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Global synthesis of bounded controls for systems with power nonlinearity

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In this work we consider the problem of global bounded control synthesis for a nonlinear system with uncontrollable first approximation. A class of bounded controls that steer the system from any initial state to the origin in some finite time is constructed based on the controllability function method.
Key words: synthesis problem, finite-time stabilization, nonlinear systems, controllability function method.

Бебія М. О., **Глобальний синтез обмежених керувань для систем зі степеневою нелінійністю.** У роботі розглядається задача глобального синтезу обмежених керувань для нелінійної некерованої за першим наближенням системи. На основі методу функції керованості побудовано клас обмежених керувань, які переводять систему із довільного начального стану у початок координат за скінченний час.
Ключові слова: задача синтезу, стабілізація за скінченний час, нелінійні системи, метод функції керованості.

Бебия М. О., **Глобальный синтез ограниченных управлений для систем со степенной нелинейностью.** В работе рассматривается задача глобального синтеза ограниченных управлений для нелинейной неуправляемой по первому приближению системы. На основе метода функции управляемости построен класс ограниченных управлений, которые переводят систему из произвольного начального состояния в ноль за конечное время.
Ключевые слова: задача синтеза, стабилизация за конечное время, нелинейные системы, метод функции управляемости.

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1. Introduction

The problem of control design for nonlinear systems has been paid much attention in recent years [1]—[12]. In the present paper we consider a class of nonlinear systems with uncontrollable first approximation. Such systems play important role in control theory since most actual dynamical systems are inherently nonlinear.

Consider the following nonlinear system

$$\begin{cases} \dot{x}_1 = u, \quad |u| \le d, \\ \dot{x}_i = c_{i-1}x_{i-1}, \quad i = 2, \dots, n-1, \\ \dot{x}_n = c_{n-1}x_{n-1}^{2k+1}, \end{cases} \tag{1}$$

where $k = \frac{p}{q}$, $p > 0$ is an integer, $q > 0$ is an odd integer, $u \in \mathbb{R}$ is a control, c_i, $i = 1, \dots, n-1$ are real numbers such that $\prod\limits_{i=1}^{n-1} c_i \neq 0$, $d > 0$ is a given number.

System (1) is not stabilizable with respect to the first approximation. The stabilization problem for system (1) with $c_i = 1$, $i = 1, \dots, n-1$, and $k \in \mathbb{N}$ was solved in [4]. In the present paper we consider the problem of global synthesis of bounded controls for system (1). For the sake of brevity this problem will be referred as the global synthesis problem.

The global synthesis problem for system (1) is to find a control $u = u(x)$ such that

(i) for every $x_0 \in \mathbb{R}^n$ there exists a number $T(x_0) < +\infty$ such that $\lim\limits_{t \to T(x_0)} x(t, x_0) = 0$, where $x(t, x_0)$ is a solution of system (1) with $u = u(x)$ that satisfies the condition $x(0, x_0) = x_0$;

(ii) the control $u(x)$ satisfies the restriction $|u(x)| \le d$ for all $x \in \mathbb{R}^n$.

The control law construction is based on the controllability function method, which was proposed by V.I. Korobov [2] for a nonlinear system of the form

$$\dot{x} = \varphi(t, x, u), \quad x \in \mathbb{R}^n, \quad u \in \Omega \subset \mathbb{R}^r, \quad 0 \in int\,\Omega, \tag{2}$$

where $\varphi(t, 0, 0) = 0$ for all $t \ge 0$.

Consider the case $\frac{\partial \varphi(t,x,u)}{\partial t} \equiv 0$ for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}$. The main idea of the controllability function method is to find a function $\Theta(x)$ ($\Theta(x) > 0$ for $x \neq 0$, $\Theta(0) = 0$) and a control $u = u(x)$ such that the following inequality holds

$$\sum_{i=1}^{n} \frac{\partial \Theta(x)}{\partial x_i} \varphi_i(x, u(x)) \le -\beta \Theta^{1-\frac{1}{\alpha}}(x), \quad \beta > 0, \quad \alpha > 0. \tag{3}$$

Denote by $x(t, x_0)$ the solution of the closed-loop system $\dot{x} = \varphi(t, x, u(x))$ that satisfies the condition $x(0, x_0) = x_0$. The last inequality ensures that the trajectory of the closed-loop system steers any initial point $x_0 \in \mathbb{R}^n$ to the origin in some finite $T(x_0)$ [1] and $x(t, x_0) = 0$ for all $t \ge T(x_0)$. Moreover, the time of motion satisfies the estimate $T(x_0) \le \frac{\alpha}{\beta} \Theta^{\frac{1}{\alpha}}(x_0)$.

It is important to note that inequality (3) guaranties that the origin is stable. In this case the control $u = u(x)$ is often called a finite-time stabilizing control; and the origin is said to be a finite-time stable equilibrium [10] of system (2) with $u = u(x)$.

The paper is organized as follows. In Section 2 we consider the case $c_i = 1$, $i = 1, \dots, n-1$. Namely, we construct a class of controls $u = u(x)$ that solve the global synthesis problem for system (1). We also show that these controls satisfy the condition $|u(x)| \le d$. In Section 3 we consider the case $\prod\limits_{i=1}^{n-1} c_i \ne 0$. Finally, the example is given to illustrate the implementability of the approach proposed.

2. Control law construction for systems with power nonlinearity

Consider the global synthesis problem for system (1) in the case $c_i = 1$, $i = 1, \dots, n-1$. In this case system (1) takes the form

$$\begin{cases} \dot{x}_1 = u, \quad |u| \le d, \\ \dot{x}_i = x_{i-1}, \quad i = 2, \dots, n-1, \\ \dot{x}_n = x_{n-1}^{2k+1}, \end{cases} \tag{4}$$

where $k = \frac{p}{q}$, $p > 0$ is an integer, $q > 0$ is an odd integer.

