $x_2 \geq \frac{11}{4}x_1^2$ then $x_3 \geq \frac{1}{18}x_2^2 - \frac{1}{18}x_1^2x_2 + \frac{19}{72}x_1^4$ or $F(x) \geq 0$.
Case 3: $(1,-1,0,1)$	if $x_2 \leq -\frac{1}{4}x_1^2$ then $x_3 \leq -\frac{5}{6}x_2^2 + \frac{5}{6}x_1^2x_2 + \frac{1}{24}x_1^4$, if $-\frac{1}{4}x_1^2 \leq x_2 \leq \frac{11}{4}x_1^2$ then $x_3 \leq \frac{1}{4}x_1^2x_2 - \frac{5}{32}x_1^4$, if $x_2 \geq \frac{11}{4}x_1^2$ then $x_3 \leq \frac{1}{18}x_2^2 - \frac{1}{18}x_1^2x_2 + \frac{19}{72}x_1^4$ and $F(x) \leq 0$.
Case 4: $(1,-1,1)$	$x_2 \leq -\frac{1}{4}x_1^2$ and $x_3 \geq -\frac{5}{6}x_2^2 + \frac{5}{6}x_1^2x_2 + \frac{1}{24}x_1^4$, if $-\frac{1}{2}x_1^2 \leq x_2 \leq -\frac{1}{4}x_1^2$ then $x_3 \leq \frac{1}{2}x_2^2 + \frac{1}{2}x_1^2x_2 - \frac{1}{8}x_1^4$, if $x_2 \leq -\frac{1}{2}x_1^2$ then $x_3 \leq -\frac{1}{2}x_2^2 + \frac{1}{2}x_1^2x_2 + \frac{1}{8}x_1^4$.
Case 5: $(1,0,-1)$	$x_2 \leq -\frac{1}{2}x_1^2$ and $-\frac{1}{2}x_2^2 + \frac{1}{2}x_1^2x_2 + \frac{1}{8}x_1^4 \leq x_3 \leq x_1^2x_2 + \frac{1}{4}x_1^4$, if $x_2 \leq c_2x_1^2$ then $x_3 \leq \frac{1}{4}x_1^4 + c_1(x_2 - \frac{1}{2}x_1^2)^2$.
Case 6: $(-1,0,-1)$	$c_2x_1^2 \leq x_2 < -\frac{1}{2}x_1^2$ and $x_3 \geq x_1^2x_2 + \frac{1}{4}x_1^4$, if $c_2x_1^2 \leq x_2 \leq rx_1^2$ then $G(x) \leq 0$, if $rx_1^2 \leq x_2 \leq -\frac{17}{2}x_1^2$ then $x_3 \leq -\frac{1}{18}x_2^2 + \frac{1}{18}x_1^2x_2 + \frac{17}{72}x_1^4$ or $G(x) \leq 0$, if $-\frac{17}{2}x_1^2 \leq x_2 \leq k_2x_1^2$ then $x_3 \leq x_1^2x_2 + \frac{17}{4}x_1^4$ or $G(x) \leq 0$, if $k_2x_1^2 \leq x_2 < -\frac{1}{2}x_1^2$ then $x_3 \leq -\frac{1}{4}x_1^4 + k_1(x_2 + \frac{1}{2}x_1^2)^2$.
Case 7: $(1,0,-1,1)$	if $x_2 \leq c_2x_1^2$ then $x_3 \geq \frac{1}{4}x_1^4 + c_1(x_2 - \frac{1}{2}x_1^2)^2$, if $c_2x_1^2 \leq x_2 \leq rx_1^2$ then $x_3 \geq -\frac{1}{4}x_1^4$ or $x_3 \geq x_2x_1^2 + \frac{1}{4}x_1^4$ and $G(x) \geq 0$, if $rx_1^2 \leq x_2 \leq -\frac{17}{2}x_1^2$ then $x_3 \geq -\frac{1}{4}x_1^4$ or $x_3 \geq -\frac{1}{18}x_2^2 + \frac{1}{18}x_1^2x_2 + \frac{17}{72}x_1^4$ and $G(x) \geq 0$, if $-\frac{17}{2}x_1^2 \leq x_2 \leq k_2x_1^2$ then $x_3 \geq -\frac{1}{4}x_1^4$ or $x_3 \geq x_1^2x_2 + \frac{17}{4}x_1^4$ and $G(x) \geq 0$, if $k_2x_1^2 \leq x_2 \leq \frac{7}{2}x_1^2$ then $x_3 \geq x_1^2x_2 + \frac{17}{4}x_1^4$, if $x_2 \geq \frac{7}{2}x_1^2$ then $x_3 \geq \frac{5}{6}x_2^2 - \frac{5}{6}x_1^2x_2 + \frac{11}{24}x_1^4$.
Case 8: $(-1,0,-1,1)$	$k_2x_1^2 \leq x_2 \leq \frac{7}{2}x_1^2$ and $x_3 \leq x_1^2x_2 + \frac{17}{4}x_1^4$, if $k_2x_1^2 \leq x_2 \leq -\frac{1}{2}x_1^2$ then $x_3 \geq -\frac{1}{4}x_1^4 + k_1(x_2 + \frac{1}{2}x_1^2)^2$, if $-\frac{1}{2}x_1^2 \leq x_2 \leq \frac{7}{2}x_1^2$ then $x_3 \geq \frac{1}{2}x_2^2 + \frac{1}{2}x_1^2x_2 - \frac{1}{8}x_1^4$.

