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The field of vertical electric dipole placed above the spiral conductive unclosed sphere

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The problem of the electromagnetic field an open spiral conductive sphere is analyzing. The method of regularization of operator tasks is applied. The integral equations with a weak singularity in the kernel is used. The infinite system of algebraic equations of type II with a compact operator in ℓ_2 is received. Some properties of electromagnetic fields are studied.
Keywords: spiral conductive sphere, vertical dipole, compact operator.

Резуненко В.О. **Поле вертикального електричного диполя, який розміщений над спірально провідною незамкненою сферою.** Досліджується задача про електромагнітне поле спірально провідної незамкненої сфери. Застосовано метод регуляризації оператора задачі, вмкористовано розв'язки інтегральних рівняннь із слабкою особливістю у ядрі. Одержано нескінченну систему алгебраїчних рівнянь II роду з компактним оператором у ℓ_2. Вивчені деякі властивості електромагнітних полів.
Ключові слова: спірально провідна сфера, вертикальний диполь, компактний оператор.

Резуненко В.А. **Поле вертикального электрического диполя, размещённого над спирально проводящей незамкнутой сферой.** Исследуется задача об электромагнитном поле спирально проводящей незамкнутой сферы. Применены метод регуляризации оператора задачи, интегральные уравнения со слабой особенностью в ядре. Получена бесконечная система алгебраических уравнений II рода с компактным оператором в ℓ_2. Изучены некоторые свойства электромагнитных полей.
Ключевые слова: спирально проводящая сфера, вертикальный диполь, компактный оператор.

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1. Introduction

The methods of regularization of matrix and integral operators of applied problems occupy a prominent place among the numerical-analytical methods [1], [2]. The variant of methods [1], [2] is used for analysis of the electrodynamic properties of the unclosed spiral conductive spherical surface. The spiral antennas and devices have small size and the lightweight. They are power saving ones. They allow to control the polarization of the radiation fields. The spiral antennas have been successfully used on mobile objects to communicate at short, medium and at very long distances [3]-[5]. We note that there are many experimental papers on this subject. The number of theoretical works is comparably small. The purpose of our work is the construction of a numerical analytical algorithm for study of fields properties of the spiral conductive unclosed sphere [1], [2]. The spiral conductive unclosed sphere is irradiated by the vertical electric dipole field. The dipole is placed above the sphere with a circular aperture on its axis of symmetry. We apply the method of regularization of problem's operator. We use the solutions of integral equations with a weak singularity in the kernels. The main part of the matrix operator is extracted and inverted. The infinite linear algebraic system of second kind with compact operator in Hilbert space l_2 is obtained. The limit cases of formulation of the problem and properties of solutions are considered.

2. Formulation of the problem

The origin of Cartesian and spherical systems of coordinates are placed in the geometrical center of the sphere of radius $r = a$. Let us cut the sphere by a horizontal plane into two parts. Consider its upper part as an unclosed sphere with a circular aperture. Let the polar angle θ of the edge of the aperture be equal θ_0. The polar angle θ on the aperture is changing from θ_0 to π. Let the vertical electrical dipole be placed on the axis of symmetry of the unclosed sphere on the axis OZ at the point $z = b > a$. We assume that the surface of an unclosed sphere is infinitely thin and spiral conductive. Let β will be the angle between the lines of conductivity of the electric current on the unclosed sphere and the lines of the meridians on the sphere. The sphere conducts the current in selected directions only. We note that the line of the conductivity on the sphere may be represented as follows: $x = sin(\eta)cos(14\eta)$, $y = sin(\eta)sin(14\eta)$, $z = 1 + cos(\eta)$, where η is a dimensionless parameter, which varies in bands $[0, \pi/2]$ (fig.1).The dipole field $\vec{E}^{(0)}$, $\vec{H}^{(0)}$ meets an unclosed sphere and creates secondary electromagnetic fields: $\vec{E}^{(1)}$, $\vec{H}^{(1)}$ in the area $0 \leqslant r < a$ and $\vec{E}^{(2)}$, $\vec{H}^{(2)}$in the area $r > a$. By definition, the total field in the area $0 \leqslant r < a$ is equal to $\vec{E}^{(1)}$, $\vec{H}^{(1)}$. According to the superposition principle of electromagnetic fields, the total field in the area $r > a$ is the sum of fields $\vec{E}^{(0)} + \vec{E}^{(2)}$ and $\vec{H}^{(0)} + \vec{H}^{(2)}$. The time dependence of the fields is taken as $exp(-i\omega t)$, where ω is the angular frequency, $\omega = 2\pi/\lambda$, λ is a wavelength of the dipole field.