In this section we construct a controllability function and a class of bounded controls that solve the global synthesis problem for system (4).

Let us introduce the following diagonal matrices

$$D(\Theta) = \operatorname{diag}\left(\Theta^{m-1}, \Theta^{m-2}, \dots, \Theta^{m-n+1}, 1\right),$$

$$H = \operatorname{diag}\left(m-1, m-2, \dots, m-n+1, 0\right),$$

where $m = 2k(n-1) + n$.

Let $a_0 > 0$ be a fixed numbed. Suppose that F is a positive definite matrix such that the matrix $F^1 = F - FH - HF$ is positive definite. The additional conditions on a_0 and F will be obtained later.

We define the controllability function $\Theta(x)$, for $x \ne 0$, as a unique positive solution of the equation

$$2a_0\Theta^{2m} = (FD(\Theta)x, D(\Theta)x). \tag{5}$$

We remark that equation (5) has a unique positive solution, for every fixed $x \ne 0$, if the matrix F^1 is positive definite. Moreover, the function $\Theta(x)$ is continuously differentiable at every point $x \ne 0$. We complete the definition of $\Theta(x)$ by putting $\Theta(0) = 0$. Thus $\Theta(x)$ satisfies the following equality

$$2a_0\Theta^{2m}(x) = (FD(\Theta(x))x, D(\Theta(x))x). \tag{6}$$

Consider the following control law

$$u(x) = \frac{1}{\Theta^m(x)}(a, D(\Theta(x))x) + a_{n+1}\frac{x_{n-1}^{2k+1}}{\Theta^{m-1}(x)}, \tag{7}$$

where $a = (a_1, a_2, \ldots, a_n)^* \in \mathbb{R}^n$. The numbers $a_i < 0$, $i = 1, \ldots, n+1$ are to be chosen later.

We use the following notation

$$A = \begin{pmatrix} a_1 & a_2 & \ldots & a_{n-2} & a_{n-1} & a_n \\ 1 & 0 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 \end{pmatrix}, \quad h_n = \begin{pmatrix} a_{n+1} \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{8}$$

Assume that the control $u = u(x)$ of the form (7) is applied to system (4). Calculating the derivative of $\Theta(x)$ along trajectories of the closed-loop system (4), from (6) we obtain

$$\begin{aligned} \dot{\Theta}(x)\Big|_{(4)} &= \frac{\big((A^*F + FA)y(\Theta(x), x), y(\Theta(x), x)\big)}{\big((2mF - FH - HF)y(\Theta(x), x), y(\Theta(x), x)\big)} \\ &+ \frac{2\big(Fh_n, y(\Theta(x), x)\big)x_{n-1}^{2k+1}\Theta(x)}{\big((2mF - FH - HF)y(\Theta(x), x), y(\Theta(x), x)\big)}, \end{aligned} \tag{9}$$

where $y(\Theta(x), x) = D(\Theta(x))x$.

We note that since the matrix A is singular, it is impossible to choose a positive definite matrix F so that the matrix $A^*F + FA$ is negative definite. So we choose the positive definite matrix F so that the matrix $A^*F + FA$ is positive semi-definite. To this end, we consider the following Lyapunov matrix equation

$$A^*F + FA = -W, \tag{10}$$

where $W = \{w_{i,j}\}_{i,j=1}^n$ ($w_{ij} = w_{ji}$, $i \neq j$) is some positive semi-definite matrix, F is an unknown matrix.

Let us introduce the following real symmetric matrix

$$W_{n-1} = \begin{pmatrix} w_{11} & \cdots & w_{1n-1} \\ \cdots & \cdots & \cdots \\ w_{1n-1} & \cdots & w_{n-1n-1} \end{pmatrix}. \tag{11}$$

Consider the case of the positive definite matrix W_{n-1}. In [4, theorem 1] it was proved that the matrix equation (10) is solvable in the class of all positive definite matrices F if and only if the matrix W has the form

$$W = \begin{pmatrix} w_{11} & \cdots & w_{1n-1} & w_{1n-1}\frac{a_n}{a_{n-1}} \\ \cdots & \cdots & \cdots & \cdots \\ w_{1n-1} & \cdots & w_{n-1n-1} & w_{n-1n-1}\frac{a_n}{a_{n-1}} \\ w_{1n-1}\frac{a_n}{a_{n-1}} & \cdots & w_{n-1n-1}\frac{a_n}{a_{n-1}} & w_{n-1n-1}\frac{a_n^2}{a_{n-1}^2} \end{pmatrix}. \tag{12}$$

Further we need the following lemma, which was proved in [4, p. 77].

Lemma 1. *The matrix W given by (12) is positive semi-definite if and only if the matrix W_{n-1} given by (11) is positive semi-definite.*

The following theorem describes the class of positive definite solutions of matrix equation (10).

Theorem 1. *Suppose that the matrices A and W are defined by (8) and (12) respectively. Furthermore, suppose that the matrix W_{n-1} defined by (11) is positive definite, and eigenvalues of the matrix*

$$A_{n-1} = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \tag{13}$$

have negative real parts. Then matrix equation (10) is solvable and its positive definite solutions have the form

$$F = \begin{pmatrix} f_{11} & \cdots & f_{1n-1} & \frac{a_n}{a_{n-1}} f_{1n-1} \\ \cdots & \cdots & \cdots & \cdots \\ f_{1n-1} & \cdots & f_{n-1n-1} & \frac{a_n}{a_{n-1}} f_{n-1n-1} \\ \frac{a_n}{a_{n-1}} f_{1n-1} & \cdots & \frac{a_n}{a_{n-1}} f_{n-1n-1} & f_{nn} \end{pmatrix}, \tag{14}$$

where elements of the matrix $F_{n-1} = \{f_{ij}\}_{i,j=1}^{n-1}$ are defined by the matrix equation

$$A_{n-1}^* F_{n-1} + F_{n-1} A_{n-1} = -W_{n-1}$$

and $f_{nn} > 0$ is an arbitrary real number such that

$$f_{nn} > \frac{a_n^2}{a_{n-1}^2} f_{n-1n-1}. \tag{15}$$

Proof. This theorem is a simple consequence of theorem 1 and theorem 2 from [4].

