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Plasticity of the unit ball of ℓ_1

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In the recent paper by Cascales, Kadets, Orihuela and Wingler it is shown that for every strictly convex Banach space X every non-expansive bijection $F : B_X \to B_X$ is an isometry. We extend this result to the space ℓ_1, which is not strictly convex.
Keywords: non-expansive map; unit ball; plastic space.

Кадець В. М., Заварзіна О. О. **Пластичність одиничної кулі простору ℓ_1.** У нещодавній статті Каскалеса, Кадеця, Оріуели та Вінглера показано, що у будь-якому строго опуклому банаховому просторі X кожна нерозтягуюча бієкція $F : B_X \to B_X$ є ізометрією. Ми розповсюджуємо отриманий результат на простір ℓ_1, який не є строго опуклим.
Ключові слова: нерозтягуюче відображення; одинична куля; пластичний простір.

Кадец В. М., Заварзина А. О. **Пластичность единичного шара пространства ℓ_1.** В недавней статье Каскалеса, Кадеца, Ориуэлы и Винглера показано, что в любом строго выпуклом банаховом пространстве X каждая нерастягивающая биекция $F : B_X \to B_X$ является изометрией. Мы распространяем полученный результат на пространство ℓ_1, не являющееся строго выпуклым.
Ключевые слова: нерастягивающее отображение; единичный шар; пластичное пространство.
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Introduction

Let E be a metric space. A map $F : E \to E$ is called non-expansive, if $\rho(F(x), F(y)) \le \rho(x, y)$ for all $x, y \in E$. The space E is called *Expand-Contract plastic* (or simply, an EC-space) if every non-expansive bijection from E onto itself is an isometry. A metric space is called totally bounded, if for every $\varepsilon > 0$ it possesses a finite ε-net.

Satz IV of [3] or Theorem 1.1 of [6] imply that every totally bounded metric space is an EC-space, but there are also examples of EC-spaces that are not totally bounded. According to [2, Theorem 2.6], the unit ball of every strictly convex Banach space is an EC-space, so in particular the closed unit ball of an infinite-dimensional Hilbert space is an example of not totally bounded EC-space. On the other hand, there are bounded closed convex sets in an infinite-dimensional Hilbert space that are not EC-spaces [2, Example 2.7].

It is a challenging question whether unit balls of all Banach spaces are EC-spaces. The question is not easy, and a possible approach to it consists in checking what happens with Expand-Contract plasticity of unit balls in classical spaces that are not strictly convex. The list of such spaces includes $C(K)$, $L_1(\Omega, \Sigma, \mu)$, ℓ_1, c_0 and many others. In this short note we do just one step in this direction. Namely, we demonstrate the EC-plasticity of the unit ball of ℓ_1.

Below, the letters X, Y are used to denote Banach spaces, and we consider only real Banach spaces. For a Banach space X we denote by S_X and B_X the unit sphere and the closed unit ball of X respectively. For a convex set $A \subset X$ denote by $\text{ext}(A)$ the set of extreme points of A; that is, $x \in \text{ext}(A)$ if $x \in A$ and for every $y \in X \setminus \{0\}$ either $x + y \notin A$ or $x - y \notin A$. A Banach space X is called strictly convex if all elements of S_X are extreme points of B_X, or in other words, S_X does not contain non-trivial line segments.

Recall also, that ℓ_1 is the space of those sequences $x = (x_1, x_2, \ldots)$ of reals which satisfy the condition $\sum_{n=1}^{\infty} |x_n| < \infty$. This space is equipped with the norm $\|x\| = \sum_{n=1}^{\infty} |x_n|$.

We conclude the introduction by listing four known results that we will use in our proof. The first one is a part of [2, Theorem 2.3].

Proposition 1 *Let $F : B_X \to B_X$ be a non-expansive bijection. Then, the following holds true.*

1. $F(0) = 0$.
2. $F^{-1}(S_X) \subset S_X$.
3. *If $F(x)$ is an extreme point of the unit ball, then $F(ax) = aF(x)$ for all $a \in (0, 1)$.*
4. *If $F(x)$ is an extreme point of B_X, then x is also an extreme point of B_X.*
5. *If $F(x)$ is an extreme point of the unit ball, then $F(-x) = -F(x)$.*

We will need also the following result by P. Mankiewicz [5].

Proposition 2 *If $A \subset X$ and $B \subset Y$ are convex with non-empty interior, then every bijective isometry $F : A \to B$ can be extended to a bijective affine isometry $\tilde{F} : X \to Y$.*

Taking into account that in the case of A, B being the unit balls every isometry maps 0 to 0, this result implies that every bijective isometry $F : B_X \to B_Y$ is the restriction of a linear isometry from X onto Y.

Another ingredient of our proof will be the Brower invariance of domain principle [1] (see also the excellent exposition written by Terry Tao in his blog https://terrytao.wordpress.com/2011/06/13/brouwers-fixed-point-and-invariance-of-domain-theorems-and-hilberts-fifth-problem/ of the less involved proof by W. Kulpa [4]).