Table 1. Description of optimal controls for points with $x_1 > 0$

Also, we obtain the solution of the optimal synthesis problem, i.e., describe the optimal control as a function on x. To this end, we take into account that the controls of cases 1, 3, 4, 5, 7 begin with $+1$ and the controls of cases 2, 6, 8 begin with -1. The value 0 corresponds to limit cases (between cases 5 and 6, between cases 7 and 8). There exist surfaces for which both values $+1$ and -1 are possible; they are described by the equations $F(x) = 0$ and $G(x) = 0$ (between cases 2 and 3, between cases 6 and 7). Fig. 27 shows the solution of the optimal synthesis problem, namely, the intersection of the plane $x_1 = 1$ with the domains in which the optimal control as a function of x equals $+1$ or -1. The intersection with surfaces corresponding to the value 0 are drawn by dotted lines; the intersection with surfaces where both values $+1$ and -1 are possible are drawn by bold lines.

Let us show that the rest part of the border (drawn by thin lines) corresponds to the value -1. In fact, the upper thin curve separates cases 1 and 2 and the lower thin curve consists of two segments: one segment separates cases 8 and 4 and the second segment separates cases 2 and 3. At all these points $A = x_1$ where A corresponds to cases 1, 4, and 3 respectively, hence, at these points $u = -1$.

