Вісник Харківського національного університету імені В.Н. Каразіна
Серія "Математика, прикладна математика і механіка"
Том 81, 2015, с.20–35

Visnyk of V.N.Karazin Kharkiv National University
Ser. "Mathematics, Applied Mathematics and Mechanics"
Vol. 81, 2015, p. 20–35

УДК 537.84

The volumetric parametric resonance in magnetizable medium

N. F. Patsegon, S. I. Potseluiev

V. N. Karazin Kharkiv National University
Svobody Sq. 4, 61022, Kharkiv, Ukraine
dptmech@univer.kharkov.ua

The stability of magnetizable medium stationary states at parametric excitation of a magnetic field is studied. Parameters of exited acoustic wave and the influence of oscillating magnetic field on the dispersion of sound and its propagation velocity are determined using asymptotic and numerical methods.
Keywords: parametric resonance, oscillating magnetic field.

Пацегон М. Ф., Поцелуєв С. І., **Об'ємний параметричний резонанс в середовищах, що намагнічуються.** Вивчається можливість втрати стійкості стаціонарних станів намагнічуваних середовищ при їх параметричному збудженні магнітним полем. Асимптотичними та чисельними методами встановлені параметри збуджуваних акустичних хвиль, вплив осцилюючої частини магнітного поля на дисперсність збуджуваного звуку та швидкість його поширення.
Ключові слова: параметричний резонанс, осцилююче магнітне поле.

Пацегон Н.Ф., Поцелуев С.И., **Объемный параметрический резонанс в намагничивающихся средах.** Изучается возможность потери устойчивости однородных состояний намагничивающихся сред при их параметрическом возбуждении магнитным полем. Асимптотическими и численными методами установлены параметры возбуждаемых акустических волн, влияние осциллирующей части магнитного поля на дисперсность возбуждаемого звука и скорость его распространения.
Ключевые слова: параметрический резонанс, осциллирующее магнитное поле.

2000 Mathematics Subject Classification: 76N15, 76W05

Introduction

Magnetic fluids (MFs) are widely used in modern acoustical devices in order to increase their capacity, selectivity of certain sound frequencies and to increase their operational resource [1]. There are possibilities to use ferrofluids as converters of acoustic oscillations [2], a study of the connection between acoustic properties of (MFs) and their structure are of the grate interest for physico-chemistry of disperse systems in order to obtain the information about the stability, reconstruction times of microstructure and irreversible phenomena in the process of structure formation [3].The known results of ferrofluid acoustics are reduced to the study of the influence of magnetic field on the propagation velocity and absorption of ultrasonic vibrations [4]. In this paper we investigate the possibility of new excitation mechanisms of acoustic vibrations in (MFs) during the loss of stability of homogeneous fluid stationary states in oscillating magnetic field. This paper continues the study, initiated in [5], and earlier studies of the stability of ferrofluid free surface in oscillating magnetic and gravitational fields [6, 7].

1. Basic equations

Magnetizable medium and electromagnetic field form closed thermodynamic system. Therefore, dynamic equations of magnetizable medium take the form of conservation laws [8]:

1. Mass conservation

$$\frac{\partial \rho}{\partial t} + div \rho \vec{v} = 0 \tag{1}$$

2. Momentum conservation

$$\frac{\partial \rho v_i}{\partial t} = -\frac{\partial}{\partial x_k}(\rho v_i v_k - p_{ik}) \tag{2}$$

3. Energy conservation

$$\frac{\partial}{\partial t}\left(\rho u + \rho \frac{v^2}{2}\right) = -div \vec{J}_e \tag{3}$$

4. Entropy balance equation

$$\frac{\partial}{\partial t}\rho\ s = -div \vec{J}_s + \sigma_s. \tag{4}$$

Here and below the following notation are introduced as: ρ is the density of medium, $\vec{v}$ is the velocity, $\{p_{ik}\}$ is the Cauchy symmetric stress tensor; u, s are the density of the internal energy and the entropy; $\vec{J}_e, \vec{J}_s$ are flux density vectors of the energy and the entropy, σ_s is internal entropy production, $div \equiv \vec{\nabla} \cdot ()$, $rot \equiv \vec{\nabla} \times ()$.

Equations (1)-(4) are supplemented by equations of quasi-stationary electrodynamics of non-conductive medium:

$$div \vec{B} = 0,\ \ rot \vec{H} = 0,\ \ \frac{\partial \vec{B}}{\partial t} = -c\, rot \vec{E},\ \ \vec{B} = \vec{H} + 4\pi \vec{M}. \tag{5}$$

In equations (5) displacement currents are neglected, which is equivalently to the basic ferrohydrodynamics assumption about the same order of the characteristic frequency and size of changes of electromagnetic and hydrodynamic quantities.