Proposition 3 *Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ be an injective continuous map, then $f(U)$ is open in $\mathbb{R}^n$.*

The next easy proposition is surely not new, but we were not able to find it in the literature. That is why we present it here with a sketch of the proof.

Proposition 4 *Let X be a finite-dimensional normed space and V be a subset of B_X with the following two properties: V is homeomorphic to B_X and $V \supset S_X$. Then $V = B_X$.*

Proof. Recall, that a topological space E has the fixed-point property (FPP for short), if every continuous map $f : E \to E$ has a fixed point. According to Brouwer's fixed point theorem, B_X has the FPP, so V also has the FPP. Now let us argue "ad absurdum". Assume that $V \neq B_X$. Then there is a point $x_0 \in B_X \setminus V$. For every point $x \in V$ consider the semiaxis $L_x = \{x_0 + tx : t \in [0, +\infty)\}$ and denote $P(x)$ the point where L_x intersects S_X. Then P is a continuous retraction from V onto S_X, so S_X is a retract of V. This leads to contradiction, because a retract of a set with FPP must also have the FPP, but S_X does not have the FPP (just consider the map $x \mapsto -x$).

The main result

Theorem 1 *The unit ball of ℓ_1 is an EC-space.*

Proof. Denote U the closed unit ball of ℓ_1, and let $e_n = (\delta_{i,n})_{i \in \mathbb{N}}$, $n = 1, 2, \ldots$ be the elements of the canonic basis of ℓ_1 (here, as usual, $\delta_{i,n} = 0$ for $n \neq i$ and $\delta_{n,n} = 1$). It is well-known and easy to check that $ext(U) = \{\pm e_n, i = 1, 2, ...\}$. Now consider a non-expansive bijection $F : U \to U$. Our goal is to demonstrate that F is an isometry.

Denote $g_n = F^{-1} e_n$. According to item (4) of Proposition 1 g_n is an extreme point of U, so it is of the form $\theta_n e_{m(n)}$, $\theta_n = \pm 1$. Moreover, by item (5) of the same Proposition 1, $m(n_1) \neq m(n_2)$ for $n_1 \neq n_2$. This means that the sequence (g_n) is equivalent to the canonic basis of ℓ_1 in the following usual sense: for every $(a_k) \in \ell_1$

$$\left\| \sum_{n \in \mathbb{N}} a_n g_n \right\| = \sum_{n \in \mathbb{N}} |a_n|.$$

One more notation: for every $N \in \mathbb{N}$ and $X_N = \operatorname{lin}\{g_k\}_{k\le N}$ denote U_N and ∂U_N the unit ball and the unit sphere of X_N respectively, i.e.

$$U_N = \left\{ \sum_{n\le N} a_n g_n : \sum_{n\le N} |a_n| \le 1 \right\}, \partial U_N = \left\{ \sum_{n\le N} a_n g_n : \sum_{n\le N} |a_n| = 1 \right\},$$

and analogously for $Y_N = \operatorname{lin}\{e_k\}_{k\le N}$ denote V_N and ∂V_N the unit ball and the unit sphere of Y_N respectively.

Claim. *For every $N \in \mathbb{N}$ and every collection $\{a_k\}_{k\le N}$ of reals with $\|\sum_{n\le N} a_n g_n\| \le 1$*

$$F\left(\sum_{n\le N} a_n g_n\right) = \sum_{n\le N} a_n e_n.$$

Proof of the Claim. We will use the induction in N. If $N = 1$, the Claim follows from items (3) and (5) of Proposition 1. Now assume the validity of the Claim for $N-1$, and let us prove it for N. At first, let us prove that

$$F(U_N) \subset V_N. \tag{1}$$

To this end, consider $x \in U_N$. If x is of the form αg_N the statement follows from Proposition 1. So we must consider $x = \sum_{i=1}^N \alpha_i g_i$, $\sum_{i=1}^N |\alpha_i| \le 1$ with $\sum_{i=1}^{N-1} |\alpha_i| \ne 0$. Denote the expansion of $F(x)$ by $F(x) = \sum_{i=1}^\infty y_i e_i$. For the element

$$x_1 = \frac{\sum_{i=1}^{N-1} \alpha_i g_i}{\sum_{i=1}^{N-1} |\alpha_i|}$$

by the induction hypothesis

$$F(x_1) = \frac{\sum_{i=1}^{N-1} \alpha_i e_i}{\sum_{i=1}^{N-1} |\alpha_i|},$$

so we may write the following inequalities:

$$\begin{aligned}
2 &= \left\| F(x_1) - \frac{\alpha_N}{|\alpha_N|} e_N \right\| \le \left\| F(x_1) - \sum_{i=1}^N y_i e_i \right\| + \left\| \sum_{i=1}^N y_i e_i - \frac{\alpha_N}{|\alpha_N|} e_N \right\| \\
&= \|F(x_1) - F(x)\| + \left\| F(x) - \frac{\alpha_N}{|\alpha_N|} e_N \right\| - 2\sum_{i=N+1}^\infty |y_i| \\
&\le \|F(x_1) - F(x)\| + \left\| F(x) - F\left(\frac{\alpha_N}{|\alpha_N|} g_N\right) \right\| \le \|x_1 - x\| + \left\| x - \frac{\alpha_N}{|\alpha_N|} g_N \right\| \\
&= \sum_{j=1}^{N-1} \left| \alpha_j - \frac{\alpha_j}{\sum_{i=1}^{N-1} |\alpha_i|} \right| + |\alpha_N| + \sum_{j=1}^{N-1} |\alpha_j| + \left| \alpha_N - \frac{\alpha_N}{|\alpha_N|} \right| \\
&= \sum_{j=1}^{N-1} |\alpha_j| \left(1 + \left| 1 - \frac{1}{\sum_{i=1}^{N-1} |\alpha_i|} \right| \right) + |\alpha_N| \left(1 + \left| 1 - \frac{1}{|\alpha_N|} \right| \right) = 2.
\end{aligned}$$

This means that all the inequalities in between are in fact equalities, so in particular $\sum_{i=N+1}^{\infty}|y_i| = 0$, i.e. $F(x) = \sum_{i=1}^{N} y_i e_i \in V_N$ and (1) is proved.

Now, let us demonstrate that

$$F(U_N) \supset \partial V_N. \tag{2}$$

Assume contrary, that there is a $y \in \partial V_N \setminus F(U_N)$. Denote $x = F^{-1}(y)$. Then, $\|x\| = 1$ (by (2) of Proposition 1) and $x \notin U_N$. For every $t \in [0,1]$ consider $F(tx)$. Let $F(tx) = \sum_{n\in\mathbb{N}} b_n e_n$ be the corresponding expansion. Then,

$$\begin{aligned} 1 &= \|0 - tx\| + \|tx - x\| \geq \|0 - F(tx)\| + \|F(tx) - y\| \\ &= 2\sum_{n>N}|b_n| + \left\|\sum_{n\leq N} b_n e_n\right\| + \left\|y - \sum_{n\leq N} b_n e_n\right\| \geq 2\sum_{n>N}|b_n| + 1, \end{aligned}$$

so $\sum_{n>N}|b_n| = 0$. This means that $F(tx) \in V_N$ for every $t \in [0,1]$. On the other hand, $F(U_N)$ contains a relative neighborhood of 0 in V_N (here we use that $F(0) = 0$ and Proposition 3), so the continuous curve $\{F(tx) : t \in [0,1]\}$ in V_N which connects 0 and y has a non-trivial intersection with $F(U_N)$. This implies that there is a $t \in (0,1)$ such that $F(tx) \in F(U_N)$. Since $tx \notin U_N$ this contradicts the injectivity of F. Inclusion (2) is proved.

Now, inclusions (1) and (2) together with Proposition 4 imply $F(U_N) = V_N$. Remark, that U_N is isometric to V_N and, by finite dimensionality, U_N and V_N are compacts. So, U_N and V_N can be considered as two copies of one the same compact metric space, and Theorem 1.1 of [6] (which we mentioned in the beginning of the Introduction) implies that every bijective non-expansive map from U_N onto V_N is an isometry. In particular, F maps U_N onto V_N isometrically. Finally, the application of Proposition 2 gives us that the restriction of F to U_N extends to a linear map from X_N to Y_N, *which completes the proof of the Claim.*

The remaining part of the proof is easy. The continuity of F and the claim imply that for every $x = (x_k)_{k\in\mathbb{N}} \in U$

$$F\left(\sum_{n=1}^{\infty} x_n g_n\right) = \sum_{n=1}^{\infty} x_n e_n = x.$$

Consequently, $\|x\| = \sum_{n=1}^{\infty}|x_n| = \|\sum_{n=1}^{\infty} x_n g_n\| = \left\|F^{-1}(x)\right\|$.

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REFERENCES

1. Brouwer L.E.J. Beweis der Invarianz des n-dimensionalen Gebiets // Mathematische Annalen, 1912. – **71**. – P. 305–315.

2. Cascales B., Kadets V., Orihuela J., Wingler E.J. Plasticity of the unit ball of a strictly convex Banach space // Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas. – 2016. – **110(2)**. – P. 723–727.

3. Freudenthal H., Hurewicz W. Dehnungen, Verkürzungen, Isometrien // Fund. Math., 1936. – **26**. – P. 120–122.

4. Kulpa, W. Poincaré and domain invariance theorem // Acta Univ. Carolin. Math. Phys., 1998. – **39** . – no. 1-2, – P. 127–136.

5. Mankiewicz P. On extension of isometries in normed linear spaces // Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys., 1972. – **20**. – P. 367 –371.

6. Naimpally S.A., Piotrowski Z., Wingler E.J. Plasticity in metric spaces // J. Math. Anal. Appl., 2006 . – **313**. – P. 38–48.

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