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Bishop-Phelps-Bollobás modulus of a uniformly non-square Banach space

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Chica, Kadets, Martín and Soloviova demonstrated recently that the Bishop-Phelps-Bollobás modulus Φ_X^S of a Banach spaces X can be estimated from above through the parameter of uniform non-squareness $\alpha(X)$: $\Phi_X^S(\varepsilon) \leq \sqrt{2\varepsilon}\sqrt{1-\frac{1}{3}\alpha(X)}$. In this short note we demonstrate that the right-hand side in the above theorem cannot be substituted by anything smaller than $\sqrt{2\varepsilon}\sqrt{1-\alpha(X)}$.
Keywords: Bishop-Phelps theorem; uniformly non-square spaces.

Соловйова М. В. **Модулі Бішопа-Фелпса-Болобаша в рівномірно неквадратних банахових простірах.** Чіка, Кадець, Мартін, Соловйова нещодавно довели, що модуль Бішопа-Фелпса-Болобаша Φ_X^S банахового простора X може бути оцінений зверху через параметр рівномірної неквадратності $\alpha(X)$: $\Phi_X^S(\varepsilon) \leq \sqrt{2\varepsilon}\sqrt{1-\frac{1}{3}\alpha(X)}$. У цій короткій статті ми покажемо, що права частина оцінки не може бути змінена на щось меньше, ніж $\sqrt{2\varepsilon}\sqrt{1-\alpha(X)}$.
Ключові слова: теорема Бішопа-Фелпса, рівномірно неквадратні простори.

Соловьева М. В. **Модули Бишопа-Фелпса-Боллобаша в равномерно неквадратных банаховых пространствах** Чика, Кадец, Мартин, Соловьёва недавно доказали, что модуль Бишопа-Фелпса-Боллобаша Φ_X^S банахового пространства X может быть оценен сверху через параметр равномерной неквадратности $\alpha(X)$: $\Phi_X^S(\varepsilon) \leq \sqrt{2\varepsilon}\sqrt{1-\frac{1}{3}\alpha(X)}$. В этой короткой статье мы покажем, что правая часть этой оценки не может быть заменена на что-то меньшее, чем $\sqrt{2\varepsilon}\sqrt{1-\alpha(X)}$.
Ключевые слова: теорема Бишопа-Фелпса, равномерно неквадратные пространства.
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Introduction

In this paper letter X stands for a real Banach space. A functional $x^* \in X^*$ *attains its norm*, if there is an $x \in S_X$ with $x^*(x) = \|x^*\|$. The classical Bishop-Phelps theorem states that the set of norm attaining functionals on a Banach space is norm dense in the dual space ([1], see also [6, Chapter 1]). A refinement of this theorem, nowadays known as the Bishop-Phelps-Bollobás theorem [2], was proved by B. Bollobás and allows to approximate at the same time a functional and a vector in which it almost attains the norm. Very recently, the following quantity have been introduced [4] which measure, for a given Banach space, what is the best possible Bishop-Phelps-Bollobás theorem in this space. Denote by S_X and B_X the unit sphere and the closed unit ball of X respectively. We will also use the notation

$$\Pi(X) := \big\{(x, x^*) \in X \times X^* \,:\, \|x\| = \|x^*\| = x^*(x) = 1\big\}.$$

Definition 1 (Bishop-Phelps-Bollobás modulus, [4])
Let X be a real Banach space. The spherical Bishop-Phelps-Bollobás modulus *of the space X is the function $\Phi_X^S : (0,2) \longrightarrow \mathbb{R}^+$ such that given $\varepsilon \in (0,2)$, $\Phi_X^S(\varepsilon)$ is the infimum of those $\delta > 0$ satisfying that for every $(x, x^*) \in S_X \times S_{X^*}$ with $x^*(x) > 1 - \varepsilon$, there is $(y, y^*) \in \Pi(X)$ with $\|x - y\| < \delta$ and $\|x^* - y^*\| < \delta$.*

It is known (see, for example, [4, Theorem 2.1]) that for every Banach space X and every $\varepsilon \in (0,2)$ one has $\Phi_X^S(\varepsilon) \le \sqrt{2\varepsilon}$. This estimate is sharp for the two-dimensional real space $\ell_1^{(2)}$ (see [2] or [4, Example 2.5]).

Uniformly non-square spaces were introduced by James [7] as those spaces whose two-dimensional subspaces are uniformly separated from $\ell_1^{(2)}$. The main result of [7] – the reflexivity of uniformly non-square spaces – was the origin of the theory of superreflexive spaces.