Now we define the matrix F and numbers a_i, $i = 0, \ldots, n+1$ so that there exists $\beta > 0$ such that $\dot{\Theta}(x)\Big|_{(4)} \leq -\beta$. This means that inequality (3) holds for $\alpha = 1$.

Suppose that the matrix W_{n-1} is a given positive definite matrix of the form (11). Then, by Lemma 1, the matrix W of the form (12) is positive semi-definite. Suppose that the numbers $a_i < 0$, $i = 1, \ldots, n-1$ are such that the matrix A_{n-1} of the form (13) is stable, i.e. eigenvalues of the matrix A_{n-1} have negative real parts. We define the matrix F as a positive definite solution of matrix equation (10). Then, according to Theorem 1, F has the form (14).

Thus, using (9), the derivative of the controllability function takes the form

$$\dot{\Theta}(x)\Big|_{(4)} = \frac{-(Wy(\Theta(x),x),y(\Theta(x),x))+2(Fh_n,y(\Theta(x),x))x_{n-1}^{2k+1}\Theta(x)}{(F^1y(\Theta(x),x),y(\Theta(x),x))}, \quad (16)$$

where $F^1 = 2mF - FH - HF$.

We introduce the following notation $I_{n,2} = \operatorname{diag}(1,\dots,1,0,0)$ is a matrix of dimension $(n\times n)$, $I_{n-1,1}$=diag$(1,\dots,1,0)$ is a matrix of dimension $(n-1)\times(n-1)$, I_{n-1} is the identity $(n-1)\times(n-1)$ matrix, $\widehat{x} = (x_1,\dots,x_{n-1})$.

Since the matrix W_{n-1} is positive definite, we have the following estimate

$$(W_{n-1}\widehat{x},\widehat{x}) \geq \lambda_{min}(\widehat{x},\widehat{x}) \quad \text{for all} \quad \widehat{x}\in\mathbb{R}^{n-1},$$

where $\lambda_{min}>0$ is the smallest eigenvalue of the matrix W_{n-1}. Therefore,

$$-((W_{n-1}-\lambda_{min}I_{n-1})\widehat{x},\widehat{x}) - \lambda_{min}x_{n-1}^2 \leq 0 \quad \text{for all} \quad \widehat{x}\in\mathbb{R}^{n-1},$$

i.e. the matrix $W_{n-1}-\lambda_{min}I_{n-1,1}$ is positive semi-definite. Then, by Lemma 1, we have

$$-((W-\lambda_{min}I_{n,2})x,x) \leq 0 \quad \text{for all} \quad x\in\mathbb{R}^n. \quad (17)$$

Introducing the notation $b=-Fh_n$, we get

$$\begin{aligned} b_i &= -(f_{1i}a_{n+1}+\frac{a_n}{a_{n-1}}f_{in-1}), \quad i=1,\dots,n-1,\\ b_n &= a_{n+1}\frac{a_n}{a_{n-1}}f_{1n-1}+f_{nn}. \end{aligned}$$

We choose a_{n+1} so that $b_n=0$. Thus we put

$$a_{n+1} = -\frac{f_{nn}}{f_{1n-1}}\cdot\frac{a_{n-1}}{a_n}. \quad (18)$$

Finally, we obtain

$$b_i = \left(f_{1i}\frac{f_{nn}}{f_{1n-1}} - f_{in-1}\frac{a_n^2}{a_{n-1}^2}\right)\frac{a_{n-1}}{a_n}, \quad i=1,\dots,n-1. \quad (19)$$

Combining (15) and (19), we deduce

$$b_{n-1} = \left(f_{nn} - f_{n-1n-1}\frac{a_n^2}{a_{n-1}^2}\right)\frac{a_{n-1}}{a_n} > 0.$$

Consider the following $(n-1)\times(n-1)$ matrix

$$W_{\lambda_{min}}(\Theta,x_{n-1}) = \begin{pmatrix} \lambda_{min} & 0 & \cdots & 0 & b_1\frac{x_{n-1}^k}{\Theta^{k(n-1)}} \\ 0 & \lambda_{min} & \cdots & 0 & \vdots \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_{min} & b_{n-2}\frac{x_{n-1}^k}{\Theta^{k(n-1)}} \\ b_1\frac{x_{n-1}^k}{\Theta^{k(n-1)}} & \cdots & \cdots & b_{n-2}\frac{x_{n-1}^k}{\Theta^{k(n-1)}} & 2b_{n-1} \end{pmatrix}.$$

For definiteness we assume that

$$W_{\lambda_{min}}(\Theta,x_1)=2b_1,\quad W_{\lambda_{min}}(\Theta,x_2)=\begin{pmatrix}\lambda_{min} & b_1\dfrac{x_2^k}{\Theta^{2k}}\\ b_1\dfrac{x_2^k}{\Theta^{2k}} & 2b_2\end{pmatrix}.$$

By direct calculation it can be shown that

$$\lambda_{min}\big(I_{n,2}y(\Theta,x),y(\Theta,x)\big)+2\big(b,y(\Theta,x)\big)x_{n-1}^{2k+1}\Theta= \big(W_{\lambda_{min}}(\Theta,x)\widehat{y}(\Theta,x),\widehat{y}(\Theta,x)\big), \tag{20}$$

where $\widehat{y}(\Theta,x)=(x_1\Theta^{m-1},\dots,x_{n-2}\Theta^{m-n+2},x_{n-1}^{k+1}\Theta^{\frac{m-n+2}{2}})$.