Accepting the hypothesis of local equilibrium, the medium is concretized by the Gibbs identity in the form

$$du = Tds - pd\frac{1}{\rho} + \frac{\vec{H}}{4\pi} d\left(\frac{\vec{B}}{\rho}\right). \tag{6}$$

Here T is the temperature, p is the pressure, $\vec{H}, \vec{E}$ are strength of magnetic and electric fields, $\vec{B}$ is magnetic induction, $\vec{M}$ is the magnetization.

It should be noted that implementation of equation (6) does not depend on the way of magnetization of the medium (isotropic or anisotropic) [9].

Using methods of non-equilibrium thermodynamics [8], expressions for unknown flows in equations (1)-(4) are obtained

$$\begin{gathered} p_{ik} = -p\delta_{ik} + \tfrac{H_i B_k}{4\pi} + \tau_{ik}; \\ J_{ek} = \rho v_k (u + \tfrac{p}{\rho} + \tfrac{v^2}{2}) + \tfrac{c}{4\pi}[\vec{E}^*, \vec{H}]_k - \tfrac{(\vec{v}\vec{H})B_k}{4\pi} - v_i \tau_{ik} + q_k; \\ \vec{J}_s = \rho s \vec{v} + \tfrac{\vec{q}}{T}; \quad \sigma_s = \tfrac{1}{T}(\tau_{ik}\tfrac{\partial v_i}{\partial x_k} - \vec{q}\nabla T). \end{gathered} \tag{7}$$

Where $\{\tau_{ik}\}$ is the tensor of viscous stresses, $\vec{q}$ is the vector of heat flux density, $\vec{E}^* = \vec{E} + \frac{1}{c}[\vec{v}, \vec{B}]$ is the electric field strength in the proper reference frame.

Satisfying the second law of thermodynamics, i.e. inequality $\sigma_s \geq 0$, in the linear approximation of the Onsager theory, constitutive equations are obtained

$$\begin{gathered} \vec{q} = -\kappa \nabla T, \quad \tau_{ik} = 2\eta v_{ik} + (\varsigma - \tfrac{2}{3}\eta) v_{ee}\delta_{ik}\,, \\ \kappa \geq 0, \quad \eta \geq 0, \quad \varsigma \geq 0, \end{gathered} \tag{8}$$

where κ, η, ς are coefficients of conductivity, shear and bulk viscosities; $\{v_{ik}\}$ is the strain rate tensor.

Equations (1)-(5) should be supplemented by equations of the thermodynamic state. To obtain them, the thermodynamic potential f is introduced as

$$f = u - Ts - \frac{\vec{B}\vec{H}}{4\pi\rho}. \tag{9}$$

From the Gibbs equation (6) follows

$$df = -sdT + \frac{p}{\rho^2} d\rho - \frac{\vec{B} d\vec{H}}{4\pi\rho}.$$

Therefore

$$s = -(\frac{\partial f}{\partial T})_{\rho,\vec{H}}; \quad p = \rho^2 (\frac{\partial f}{\partial \rho})_{T,\vec{H}}; \quad \vec{B} = 4\pi\rho(\frac{\partial f}{\partial \vec{H}})_{\rho,T},$$

thus

$$s = s(\rho, T, \vec{H}); \quad p = p(\rho, T, \vec{H}); \quad \vec{B} = \vec{B}(\rho, T, \vec{H})$$

is the most common form of equations of the thermodynamic state. The final equation determines the magnetization law of the medium.

For isotropic magnetizable medium equations of state have following form:

$$\begin{aligned}
&\vec{B} = \mu\vec{H}; \mu = \mu(\rho, T, H);\\
&f = f^0(\rho, T) - \tfrac{1}{4\pi}\int_0^H \mu(\rho, T, H) H dH;\\
&p = p^0(\rho, T) + \psi \ ; \ \ s = s^0(\rho, T) + s^{(m)}\\
&\psi \ = \ \tfrac{1}{4\pi}\int_0^H [\mu - \rho(\tfrac{\partial\mu}{\partial\rho})_{T,H}] H dH \ ;\\
&s^{(m)} \ = \ \tfrac{1}{4\pi\rho}\int_0^H (\tfrac{\partial\mu}{\partial T})_{\rho,H} H dH;\\
&u = u^0(\rho, T) + \tfrac{BH}{4\pi\rho} - \tfrac{1}{4\pi}\int_0^H (\mu - T\mu_T) H dH.
\end{aligned} \tag{10}$$