Recall that a Banach space X is *uniformly non-square* if and only if there is $\alpha > 0$ such that

$$\frac{1}{2}(\|x + y\| + \|x - y\|) \le 2 - \alpha$$

for all $x, y \in B_X$. The *parameter of uniform non-squareness* of X, which we denote $\alpha(X)$, is the best possible value of α in the above inequality. In other words,

$$\alpha(X) := 2 - \sup_{x,y \in B_X} \left\{ \frac{1}{2}(\|x + y\| + \|x - y\|) \right\}.$$

With this notation X is uniformly non-square if and only if $\alpha(X) > 0$. In a uniformly non-square space the estimate $\Phi_X^S(\varepsilon) \le \sqrt{2\varepsilon}$ can be improved.

Theorem 1 (Theorem 3.3 of [5]) *Let X be a Banach space with $\alpha(X) > 0$. Then,*

$$\Phi_X^S(\varepsilon) \le \sqrt{2\varepsilon}\sqrt{1 - \frac{1}{3}\alpha(X)} \quad \textit{for} \quad 0 < \varepsilon < \frac{1}{2} - \frac{1}{6}\alpha(X).$$

Although we don't know whether the above estimate of $\Phi_X^S(\varepsilon)$ through $\alpha(X)$ is sharp, we are able to demonstrate (and this is the goal of this short article) that this result cannot be improved too much. Namely, we demonstrate that the unknown optimal estimate of $\Phi_X^S(\varepsilon)$ through $\alpha(X)$ cannot be better than $\sqrt{2\varepsilon}\sqrt{1-\alpha(X)}$.

The main result

We will make a use of "hexagonal spaces" X_ρ introduced in [8] and the description of $\Pi(X_\rho)$ from that paper. Fix a $\rho > \frac{1}{2}$ and denote X_ρ the linear space $\mathbb{R}^2$ equipped with the norm

$$\|(x_1,x_2)\| = \|(x_1,x_2)\|_\rho = \max\left\{|x_1 - \frac{1-\rho}{\rho}x_2|, |x_2 - \frac{1-\rho}{\rho}x_1|, |x_1+x_2|\right\}.$$

In other words,

$$\|(x_1,x_2)\| = \begin{cases} |x_1+x_2|, & \text{if } x_1x_2 \geq 0; \\ |x_1 - \frac{1-\rho}{\rho}x_2|, & \text{if } x_1x_2 < 0 \text{ and } |x_1| > |x_2|; \\ |x_2 - \frac{1-\rho}{\rho}x_1|, & \text{if } x_1x_2 < 0 \text{ and } |x_1| \leq |x_2|. \end{cases}$$

and the unit ball B_ρ of X_ρ is the hexagon $abcdef$, where $a = (1,0); b = (0,1)$; $c = (-\rho,\rho); d = (-1,0); e = (0,-1)$; and $f = (\rho,-\rho)$.

The dual space to X_ρ is $\mathbb{R}^2$ equipped with the polar to B_ρ as its unit ball. So the norm on X_ρ^* is given by the formula

$$\|(x_1,x_2)\|^* = \|(x_1,x_2)\|_\rho^* = \max\{|x_1|,|x_2|,\rho|x_1-x_2|\},$$

and the unit ball B_ρ^* of X_ρ^* is the hexagon $a^*b^*c^*d^*e^*f^*$, where $a^* = (1,1)$; $b^* = \left(-\frac{1-\rho}{\rho},1\right)$; $c^* = \left(-1,\frac{1-\rho}{\rho}\right)$; $d^* = (-1,-1)$; $e^* = \left(\frac{1-\rho}{\rho},-1\right)$; and $f^* = \left(1,-\frac{1-\rho}{\rho}\right)$. The corresponding spheres S_ρ and S_ρ^* are shown on Fig. 1 and 2 respectively.

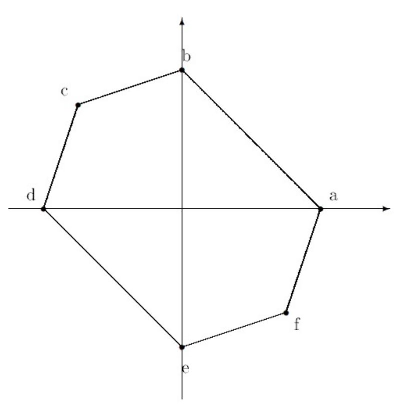

Fig. 1: Unit sphere of X_ρ.

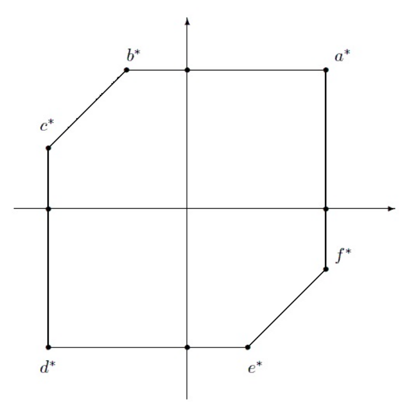

Fig. 2: Unit sphere of X_ρ^*.