For $n=2$ equality (20) reads as

$$\lambda_{min}\big(I_{2,2}y(\Theta,x),y(\Theta,x)\big)+2\big(b,y(\Theta,x)\big)x_1^{2k+1}\Theta=2b_1x_1^{2k+2}\Theta^m.$$

Using equality (20), we rewrite $\dot\Theta(x)\Big|_{(4)}$ in the form

$$\dot\Theta(x)\Big|_{(4)}=-\frac{\big((W-\lambda_{min}I_{n,2})y(\Theta(x),x),y(\Theta(x),x)\big)}{(F^1y(\Theta(x),x),y(\Theta(x),x))} -\frac{\big(W_{\lambda_{min}}(\Theta(x),x_{n-1})\widehat{y}(\Theta(x),x),\widehat{y}(\Theta(x),x)\big)}{(F^1y(\Theta(x),x),y(\Theta(x),x))}, \tag{21}$$

where $F^1=2mF-FH-HF$.

Lemma 2. *Let $\widehat{\lambda}_{min}(\Theta,x_{n-1})$ be the smallest eigenvalue of the matrix $W_{\lambda_{min}}(\Theta,x_{n-1})$. Then*

$$\widehat{\lambda}_{min}(\Theta,x_{n-1})=\frac{1}{2}\left(\lambda_{min}+2b_{n-1}-\sqrt{(\lambda_{min}-2b_{n-1})^2+4\frac{x_{n-1}^{2k}}{\Theta^{2k(n-1)}}\sum_{i=1}^{n-2}b_i^2}\right)$$

for $n\ge 3$.

Proof. Denote by $\chi_A(\lambda)$ the characteristic polynomial of the matrix $W_{\lambda_{min}}(\Theta,x_{n-1})$. It is not difficult to establish by induction that

$$\chi_A(\lambda)=(\lambda_{min}-\lambda)^{n-3}\left(\lambda^2-(2b_{n-1}+\lambda_{min})\lambda-\frac{x_{n-1}^{2k}}{\Theta^{2k(n-1)}}\sum_{i=1}^{n-2}b_i^2+2b_{n-1}\lambda_{min}\right).$$

By direct calculation, it is easy to verify that the smallest root of this equation is $\widehat{\lambda}_{min}(\Theta,x_{n-1})$. Thus the lemma is proved.

Lemma 3. *Suppose that a_0 satisfies the inequality*

$$0<a_0<\frac{1}{2}\lambda_{min}(F)\left(\frac{2b_{n-1}\lambda_{min}}{b_1^2+b_2^2+\dots+b_{n-2}^2}\right)^{\frac{1}{k}}. \tag{22}$$

Then the matrix $W_{\lambda_{min}}(\Theta(x), x_{n-1})$ is positive definite for every fixed $x \neq 0$.

Proof. The matrix F is positive definite. Then, from (6), we obtain

$$2a_0\Theta^{2m}(x) \geq \lambda_{min}(F)\|y(\Theta(x), x)\|^2, \tag{23}$$

where $\lambda_{min}(F) > 0$ is the smallest eigenvalue of the matrix F. Since

$$\|y(\Theta, x)\|^2 \geq x_i^2\Theta^{2(m-i)}, \quad i = 1, \dots, n-1 \quad \text{and} \quad \|y(\Theta, x)\|^2 \geq x_n^2,$$

it follows from (22) that

$$\frac{x_i^2}{\Theta^{2i}(x)} \leq \frac{2a_0}{\lambda_{min}(F)}, \quad i = 1, \dots, n-1, \quad \frac{x_n^2}{\Theta^{2m}(x)} \leq \frac{2a_0}{\lambda_{min}(F)}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$. In particular

$$\frac{x_{n-1}^2}{\Theta^{2(n-1)}(x)} \leq \frac{2a_0}{\lambda_{min}(F)}. \tag{24}$$

Combining (22) and (24), we obtain

$$\frac{x_{n-1}^{2k}}{\Theta^{2k(n-1)}(x)} < \frac{2b_{n-1}\lambda_{min}}{b_1^2 + b_2^2 + \dots + b_{n-2}^2}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$. This inequality implies that

$$\begin{aligned}\widehat{\lambda}_{min}(\Theta(x), x_{n-1}) &> \tfrac{1}{2}\left(\lambda_{min} + 2b_{n-1} - \sqrt{(\lambda_{min} - 2b_{n-1})^2 + 8b_{n-1}\lambda_{min}}\right) \\ &= \tfrac{1}{2}\left(\lambda_{min} + 2b_{n-1} - \sqrt{(\lambda_{min} + 2b_{n-1})^2}\right) = 0.\end{aligned}$$

Therefore the matrix $W_{\lambda_{min}}(\Theta(x), x_{n-1})$ is positive definite for every fixed $x \neq 0$. This concludes the proof.

First we prove that $\dot{\Theta}(x) < 0$ for any a_0 that satisfies condition (22). So suppose a_0 satisfies condition (22). Let us introduce the following notation

$$\widehat{\lambda} = \frac{1}{2}\left(\lambda_{min} + 2b_{n-1} - \sqrt{(\lambda_{min} - 2b_{n-1})^2 + 4L^k\sum_{i=1}^{n-2} b_i^2}\right),$$

where $L = \dfrac{2a_0}{\lambda_{min}(F)}$. Then, by inequality (24), we obtain that the smallest eigenvalue of the matrix $W_{\lambda_{min}}(\Theta(x), x)$ satisfies the following inequality

$$\widehat{\lambda}_{min}(\Theta(x), x_{n-1}) \geq \widehat{\lambda} > 0. \tag{25}$$

The last inequality implies that

$$\big(W_{\lambda_{min}}(\Theta(x), x_{n-1})\widehat{y}(\Theta(x), x), \widehat{y}(\Theta(x), x)\big) \geq \widehat{\lambda}\|\widehat{y}(\Theta(x), x)\|^2. \tag{26}$$

Due to positive definiteness of the matrix F^1 we have the following estimate

$$(F^1 y(\Theta(x),x), y(\Theta(x),x)) \le \lambda_{max}(F^1)\|y(\Theta(x),x)\|^2, \tag{27}$$

where $\lambda_{max}(F^1) > 0$ is the largest eigenvalue of the matrix F^1.