Here μ is a magnetic permeability of the medium; the expression $f_\psi := \partial f/\partial\psi$ denotes the corresponding partial derivative, index " 0 " at the top marked thermodynamic functions of the medium in the absence of the field. These functions, which assumed known, satisfy the Gibbs equation in the absence of the field

$$du^0 = T ds^0 - p^0 d\frac{1}{\rho}.$$

Equations (1)-(5), (7)-(10) form a closed system of equations of the medium dynamics with the equilibrium magnetization and written as [11],[15]:

$$\begin{gathered}
\tfrac{d\rho}{dt} + \rho\, div\vec{v} = 0,\\
\rho\tfrac{d\vec{v}}{dt} = -\nabla p + M\nabla H + \eta\Delta\vec{v} + \left(\varsigma + \tfrac{1}{3}\eta\right)\nabla div\,\vec{v},\\
\rho T\tfrac{ds}{dt} = \kappa\Delta T + 2\eta v_{ik} v_{ik}\,,\\
div\vec{B} = 0, \quad rot\vec{H} = 0\,, \quad \vec{B} = \mu\vec{H}\,, \quad \mu = \mu(\rho, T, H)\,,\\
p = p^0(\rho, T) + \psi\,, \quad s = s^0(\rho, T) + s^{(m)}\,.
\end{gathered} \tag{11}$$

By virtue of (6), instead of the entropy equation in this system can be used the energy equation in the form

$$\begin{aligned}
&\tfrac{\partial}{\partial t}\left(\rho u + \rho\tfrac{v^2}{2}\right) = -\tfrac{\partial}{\partial x_k}\left[\rho v_k\left(u + \tfrac{p}{\rho} + \tfrac{v^2}{2} - \right.\right.\\
&\left.\left. -\tfrac{(\vec{B}\vec{H})}{4\pi\rho}\right) + \tfrac{c}{4\pi}\left[\vec{E}\times\vec{H}\right]_k - \kappa\,\tfrac{\partial T}{\partial x_k} - v_i\tau_{ik}\right].
\end{aligned} \tag{12}$$

2. Effective nonmagnetic medium, corresponding to magnetizable medium

One-dimensional unsteady motion of a magnetizable medium along x axis is considered. Then $v_x = v, v_y \equiv 0, v_z \equiv 0$ and besides

$$v = v(x,t), \quad \rho = \rho(x,t), \quad T = T(x,t), \quad \vec{H} = \vec{H}(x,t).$$

From equations of electrodynamics (5) follows

$$B_x = B_x(t), \quad H_y = H_y(t), \quad H_z = H_z(t).$$

Denote

$$B_x(t) = \chi_1(t), \; H_y(t) = \chi_2(t), \; H_z(t) = \chi_3(t).$$

Functions $\chi_i(t)$ are determined by boundary conditions.

Equations of motion (2) are reduced to the form:

$$\begin{gathered} \frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}\rho v = 0, \\ \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p_e}{\partial x} + \left(\zeta + \frac{4}{3}\eta\right)\frac{\partial^2 v}{\partial x^2}, \\ \rho T\frac{ds}{dt} = \kappa\frac{\partial^2 T}{\partial x^2} + 2\eta\left(\frac{\partial v}{\partial x}\right)^2. \end{gathered} \tag{13}$$

Taking into account that

$$div(\vec{E}\times\vec{H}) = \vec{H}rot\vec{E} - \vec{E}rotH = -\frac{1}{c}\vec{H}\frac{\partial\vec{B}}{\partial t}$$

the energy equation (12) is written as

$$\frac{\partial}{\partial t}\left(\rho u_e + \rho\frac{v^2}{2}\right) = -\frac{\partial}{\partial x}\left[\rho v(u_e + \frac{p_e}{\rho} + \frac{v^2}{2}) - \kappa\frac{\partial T}{\partial x} - v\tau_{11}\right] + \rho q.$$

Here the following notations are introduced:

$$\begin{gathered} p_e = p - \frac{\chi_1^2}{4\pi\mu}; \\ u_e = u - \frac{\mu}{4\pi\rho}(\chi_2^2 + \chi_3^2); \\ \rho q = \frac{1}{8\pi\mu}\frac{d}{dt}\chi_1^2 - \frac{\mu}{8\pi}\frac{d}{dt}(\chi_2^2 + \chi_3^2). \end{gathered} \tag{14}$$

Thus, equations of one-dimensional motion of magnetizable medium are reduced to equations of one-dimensional gas dynamics with special equations of state.