The total electromagnetic fields outside of the unclosed sphere satisfy the

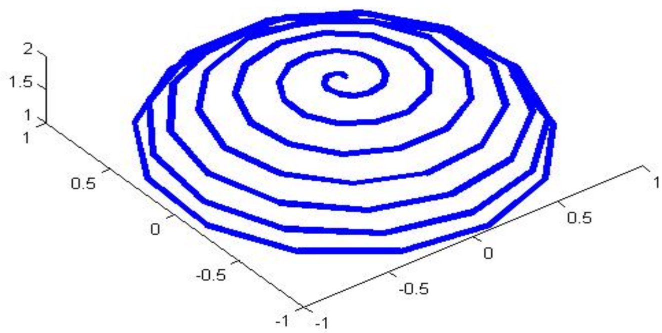

Fig.1: The line of spiral conductivity of the electric current.

following conditions: 1) the Maxwell and material equations:

$$rot\,\vec{E} = ik\vec{H}, \quad rot\,\vec{H} = -ik\vec{E}, \quad div\,\vec{D} = \rho, \quad div\,\vec{B} = 0, \tag{1}$$

$$\vec{D} = \varepsilon\vec{E}, \quad \vec{B} = \mu\vec{H}, \quad \vec{J} = \sigma\vec{E};$$

where $k = \omega\sqrt{\epsilon\mu}c^{-1}$, ϵ, μ and σ are the dielectric permittivity, the magnetic permeability and the conductivity of the medium, ρ is the charge density, c is the speed of light in vacuum; 2) the energy boundedness in any restricted volume A in R^3:

$$\int_A \left(\varepsilon|\vec{E}|^2 + \mu|\vec{H}|^2\right) dx\,dy\,dz < \infty, \tag{2}$$

where the volume A may contain the edge of the unclosed sphere; 3) the condition of fields radiation on infinity:

$$\lim r[\frac{\partial\Psi}{\partial r} - ik\Psi] = 0, r \to \infty,$$

where Ψ is any component $\vec{E}$ or $\vec{H}$.

3. Boundary conditions

In addition to conditions 1)-3), the total fields satisfy the boundary conditions. We write the conditions for the field's components

$$\vec{E}(E_r, E_\theta, E_\varphi), \quad \vec{H}(H_r, H_\theta, H_\varphi). \tag{3}$$

in the spherical coordinate system.

B1) the field's components on the surface of the unclosed sphere $\{r = a, 0 \le \theta < \theta_0, \phi \in [0, 2\pi]\}$ satisfy the conditions of the spiral conductivity:

$$
\begin{gathered}
(H_\theta^{(2)} - H_\theta^{(1)}) + (H_\varphi^{(2)} + H_\varphi^{(0)} - H_\varphi^{(1)})\, tg\,\beta = 0, \\
E_\theta^{(0)} + E_\theta^{(2)} = E_\theta^{(1)}, \quad E_\varphi^{(2)} = E_\varphi^{(1)}, \\
E_\theta^{(1)} - E_\theta^{(1)}\, tg\,\beta = 0; \qquad (4)
\end{gathered}
$$

B2) the total fields are continuous on the aperture of the unclosed sphere $\{r = a, \theta_0 < \theta \le \pi, \phi \in [0, 2\pi]\}$:

$$
\vec{E}^{(2)} + \vec{E}^{(0)} = \vec{E}^{(1)}, \quad \vec{H}^{(2)} + \vec{H}^{(0)} = \vec{H}^{(1)}. \qquad (5)
$$

The total fields satisfy the requirement for singularity in the dipole placement point. The problem (1) - (5) has a unique solution [8].