From (21), using (26) and (27), we obtain

$$\dot\Theta(x)\Big|_{(4)} \le -\frac{\big((W-\lambda_{min} I_{n,2})y(\Theta(x),x), y(\Theta(x),x)\big) + \widehat{\lambda}\cdot\|\widehat{y}(\Theta(x),x)\|^2}{\lambda_{max}(F^1)\|y(\Theta(x),x)\|^2}, \tag{28}$$

where $\widehat{y}(\Theta(x),x) = (x_1\Theta^{m-1}(x),\dots,x_{n-2}\Theta^{m-n+2}(x), x_{n-1}^{k+1}\Theta^{\frac{m-n+2}{2}}(x))$.

Inequality (28) implies that

$$\dot\Theta(x)\Big|_{(4)} < 0 \quad \text{for all} \quad x\in\mathbb{R}^n.$$

Indeed, for $\|\widehat{y}(\Theta(x),x)\| \ne 0$ the last inequality is true since inequalities (17) and (25) hold. For $\|\widehat{y}(\Theta(x),x)\| = 0$, from (28), we have

$$\dot\Theta(x)\Big|_{(4)} \le -\frac{w_{n-1n-1}a_n^2}{\lambda_{max}(F^1)a_{n-1}^2} < 0,$$

where $\lambda_{max}(F^1) > 0$, $w_{n-1n-1}>0$.

Thus the origin $x = 0$ is a globally asymptotically stable equilibrium of the closed-loop system (4). Now we prove that there exists $\beta > 0$ such that

$$\dot\Theta(x)\Big|_{(4)} \le -\beta.$$

Suppose that x_i^0, $i = 1,\dots,n$ are real numbers such that $\sum\limits_{i=1}^{n}|x_i^0| \ne 0$. Consider a family of curves defined by

$$\begin{cases} x_1 = x_1^0|x_n^0|^{-\frac{1}{m}} sign(x_n^0)|x_n|^{\frac{1}{m}} sign(x_n), \\ x_2 = x_2^0|x_n^0|^{-\frac{2}{m}} sign(x_n^0)|x_n|^{\frac{2}{m}} sign(x_n), \\ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ . \\ x_{n-1} = x_{n-1}^0|x_n^0|^{-\frac{n-1}{m}} sign(x_n^0)|x_n|^{\frac{n-1}{m}} sign(x_n) \\ x_n = x_n. \end{cases} \tag{29}$$

We note that for every fixed point $x^0 = (x_1^0,\dots,x_n^0) \in \mathbb{R}^n\setminus\{0\}$ such that $x_n^0 \ne 0$ there is exactly one curve from the family passing through x^0.

Suppose that the point $x\in\mathbb{R}^n$ lies on the curve (29) for some fixed $x^0 \ne 0$. By direct calculation, it is easy to verify that

$$\Theta(x) = \Theta(x_0)|x_n^0|^{-\frac{1}{m}}|x_n|^{\frac{1}{m}}. \tag{30}$$

Now we estimate $\dot{\Theta}(x)\Big|_{(4)}$ for every point $x \in \mathbb{R}^n$ that lies on the curve (29) with some fixed $x_0 \neq 0$. From (28), using (29) and (30), we obtain

$$\dot{\Theta}(x)\Big|_{(4)} \leq -\frac{\big((W - \lambda_{min} I_{n,2})z, z\big) + \widehat{\lambda} \cdot \left(\sum\limits_{i=1}^{n-2} z_i^2 + z_{n-1}^{2k+2}\Big(\dfrac{2a_0}{(Fz,z)}\Big)^k\right)}{\lambda_{max}(F^1)\|z\|^2}, \qquad (31)$$

where

$$z = (z_1, \ldots, z_n) = \left(\frac{x_1^0}{x_n^0}\Theta^{m-1}(x_0), \ldots, \frac{x_{n-1}^0}{x_n^0}\Theta^{m-n+1}(x_0), 1\right).$$

We will show that the right-hand side of (31) is bounded from zero. Consider the function $G(\widehat{z})$ defined by

$$G(\widehat{z}) = -\frac{\big((W - \lambda_{min} I_{n,2})z, z\big) + \widehat{\lambda} \cdot \left(\sum\limits_{i=1}^{n-2} z_i^2 + z_{n-1}^{2k+2}\Big(\dfrac{2a_0}{(Fz,z)}\Big)^k\right)}{\lambda_{max}(F^1)\|z\|^2}, \qquad (32)$$

where $\widehat{z} = (z_1, \ldots, z_{n-1})$. Let R be an arbitrary number such that

$$0 < R < \frac{1}{2} \cdot \frac{a_n}{a_{n-1}} \cdot \frac{w_{n-1n-1}}{\sqrt{\sum\limits_{i=1}^{n-1} w_{in-1}^2}}. \qquad (33)$$