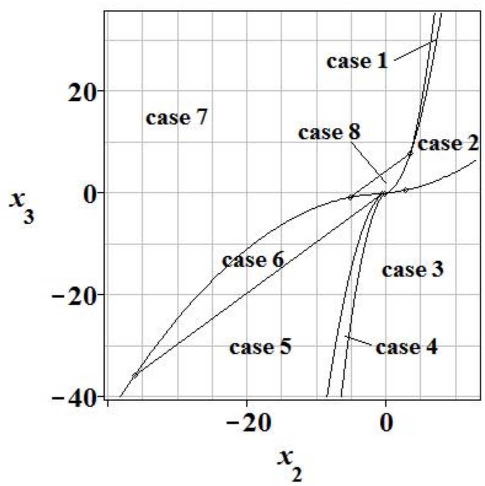

Fig. 26. Optimal controls on the plane $x_1 = 1$

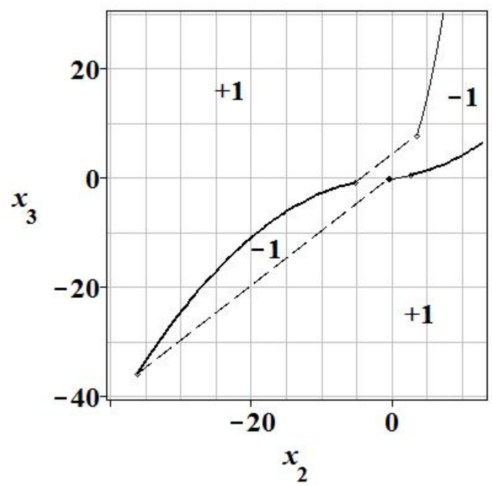

Fig. 27. Optimal synthesis on the plane $x_1 = 1$

For the points with $x_1 < 0$ we use the symmetry arguments. Namely, let us solve the time-optimal control problem for the point $-x$; suppose $\widehat{u}(t,-x)$ is the optimal control and $\widehat{\theta}(-x)$ is the optimal time. Then the optimal control and the optimal time for the initial point equal $\widehat{u}(t,x) = -\widehat{u}(t,-x)$ and $\widehat{\theta}(x) = \widehat{\theta}(-x)$.

Finally, let us find optimal controls for points with $x_1 = 0$. In this case the analysis of possible types of control is shorter since cases 6 and 8 are impossible. Since $x_1 = 0$, controls of cases 1, 3, 4 and 7 can be chosen in two forms; as an example, two forms of the control of case 3 are shown in Fig. 28.

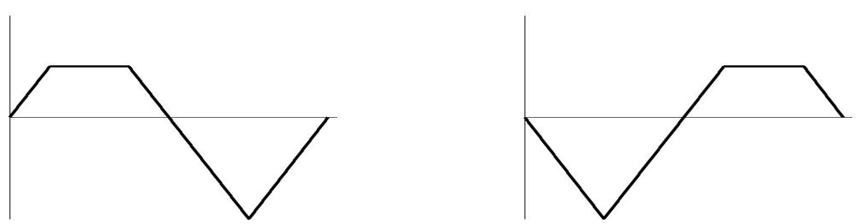

Fig. 28. Graph of $x_1(t)$ for two variants of the optimal control of case 3

Moreover, domains corresponding to cases 1 and 4 are symmetric to each other; the same holds for cases 2 and 5 and for cases 3 and 7. We notice that, from the point of view of the synthesis problem, in these cases the both values $+1$ and -1 are possible.

Arguing analogously to the previous sections, one can find the domains in which controls corresponding to these cases exist, and analyze the overlapping domains. We give the final answer only, see Table 2, Fig. 29 and Fig. 30.

Case 1:	$x_2 \geq 0$ and $\frac{1}{2}x_2^2 \leq x_3 \leq \frac{5}{6}x_2^2$.
Case 2:	$x_2 \geq 0$ and $-c_1x_2^2 \leq x_3 \leq \frac{1}{2}x_2^2$,
Case 3:	if $x_2 \geq 0$ then $x_3 \leq -c_1x_2^2$, if $x_2 \leq 0$ then $x_3 \leq -\frac{5}{6}x_2^2$.
Case 4:	$x_2 \leq 0$ and $-\frac{5}{6}x_2^2 \leq x_3 \leq -\frac{1}{2}x_2^2$.
Case 5:	$x_2 \leq 0$ and $-\frac{1}{2}x_2^2 \leq x_3 \leq c_1x_2^2$.
Case 7:	if $x_2 \leq 0$ then $x_3 \geq c_1x_2^2$, if $x_2 \geq 0$ then $x_3 \geq \frac{5}{6}x_2^2$.