To solve the problem (1) - (5), we use the methods of regularization of the auxiliary integral and matrix operators. First, using the Debye u electric and v magnetic scalar potentials, the components of the field (3) are written. The fields components are uniquely expressed by the Debye potentials. The scalar potentials u, v satisfy the Helmholtz equation, which follows from the Maxwell equations (1), in particular, $\Delta u + k^2 u = 0$. We write the Helmholtz equation in the spherical coordinate system and separate the variables in the equation. The potentials are represented by the Fourier series. We note that the magnetic potential of the vertical electric dipole, placed on the axis OZ, is equal to zero: $v^{(0)} = 0$. The electric potential of the dipole is present by the series of eigenfunctions of the auxiliary the Sturm-Liouville problem as follows

$$
u^{(0)} = \sum_{n=1}^{\infty} F(n) \frac{P_n(\cos\theta)}{k^3 r b^2} \left\{ \begin{array}{ll} \psi_n(kr)\xi_n(kb), & r < b, \\ \psi_n(kb)\xi_n(kr), & r > b, \end{array} \right. \quad F(n) = 2n + 1. \qquad (6)
$$

We note that the modulus of dipole moment $\vec{P}$, which is directed along the axis OZ is equal to unity in the expression (6). We also take into account that the dipole is placed in the upper half space on the axis OZ above the unclosed sphere. In (6) $\psi_n(x), \xi_n(x)$ are spherical Bessel and Hankel functions in the Debye's notation of the first and 3-th kinds, respectively, of the n-th order of argument x; $P_n(\cos\theta)$ are Legendre polynomials of the first kind of the n-th power and zero order of the argument $\cos\theta$. We look for the secondary potentials (7), (8) in the form of series (6):

$$
\left. \begin{array}{l} u^{(1)} \\ \nu^{(1)} \end{array} \right\} = \sum_{n=1}^{\infty} F(n) \frac{P_n(\cos\theta)}{kr} \left\{ \begin{array}{ll} A_n \psi_n(kr), & r < a, \\ B_n \xi_n(kr), & r > a, \end{array} \right. \qquad (7)
$$

$$
\left. \begin{array}{l} u^{(2)} \\ \nu^{(2)} \end{array} \right\} = \sum_{n=1}^{\infty} F(n) \frac{P_n(\cos\theta)}{kr} \left\{ \begin{array}{ll} C_n \psi_n(kr), & r < a, \\ D_n \xi_n(kr), & r > a. \end{array} \right. \qquad (8)
$$

Here in (7), (8) we take into account the occurrence of magnetic potentials in the secondary fields, which are scattered by the spiral conductive unclosed sphere. The unknown coefficients A_n, B_n, C_n, D_n of the series (7) and (8) belong to the Hilbert space (see (2)) with certain weights, which are different for different coefficients.

4. The paired functional equations containing the associated Legendre functions

Using the boundary conditions (4), (5), we get the three linear equations of connection for four unknown coefficients A_n, B_n, C_n, D_n for each $n = 1, 2, 3, \ldots$. We used here the orthogonality of the associated Legendre functions of the first kind of the first order with the weight $sin\theta$ on the segment $[0, \pi]$. The coefficients A_n, B_n and D_n are expressed in terms of the coefficients C_n by the three equations of connection. To find coefficients C_n we deduce the paired functional equations. We use all the boundary conditions (4), (5) for all the components of the unknown fields $\vec{E}^{(1)}, \vec{H}^{(1)}, \vec{E}^{(2)}, \vec{H}^{(2)}$ from (3). As a result, we obtain the system of paired functional equations, which allows to find the coefficients (8) of potential $u^{(2)}$:

$$\sum_{n=1}^{\infty} C_n F(n) \frac{1}{\psi_n'(ka)} P_n^1(\cos\theta) = 0, \qquad \theta_0 < \theta \leq \pi, \tag{9}$$

$$\sum_{n=1}^{\infty} C_n \frac{F(n)}{\psi_n'(ka)} \left\{ (tg\,\beta)^2 \psi_n(ka)\xi_n(ka) + \psi_n'(ka)\xi_n'(ka) \right\} P_n^1(\cos\theta) =$$

$$-(kb)^{-2} \sum_{n=1}^{\infty} F(n)\psi_n'(ka)\xi_n(kb)\psi_n(kb)P_n^1(\cos\theta), \qquad 0 \leq \theta < \theta_0, \tag{10}$$

where the prime of the functions $\psi_n(\cdot), \xi_n(\cdot)$ means the differentiation with respect to the argument. To find all coefficients of the potential (7), (8) there is only one paired system of functional equations. In contrast to [11], the questions of the division of polarization fields and search for additional constants of integration do not arise. The system of paired functional equations (9)-(10) is of the first kind with complicated kernels, which involve various spherical functions. The multipliers of the unknown coefficients C_n in (9) and (10) are different and have different rates of decrease as $n \to \infty$. Even taking into account the orthogonality of the associated Legendre functions with the weight $sin\theta$ in $L_2(0, \pi)$, such systems can not be solved analytically. The systems of this type appear in many problems of fields diffraction on open structures. There are many direct numerical methods developed for their approximate solution. These methods are more general than the analytical ones. However, such methods do not allow to evaluate the accuracy of the solutions. This fact is important in the analysis, e.g., resonance oscillations of the investigated fields. In addition, the direct numerical methods also require the use of considerable computing resources. We apply analytical method for the regularization of the system (9), (10) [7,9-15,19,20]. As a result, we obtain the infinite system of linear algebraic equations of the second kind, which is successfully solved numerically and analytically.