First we estimate the function $G(\widehat{p})$ for every point $\widehat{z} = (z_1, \ldots, z_{n-1})$ such that $z_1^2 + \cdots + z_{n-1}^2 \leq R^2$. From (32) and (33) we deduce that

$$\begin{aligned} G(\widehat{z}) &= -\frac{\big((W_{n-1} - I_{n-1}\lambda_{min})\widehat{z}, \widehat{z}\,\big) + \frac{a_n^2}{a_{n-1}^2}\, w_{n-1n-1} + 2\frac{a_n}{a_{n-1}} \sum\limits_{i=1}^{n-1} w_{in-1} z_i}{\lambda_{max}(F^1)\|z\|^2} \\ &\leq -\frac{\frac{a_n^2}{a_{n-1}^2}\, w_{n-1n-1} - 2\frac{a_n}{a_{n-1}} \sqrt{\sum\limits_{i=1}^{n-1} w_{in-1}^2}\sqrt{\sum\limits_{i=1}^{n-1} z_i^2}}{\lambda_{max}(F^1)\|p\|^2} \\ &\leq -\frac{\frac{a_n^2}{a_{n-1}^2}\, w_{n-1n-1} - 2\frac{a_n}{a_{n-1}} \sqrt{\sum\limits_{i=1}^{n-1} w_{in-1}^2} \cdot R}{\lambda_{max}(F^1)(R^2+1)} \equiv -M_1(R) < 0. \qquad (34) \end{aligned}$$

Second we estimate the function $G(\widehat{z})$ for every point $\widehat{z} = (z_1, \ldots, z_{n-1})$ such that $z_1^2+\cdots+z_{n-1}^2 \ge R^2$. From (32) and (33) we deduce that

$$\begin{aligned} G(\widehat{z}) &\le -\frac{\widehat{\lambda}\left(z_1^2+\cdots+z_{n-2}^2+\left(\frac{2a_0}{(Fz,z)}\right)^k z_{n-1}^{2k+2}\right)}{\lambda_{max}(F^1)\|z\|^2} \\ &\le -\frac{\widehat{\lambda}\,\min\left\{1,\left(\frac{2a_0}{\lambda_{max}(F)}\right)^k\right\}}{\lambda_{max}(F^1)}\cdot\frac{\left(\|z\|^{2k}(z_1^2+\cdots+z_{n-2}^2)+z_{n-1}^{2k+2}\right)}{\|z\|^{2k+2}} \\ &\le -\frac{\widehat{\lambda}\,\min\left\{1,\left(\frac{2a_0}{\lambda_{max}(F)}\right)^k\right\}}{\lambda_{max}(F^1)}\cdot\frac{\left(z_1^{2k+2}+\cdots+z_{n-2}^{2k+2}+z_{n-1}^{2k+2}\right)}{\|z\|^{2k+2}} \\ &\le -\frac{\widehat{\lambda}\,\min\left\{1,\left(\frac{2a_0}{\lambda_{max}(F)}\right)^k\right\}}{\lambda_{max}(F^1)}\cdot\frac{2^{(2-n)k}\left(z_1^2+\cdots+z_{n-2}^2+z_{n-1}^2\right)^{k+1}}{\|z\|^{2k+2}} \\ &\le -\frac{\widehat{\lambda}\,\min\left\{1,\left(\frac{2a_0}{\lambda_{max}(F)}\right)^k\right\}}{\lambda_{max}(F^1)2^{(n-2)k}}\cdot\frac{R^{2k+2}}{(R^2+1)^{k+1}} \equiv -M_2(R) < 0. \end{aligned} \tag{35}$$

Thus, from (34) and (35), we obtain

$$G(\widehat{z}) \le -\min\{M_1(R), M_2(R)\} < 0 \quad \text{for all} \quad \widehat{z} \in \mathbb{R}^{n-1}.$$

The last inequality implies that $\dot{\Theta}(x)\Big|_{(4)}$ is bounded from zero for every point $x \in \mathbb{R}^n$ such that $x_n \ne 0$. Since $\dot{\Theta}(x)\Big|_{(4)}$ is continuous at every point $x \in \mathbb{R}^n \setminus \{0\}$, we have the following estimate

$$\dot{\Theta}(x)\Big|_{(4)} \le -\min\{M_1(R), M_2(R)\} \quad \text{for all} \quad x \in \mathbb{R}^n \setminus \{0\}. \tag{36}$$

Thus inequality (3) is satisfied for $\alpha = 1$ and $\beta = \min\{M_1(R), M_2(R)\} > 0$. Therefore the equilibrium point $x = 0$ of the closed-loop system (4) is finite-time stable.

We proceed now to establish conditions under which the control $u = u(x)$ defined by (7) satisfies the estimate $|u(x)| \le d$.

Lemma 4. *Suppose a_0^* is a unique positive root of the equation*

$$\sqrt{\frac{2a_0^*}{\lambda_{min}(F)}}\left(\|a\| - a_{n+1}\Big(\frac{2a_0^*}{\lambda_{min}(F)}\Big)^k\right) = d, \tag{37}$$

where $a = (a_1, \ldots, a_n)$, $a_{n+1} < 0$, $\lambda_{min}(F) > 0$ is the smallest eigenvalue of the matrix F. If a_0 satisfies the inequality

$$0 < a_0 \le a_0^*,$$

then the control $u = u(x)$ defined by (7) satisfies the restriction $|u(x)| \leq d$ for all $x \in \mathbb{R}^n$.

Proof. Consider the function

$$\Phi(a_0) = \sqrt{\frac{2a_0}{\lambda_{min}(F)}} \left(\|a\| - a_{n+1} \Big(\frac{2a_0}{\lambda_{min}(F)} \Big)^k \right).$$

The function $\Phi(a_0)$ is continuous and strictly increasing. Moreover, $\Phi(a_0) > 0$ for all $a_0 > 0$. It is clear that

$$\Phi(0) = 0, \quad \text{and} \quad \Phi(a_0) \longrightarrow +\infty \quad \text{as} \quad a_0 \longrightarrow +\infty.$$

Then there exists a unique number $a_0^* > 0$ such that $\Phi(a_0^*) = d$.