In the case of $\rho = \frac{1}{2}$ the sphere of X_ρ reduces to the square $abde$, and consequently $X_{1/2}$ is isometric to the spaces $\ell_1^{(2)}$ and $\ell_\infty^{(2)}$. When $\rho > \frac{1}{2}$, the space X_ρ is not isometric to $\ell_\infty^{(2)}$. Let us calculate the parameter of uniform non-squareness for X_ρ.

Lemma 1 *Let $\rho \in [1/2, 1]$. Then, in the space $X = X_\rho$,*

$$\alpha(X_\rho) = 1 - \frac{1}{2\rho}. \tag{1}$$

Proof. Consider $\varphi(x, y) = \frac{1}{2}(\|x+y\| + \|x-y\|)$. Then $\alpha(X) = 2 - \sup\{\varphi(x, y) : (x, y) \in B_{X_\rho} \times B_{X_\rho}\}$. Since $\varphi : B_{X_\rho} \times B_{X_\rho} \to \mathbb{R}$ is a convex function, it attains its maximum at some extreme point of $S_{X_\rho} \times S_{X_\rho}$, i.e. at a point of the form (x, y) with $x, y \in \{a, b, c, d, e, f\}$. Also, $\varphi(x, y) = \varphi(y, x) = \varphi(x, -y)$, so by symmetry of the function and symmetry of the ball, is sufficient to check values of functions φ for the following two pairs (x, y): $x = a, y = b$ and $x = a, y = c$.

If $x = a = (1, 0), y = b = (0, 1)$, then $\|x + y\| = \|(1, 1)\| = 2$, $\|x - y\| = \|(1, -1)\| = 1 + \frac{1-\rho}{\rho} = \frac{1}{\rho}$. So, $\varphi(a, b) = 1 + \frac{1}{2\rho}$.

If $x = a = (1, 0), y = c = (-\rho, \rho)$, then $\|x + y\| = \|(1 - \rho, \rho)\| = 1 - \rho + \rho = 1$, $\|x - y\| = \|(1 + \rho, -\rho)\| = 1 + \rho + 1 - \rho = 2$. So, $\varphi(a, c) = 1 + \frac{1}{2} \le 1 + \frac{1}{2\rho}$.

Therefore $\max\{\varphi(x, y) : (x, y) \in B_{X_\rho} \times B_{X_\rho}\} = 1 + \frac{1}{2\rho}$, and consequently $\alpha(X_\rho) = 1 - \frac{1}{2\rho}$. The lemma is proved.

The set $\Pi(X_\rho)$ is the following polygon in $\mathbb{R}^2 \times \mathbb{R}^2$:
$\Pi(X_\rho) = \{(a, x^*) : x^* \in [f^*, a^*]\} \cup \{(x, a^*) : x \in [a, b]\} \cup \{(b, x^*) : x^* \in [a^*, b^*]\}$
$\cup \{(x, b^*) : x \in [b, c]\} \cup \{(c, x^*) : x^* \in [b^*, c^*]\} \cup \{(x, c^*) : x \in [c, d]\}$
$\cup \{(d, x^*) : x^* \in [c^*, d^*]\} \cup \{(x, d^*) : x \in [d, e]\} \cup \{(e, x^*) : x^* \in [d^*, e^*]\}$
$\cup \{(x, e^*) : x \in [e, f]\} \cup \{(f, x^*) : x^* \in [e^*, f^*]\} \cup \{(x, f^*) : x \in [f, a]\}$,
where we use brackets like $[\cdot, \cdot]$, $[\cdot, \cdot[$ to denote line segments in a linear space, for example, $[a, b] = \{\lambda b + (1 - \lambda)a : 0 \le \lambda \le 1\}$; and parenthesis $(\cdot, \cdot)$ are reserved to denote an element of a Cartesian product.

Theorem 2 *For every $\alpha \in [0, 1/2]$ there is a Banach space X with $\alpha(X) = \alpha$ such that*

$$\Phi_X^S(\varepsilon) \ge \sqrt{2\varepsilon}\sqrt{1 - \alpha(X)} \tag{2}$$

for all $0 < \varepsilon < 1$.

Proof. Let us demonstrate that the space $X = X_\rho$ with $\rho = \frac{1}{2(1-\alpha)}$ is what we are looking for. The direct application of lemma 1 gives $\alpha(X) = \alpha$, so what remains to show is (2).

Denote $x = (1 - \sqrt{\varepsilon\rho}, \sqrt{\varepsilon\rho}), x^* = (1, 1 - \sqrt{\varepsilon/\rho})$. Then, $x \in]a, b[$, $x^* \in]a^*, f^*[$ and $x^*(x) = 1 - \varepsilon$. In order to demonstrate (2) it is sufficient to prove the absence of such a pair $(y, y^*) \in \Pi(X)$ that $\max\{\|x - y\|, \|x^* - y^*\|\} < \sqrt{2\varepsilon}\sqrt{1 - \alpha}$.