5. The infinite system of linear algebraic equations of the second kind

We transform the system (9), (10) into an equivalent system of functional equations, which include the trigonometric functions instead of Legendre functions. For this purpose we use the convergence of the series (10) in $L_2(0,\pi)$ and integrate the equation (10) term by term. Here we use the equality $P_n^1(\cos\theta) = -[P_n(\cos\theta)]'$. Here the constant $T^{(0)}$ of integration arises. We find the constant $T^{(0)}$ below in (16). The Meller-Dirichlet integral representation for the Legendre polynomials (11)

$$P_n(\cos\theta) = \pi^{-1}\sqrt{2}\int_0^\theta (\cos\phi - \cos\theta)^{-0.5}\cos(n+0.5)\phi\, d\phi \tag{11}$$

is substituted into the integrated equation (10). Then the integral representation of the type Meller - Dirichlet for the associated Legendre functions (12)

$$P_n^1(\cos\theta)=[\pi sin\theta]^{-1}\frac{n(n+1)}{2n+1}\sqrt{2}\int_\theta^\pi (\cos\theta - \cos\phi)^{-0.5}\cos(n+0.5)\phi\cdot\sin\phi\, d\phi \tag{12}$$

is substituted into equation (9). Using the convergence of the series in $L_2(0,\pi)$, the order of integration and summation in both equations (9) and (10) changes. In this case we have two integral equations of the first kind with the weak singularities in the kernels. The singularities are due to the presence of radicals in (11) and (12). So, we get from (10) the integral equation $\int_0^\theta(\cos\phi - \cos\theta)^{-0.5} f_1(\phi)\, d\phi = 0$, where

$$f_1(\phi) = \sum_{n=1}^\infty F(n)\left\{C_n[\psi_n'(ka)]^{(-1)}[(tg\,\beta)^2\psi_n(ka)\xi_n(ka) + \psi_n'(ka)\xi_n'(ka)] - \right.$$

$$\left.\psi_n'(ka)\xi_n(kb)\,\psi_n(kb)/(kb)^2\right\}\cdot\cos(n+0.5)\phi - T^{(0)}\cos(0.5)\phi.$$

The solution of the integral equation is found in $L_2(0,\pi)$ by using the composition with the kernel of the equation [6, 7, 15]. We receive the unique trivial solution: $f_1(\phi) = 0, \phi\in(0,\theta_0)$.

Similarly, from the equation (9), we obtain the integral equation $\int_\theta^\pi(\cos\theta - \cos\phi)^{-0.5} f_2(\phi)\, d\phi = 0$, where $f_2(\phi)$ is represented by the trigonometric Fourier series. That integral equation also has the unique trivial solution in $L_2(0,\pi)$: $f_2(\phi) = 0, \phi\in(\theta_0,\pi)$. We receive a new system of functional equations of the first kind. Next, we transform the system of the first kind into the system of the second kind.

For this purpose we apply the methods [1,2,7,9-15,19,20] and use the properties of the Bessel and the Hankel functions [21]. Then we do some linear transformations of the system of functional equations and find the main part of the system. Next, we relabel the coefficients C_n to the new coefficients y_n (13) and introduce the small parameters ε_n (14):

$$y_n = C_n F(n) n(n+1)\left[\psi_n^1(ka)(2n+1)\right]^{-1}, \tag{13}$$

$$\varepsilon_n = 1 + ika\frac{2n+1}{(n(n+1))}\{\psi_n'(ka)\xi_n'(ka) + (tg\,\beta)^2\psi_n(ka)\xi_n(ka)\}. \tag{14}$$