Now we estimate the control $u = u(x)$ defined by (7). Since $0 < a_0 \leq a_0^*$, using (23) and (24), we have

$$\begin{aligned} |u(x)| &= \frac{\|a\| \cdot \|D(\Theta(x))\|}{\Theta(x)^m} - a_{n+1} \frac{x_{n-1}^{2k}}{\Theta^{m-n}(x)} \cdot \frac{|x_{n-1}|}{\Theta^{n-1}(x)} \\ &\leq \sqrt{\frac{2a_0}{\lambda_{min}(F)}} \left(\|a\| - a_{n+1} \Big(\frac{2a_0}{\lambda_{min}(F)} \Big)^k \right) \leq \Phi(a_0^*) = d. \end{aligned}$$

This completes the proof.

Finally, we summarize our discussion, and formulate the main result of this section. The next theorem provides a solution of the global synthesis problem for nonlinear system (4).

Theorem 2. *Suppose that the numbers $a_i < 0$, $i = 1, \ldots, n-1$ are such that the matrix A_{n-1} defined by (13) is stable, a_n is an arbitrary negative number, the matrix W_{n-1} defined by (11) is an arbitrary positive definite matrix. Let the matrix F of the form (14) be a positive definite solution of equation (10) with right-hand side (12). Choose f_{nn} by (15), and a_{n+1} by (18). Furthermore, suppose that the matrix $F^1 = 2mF - FH - HF$ is positive definite. Choose a_0 such that*

$$0 < a_0 < \min \left\{ \frac{1}{2} \lambda_{min}(F) \left(\frac{2b_{n-1}\lambda_{min}}{b_1^2 + b_2^2 + \cdots + b_{n-2}^2} \right)^{\frac{1}{k}}, a_0^* \right\},$$

where $\lambda_{min}(F)$ is the smallest eigenvalue of the matrix F, λ_{min} is the smallest eigenvalue of the matrix W_{n-1}, b_i is defined by (19), and a_0^* is a unique positive root of equation (37). Let the controllability function $\Theta(x)$, for every $x \in \mathbb{R}^n$, be the positive solution of equation (5). Then the control $u = u(x)$ defined by (7) solves the global synthesis problem for system (4). Moreover, the time of motion $T(x_0)$ from an arbitrary point $x_0 \in \mathbb{R}^n$ to the origin satisfies the estimate

$$T(x_0) \leq \frac{1}{\min \{M_1(R), M_2(R)\}} \Theta(x_0),$$

where $M_1(R)$ and $M_2(R)$ are defined by (34) and (35) respectively.

Proof. According to (36) the inequality (3) is satisfied for $\alpha = 1$ and $\beta = \min\{M_1(R), M_2(R)\}$. Then, by theorem 1 from [2], the control $u = u(x)$ of the form (7) solves the global synthesis problem for system (4), and $T(x_0)$ satisfies the estimate

$$T(x_0) \le \frac{\alpha}{\beta}\Theta(x_0)^{\frac{1}{\alpha}} = \frac{1}{\min\{M_1(R), M_2(R)\}}\Theta(x_0).$$

Moreover, by Lemma 4, the control $u = u(x)$ satisfies the restriction $|u(x)| \le d$. This concludes the proof.

3. Global synthesis of bounded controls for systems with power nonlinearity in the case $\prod_{i=1}^{n-1} c_i \neq 0$

Now we solve the global synthesis problem for system (1) in the case c_i, $i = 1, \dots, n-1$ are some known numbers such that $\prod_{i=1}^{n-1} c_i \neq 0$. So consider the following nonlinear system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_i = c_{i-1}x_{i-1}, \quad i = 2, \dots, n-1, \\ \dot{x}_n = c_{n-1}x_{n-1}^{2k+1}, \end{cases} \tag{38}$$

where $k = \frac{p}{q}$, $p > 0$ is an integer, $q > 0$ is an odd integer.

Using the results obtained in the previous section, we formulate the following theorem, which provides the solution of the global synthesis problem for nonlinear system (38).

Theorem 3. *Suppose that the conditions of Theorem 2 hold. Let the numbers $\widehat{c}_i$, $i = 1, \dots, n$ be defined by*

$$\widehat{c}_1 = 1, \quad \widehat{c}_i = c_{i-1}\widehat{c}_{i-1}, \quad i = 2, \dots, n-1, \quad \widehat{c}_n = c_{n-1}\widehat{c}_{n-1}^{\,2k+1}.$$

Let the controllability function $\Theta(x)$, for every $x \in \mathbb{R}^n$, be the positive solution of the equation

$$2a_0\Theta^{2m} = (\widehat{C}^{-1}F\widehat{C}^{-1}D(\Theta)x, D(\Theta)x), \tag{39}$$

where $\widehat{C} = \operatorname{diag}(\widehat{c}_1, \dots, \widehat{c}_n)$ is an $n \times n$ diagonal matrix. Then the control

$$u(x) = \frac{1}{\Theta^m(x)}\left(a, D(\Theta(x))\widehat{C}^{\,-1}x\right) + \frac{a_{n+1}}{\widehat{c}_{n-1}^{\,2k+1}} \cdot \frac{x_{n-1}^{2k+1}}{\Theta^{m-1}(x)} \tag{40}$$

solves the global synthesis problem for system (38). Moreover, the time of motion $T(x_0)$ from an arbitrary point $x_0 \in \mathbb{R}^n$ to the origin satisfies the estimate

$$T(x_0) \le \frac{1}{\min\{M_1(R), M_2(R)\}}\Theta(x_0), \tag{41}$$

where $M_1(R)$ and $M_2(R)$ are defined by (34) and (35) respectively.