Table 2. Description of optimal controls for points with $x_1 = 0$

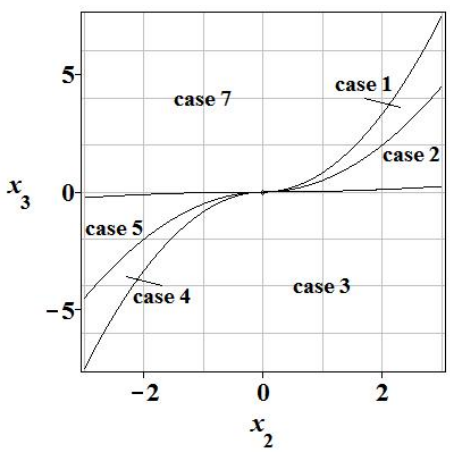

Fig. 29. Optimal controls on the plane $x_1 = 0$

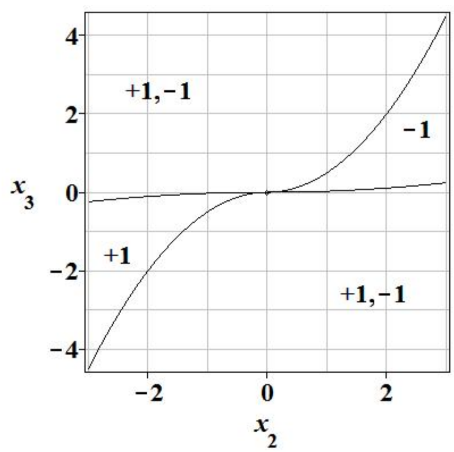

Fig. 30. Optimal synthesis on the plane $x_1 = 0$

Example. As was shown above, for some points there exist two different optimal controls. As an example, let us consider the point x with $x_1 = 1$ and $x_2 = -8$, then $-\frac{17}{2}x_1^2 \leq x_2 \leq k_2x_1^2$ (recall that $k_2 \approx -5.12$). Let us find x_3 so that $\theta_6 = \theta_7$. To this end we solve the equation $G(x) = G(1, -8, x_3) = 0$ on the interval $x_3 \in [x_1^2x_2 + \frac{17}{4}x_1^4, -\frac{1}{4}x_1^4) = [-\frac{15}{4}, -\frac{1}{4})$ and get $x_3 \approx -1.879$. For this point both controls of cases 6 and 7 are optimal. Fig. 31 and 32 show the components of the optimal trajectories corresponding to these optimal controls; the time of motion equals $\theta_6 = \theta_7 \approx 17.092$.

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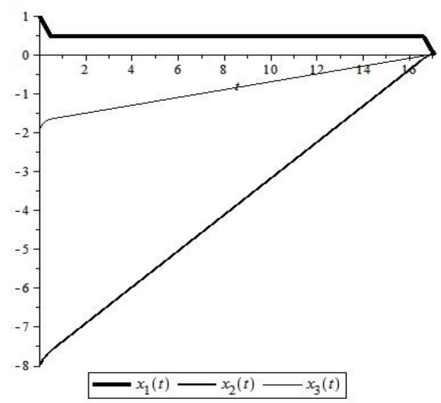

Fig. 31. Components of the optimal trajectory for the point $x = (1, -8, -1.879)$, case 6

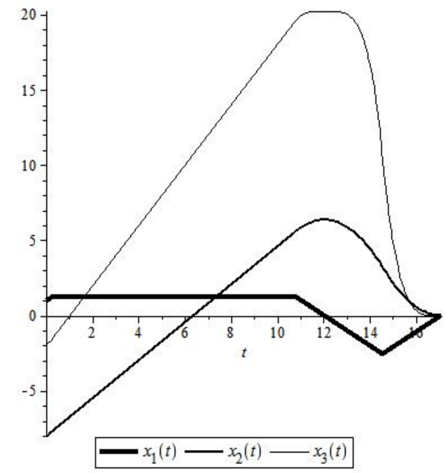

Fig. 32. Components of the optimal trajectory for the point $x = (1, -8, -1.879)$, case 7

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