Now we inverse analytically the main part of the functional equations of the second kind. For this, the methods [1,2,7,9–15, 19, 20] and the method of discrete Fourier transform are used. As a result, we obtain the infinite system of linear algebraic equations of the second kind:

$$\begin{gathered} y_n = (\pi)^{-1}\sum_{m=1}^{\infty} y_m\varepsilon_m q_{n,m}(\theta_0) - \frac{ika}{\pi}T^{(0)}q_{n,0}(\theta_0) - \\ \frac{ia}{k\pi b^2}\sum_{m=1}^{\infty} F(m)\psi_m^1(ka)\xi_m(kb)q_{n,m}(\theta_0). \end{gathered} \tag{15}$$

Also, we find the integration constant $T^{(0)}$ for the equation (9) and substitute it in (14) as :

$$T^{(0)} = \frac{i}{ka}\sum_{m=1}^{\infty} y_m\varepsilon_m \left\{1 + ika\,\psi_m^1(ka)\,\xi_m(kb)F(m)\cdot(kb)^{-2}\right\}\frac{q_{m,0}(\theta_0)}{q_{0,0}(\theta_0)}, \tag{16}$$

where

$$q_{n,m}(\theta_0) = 2\int_0^{\theta_0}[\cos(n+0.5)\phi][cos(m+0.5)\phi]d\phi. \tag{17}$$

Consider the properties of the resulting system (15). For any values ka and $\beta \in [0, \frac{\pi}{2})$ the small parameter (14) vanishes comparably quickly, proportionally to n^{-2}, when $n \to \infty$. The auxiliary values $q_{n,m}(\theta_0)$ in (17) are uniformly bounded by 2π for any $n, m \geq 1$ and any $\theta_0 \in [0, \pi]$. In addition, the values $q_{n,m}(\theta_0)$ for fixed $n = n_0$ vanish proportionally to n^{-1}, when $m \to \infty$. Similarly, the values $q_{n,m}(\theta_0)$ for fixed $m = m_0$ vanish, proportionally to n^{-1}, when $n \to \infty$. The matrix elements $\{G_{n,m}\}_{n,m=1}^{\infty}$ of the system $Y = GY + Q$ (15) for fixed $n = n_0$ vanish when $m \to \infty$ and they vanish for fixed $m = m_0$, when $n \to \infty$. The eigenvalues of the system's matrix operator differ from the unity. The right column of the system (15) belongs to l_2. The system (15) has the compact matrix operator in l_2 and a unique solution in l_2. It is solved numerically for arbitrary geometric and frequencies parameters of the problem. The system is solved analytically, in particular, by the method of successive approximations for the large apertures in the sphere ($0 \leq \theta_0 \ll 1$). This follows from the fact that the norm in l_2 of the matrix G of the system (15) is proportional to θ_0 for small θ_0. This method can be applied successfully for small apertures in the sphere ($0 \leq \pi - \theta_0 \ll 1$) after simple linear transformations of the system (15).

6. Conclusions

1. The system (15) is constructed for the study of electromagnetic fields in the case of placing of an electrical dipole in the point $z = b > a$ on the axis OZ.

The system (15) can be modified for the case of the dipole placed in the point $z = -b$ on the axis OZ. For this it is necessary to relabel the coefficients F_n in (6)-(8) as follows: $F_n^{(1)} = (-1)^{n+1} F(n), n \geq 1$.

2. Introducing the new coefficients $y_n^{(1)} = y_n n^{-2}, n \geq 1$, instead of the coefficients y_n (13), the speed of convergence of the analytical and numerical methods for solving the system (15) can be increased.

3. The polarization of the electromagnetic field of structure varies non monotonically from a linear to elliptical and almost circular with a change of the angle β of the spiral conductivity of the sphere and with an increase of the angle θ_0 between zero and π.

4. The reduced resonant frequencies $\chi_{n,m}, n, m > 1$ of the structure for small $\theta_1 = \pi - \theta_0 \ll 1$ and small β differ from the ones of the closed sphere $\chi_{n,m}^{(0)}$ on the coefficient, which is proportional to θ_1: $\chi_{n,m} = \chi_{n,m}^{(0)} + O(\theta_1)$, when $\theta_1 \to 0$ [10,16-20].

5. The sphere disappears completely when $\theta_0 \to 0$ and it turns into a closed spiral conductive sphere when $\theta_0 \to \pi$. The unclosed sphere becomes almost perfectly conductive, if β decreases from $\pi/2$ to zero. The electromagnetic field penetrates almost completely through the spiral conductive sphere when $\beta \to \pi/2$.

6. The constructed numerical-analytical algorithm can be generalized, for example, to calculate the electromagnetic fields of the horizontal dipole in the presence of a spiral conductive unclosed sphere.

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