Proof. Assume that the control $u = u(x)$ is applied to system (38). The change of variables $x_i = \widehat{c}_i z_i$, $i = 1, \ldots, n$ $(x = \widehat{C}z,\ z \in \mathbb{R}^n)$ maps the closed-loop system (38) to the system

$$\begin{cases} \dot{z}_1 = v(z) \\ z_i = z_{i-1}, \quad i = 2, \ldots, n-1, \\ z_n = z_{n-1}^{2k+1}, \end{cases} \tag{42}$$

where $v(z) = u(\widehat{C}z)$. According to (39) and (40) we have

$$v(z) = u(\widehat{C}z) = \frac{1}{\widetilde{\Theta}^m(z)}\,(a, D(\widetilde{\Theta}(z))z) + a_{n+1}\frac{z_{n-1}^{2k+1}}{\widetilde{\Theta}^{m-1}(z)},$$

where the function $\widetilde{\Theta}(z)$, for every $z \in \mathbb{R}^n$, satisfies the equation

$$2a_0\widetilde{\Theta}^{2m} = (FD(\widetilde{\Theta})z, D(\widetilde{\Theta})z).$$

It is clear that $\widetilde{\Theta}(z) = \Theta(\widehat{C}z)$. By Lemma 4, we deduce that the control $v(z)$ satisfies the estimate $|v(z)| \leq d$ for all $z \in \mathbb{R}^n$. This implies that the control $u(x)$ is bounded by the same constant $d > 0$ for all $x \in \mathbb{R}^n$.

Denote by $z(t, z_0)$ the solution of the closed-loop system (42) that satisfies the initial condition $z(0, z_0) = z_0$. Thus, by Theorem 2, we obtain that for every fixed $z_0 \in \mathbb{R}^n$ there exists a number $T(z_0) < +\infty$ such that $\lim\limits_{t \to T(z_0)} z(t, z_0) = 0$ and $z(t, z_0) = 0$ for all $t \geq T(z_0)$. Moreover, $T(z_0)$ satisfies the estimate

$$T(z_0) \leq \frac{1}{\min\{M_1(R), M_2(R)\}}\widetilde{\Theta}(z_0)$$

for every $z_0 \in \mathbb{R}^n$.

Denote by $x(t, x_0)$ the solution of the closed-loop system (38) that satisfies the condition $x(0, x_0) = x_0$. Since the matrix $\widehat{C}$ is nonsingular, we obtain

$$\lim_{t \to \widetilde{T}(x_0)} x(t, x_0) = 0 \quad \text{and} \quad x(t) = 0 \quad \text{for all} \quad t \geq \widetilde{T}(x_0),$$

where $\widetilde{T}(x_0) = T(\widehat{C}^{\,-1}x_0)$.

This means that the control $u = u(x)$ of the form (40) solves the global synthesis problem for system (38) and the time of motion $T(x_0)$ from an arbitrary point $x_0 \in \mathbb{R}^n$ to the origin satisfies the estimate (41). This concludes the proof.

Example 1. We solve the global synthesis problem for system (38) in the case $n = 4$, $d = 1$, $c_1 = -1$, $c_2 = \frac{1}{3}$, $c_3 = 2$, $k = 1$. So system (38) takes the form

$$\begin{cases} \dot{x}_1 = u, \quad |u| \leq 1, \\ \dot{x}_2 = -x_1, \\ \dot{x}_3 = \dfrac{1}{3}x_2, \\ \dot{x}_4 = 2x_3^3. \end{cases} \tag{43}$$

We choose negative real numbers a_1, a_2, a_3 so that the matrix A_3 defined by (13) is stable. For example, we put $a_1 = -3$, $a_2 = -3$, $a_3 = -1$. The matrix W_3 and the negative number $a_4 < 0$ may be chosen arbitrarily. We define W_3 by

$$W_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

and put $a_4 = -1$. Then, according to Theorem 1, the positive definite solution of the matrix equation (10), for $f_{44} = 7$, is given by

$$F = \begin{pmatrix} \frac{11}{16} & \frac{25}{16} & \frac{1}{2} & \frac{1}{2} \\ \frac{25}{16} & \frac{25}{4} & \frac{35}{16} & \frac{35}{16} \\ \frac{1}{2} & \frac{35}{16} & \frac{49}{16} & \frac{49}{16} \\ \frac{1}{2} & \frac{35}{16} & \frac{49}{16} & 7 \end{pmatrix}.$$

Using (18), we have $a_5 = -14$.

According to (39) we define the controllability function $\Theta(x)$ as a unique positive definite solution of the equation

$$2a_0\Theta^{20} = (\widehat{C}^{-1}F\widehat{C}^{-1}D(\Theta)x, D(\Theta)x),$$

where

$$D(\Theta) = \begin{pmatrix} \Theta^9 & 0 & 0 & 0 \\ 0 & \Theta^8 & 0 & 0 \\ 0 & 0 & \Theta^7 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \widehat{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & -\frac{2}{27} \end{pmatrix}.$$

Put $a_0 = 0.00178$. Then, by Theorem 3, the control

$$u(x) = -3\frac{x_1}{\Theta(x)} + 3\frac{x_2}{\Theta(x)^2} + 3\frac{x_3}{\Theta(x)^3} + \frac{27}{2}\frac{x_4}{\Theta(x)^{10}} + 378\frac{x_3^3}{\Theta(x)^9}$$

solves the global synthesis problem for system (43). Moreover, $u(x)$ satisfies the restriction $|u(x)| \le 1$ for all $x \in \mathbb{R}^n$.

Assume that the control $u = u(x)$ is applied to system (43). For instance, we take $x_0 = (-0.1, 0.1, -0.4, 0.3)$ as an initial point. By numerical simulation, for a solution $x(t)$ $(x(0) = x_0)$ of the closed-loop system (43), we have the following results: $\|x(100)\| = 0.051\ldots$, $\|x(5000)\| = 0.0079\ldots$, $\|x(11000)\| = 0.00064\ldots$, $\|x(15700)\| = 0.1142\ldots \times 10^{-21}$.

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