Denote $r = \sqrt{2\varepsilon}\sqrt{1-\alpha}$ and consider the set U of those $y \in S_X$ that $\|x-y\| < r$. U is the intersection of S_X with the open ball of radius r centered in x (U is the bold line in Fig. 3). The radius of the ball equals to the distance from x to a:

$$\|x-a\| = \|(-\sqrt{\varepsilon\rho}, \sqrt{\varepsilon\rho})\| = \sqrt{\varepsilon\rho} + \frac{1-\rho}{\rho}\sqrt{\varepsilon\rho} = \sqrt{\varepsilon/\rho} = \sqrt{2\varepsilon}\sqrt{1-\alpha} = r,$$

which explains the picture for small r. Also for bigger values of r the set U can contain points b and c, but it never contains any point of $[d,e],[e,f]$ and $[f,a]$. Observe that the open ball of radius $1/\rho$ centered in b contains the set U, as if $h \in U$, we have $\|b-h\| \le \|b-x\| + \|x-h\| < \|b-x\| + \|x-a\| = \|b-a\| = 1/\rho$. Therefore it is sufficient to check that the distance from b to every point of $[d,e],[e,f]$ and $[f,a]$ is no less than $1/\rho$. Indeed, if $s = (-w, w-1)$ is a point of $[d,e]$ $(0 \le w \le 1)$, then

$$\|b-s\| = \|(w, 2-w)\| = w + 2 - w = 2 \ge 1/\rho.$$

If $s = (w, \frac{1-\rho}{\rho}w - 1)$ is a point of $[e,f]$, $0 \le w \le \rho$, and so

$$\|b-s\| = \|(-w, 1 - \frac{1-\rho}{\rho}w + 1)\| = \frac{1-\rho}{\rho}w + 2 - \frac{1-\rho}{\rho}w = 2 \ge 1/\rho.$$

If s is a point of $[f,a]$, $\rho \le w \le 1$, we shall consider cases $\rho < 1$ and $\rho = 1$ separately. For $\rho < 1$ we have $s = (w, -\frac{\rho}{1-\rho}(1-w))$, then

$$\|b-s\| = \|(-w, 1 + \frac{\rho}{1-\rho}(1-w))\| = \frac{\rho}{1-\rho}w + 1 + \frac{\rho}{1-\rho} - \frac{\rho}{1-\rho}w \ge 2 \ge 1/\rho.$$

And for $\rho = 1$ we have $s = (1, -w)$, $0 \le w \le 1$. Hence

$$\|b-s\| = \|(-1, 1+w)\| = max\{1, 1+w\} \ge 1 = 1/\rho.$$

So, $U \subset]a,b] \cup [b,c] \cup [c,d[$.

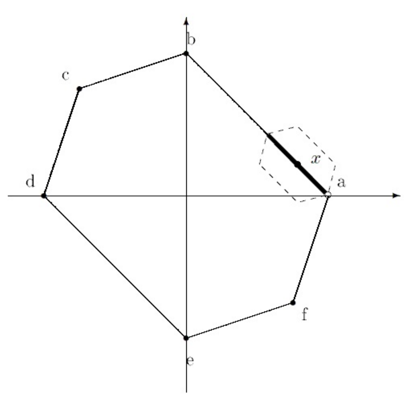

Fig. 3: The set U.

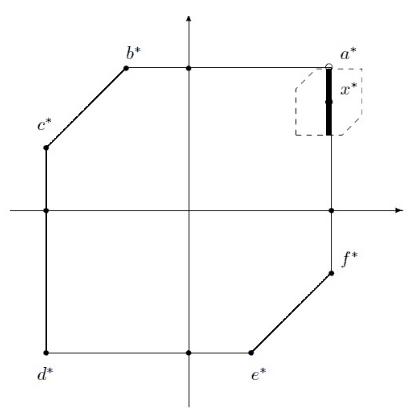

Fig. 4: The set V.

Consider also the set V of those $y^* \in S_{X^*}$ that $\|x^* - y^*\| < r$. V is the intersection of S_{X^*} with the open ball of radius r centered in x^* (the bold line in Fig. 4). The radius of the ball equals to the distance from x^* to a^*: $\|x^* - a^*\| = \|(0, -\sqrt{\varepsilon/\rho})\| = \sqrt{\varepsilon/\rho} = r$.

What remains to show is that $(y, y^*) \notin \Pi(X)$ for every $y \in U$ and every $y^* \in V$. The latter fact follows immediately form the above descriptions of the sets $\Pi(X_\rho)$ and U together with the fact that $V \subset]d^*, e^*] \cup [e^*, f^*] \cup [f^*, a^*[$.

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