Equations have this form regardless from the way of magnetization (isotropic or anisotropic). It affects only on the equation of state, i.e. function p_e, u_e. The energy equation (14) differs from the ordinary equation of gas dynamics by the presence of term ρq on the right-hand side. Note that $q = 0$ if $\chi_i = const$ and this case was considered in [10]. The value $q \neq 0$ can be interpreted as a mass density of energy sources in the medium. This especially becomes clear from the Gibbs equation (6), which for one-dimensional motions of magnetizable media can be written as

$$du_e = Tds - p_e d\frac{1}{\rho} + qdt. \tag{15}$$

In the equation (15) the magnetic field strength is excluded. If $q = 0\,(\chi_i = const)$ this corresponds to a two-parametric medium with constitutive parameters: ρ and s, mass density of internal energy u_e and pressure p_e, besides

$$u_e = u_e(\rho, s); \quad p_e = p_e(\rho, s).$$

At $q \neq 0$ functions $\chi_i = \chi_i(t)$ are given by appropriate boundary conditions. They determine the energy exchange between the nonmagnetic medium and external bodies. They can be considered as external control of nonmagnetic medium from the external system, which is the magnetic field.

Nonmagnetic medium, defined by equations of state (14), below will be called an effective medium, corresponding to the initial magnetizable medium.

Equations (14) can be written in the form:

$$p_e(\rho, s, t) = p^0(\rho, T) - \frac{\chi_1^2}{4\pi\mu} + \frac{1}{4\pi}\int_0^H (\mu - \rho\mu_\rho)HdH,$$

$$u_e(\rho, s, t) = u^0(\rho, T) + \frac{\chi_1^2}{4\pi\rho\mu} - \frac{1}{4\pi\rho}\int_0^H (\mu - T\mu_T)HdH.$$

The temperature and the magnetic field strength in the right-hand side of equations must be excluded using relations:

$$T = T(\rho, s, \chi_i); \quad H = H(\rho, s, \chi_i)\,;$$

$$\chi_i := B_x(t)\,,\, H_y\,(t)\,,\, H_z\,(t).$$

To obtain them it is necessary to solve for T , H the following system of nonlinear functional equations:

$$\Phi = \mu(\rho, T, H)H - [\chi_1^2 + \mu^2(\rho, T, H)(\chi_2^2 + \chi_3^2)]^{\frac{1}{2}} = 0,$$

$$\Psi = s - s^0(\rho, T) - \frac{1}{4\pi}\int_0^H \mu_T HdH = 0.$$

Conditions for the solvability of this system of equation for the $T,\ H$ consist of the inequality

$$\frac{\partial(\Phi, \Psi)}{\partial(T, H)} \neq 0,$$

which assumed to be satisfied.

Thus, in the case of a linear isotropic magnetization, taking into account the magnetocaloric effect ($\mu = \mu(\rho, T)$), we have:

$$s = s^0(\rho, T) + \frac{1}{8\pi\mu^2\rho}[\mu^2(\chi_2^2 + \chi_3^2) + \chi_1^2]\mu_T,$$
$$H = [\chi_2^2 + \chi_1^2 + \mu^{-2}\chi_1^2]^{\frac{1}{2}}.$$

Then from the first equation the dependence $T = T(\rho, s, \chi_i)$ can be determined and the second equation gives necessary relation $H = H(\rho, s, \chi_i)$.

After that, equations of state of an effective medium are determined:

$$p_e = p_e(\rho, s, \chi_i) = p^0(\rho, T) + \frac{1}{8\pi\mu^2}[\mu^2(\mu -$$
$$-\rho\mu_\rho)(\chi_2^2 + \chi_3^2) - (\mu - \rho\mu_\rho)\chi_1^2],$$
$$u_e = u_e(\rho, s, \chi_i) = u^0(\rho, T) - \frac{1}{8\pi\mu^2}[\mu^2(\mu -$$
$$-T\mu_\rho)(\chi_2^2 + \chi_3^2) - (\mu - T\mu_\rho)\chi_1^2].$$

If a non-linear law of magnetization is considered and magnetocaloric effect can be neglected, i.e. $\mu = \mu(\rho, H)$, then $s = s^0(\rho, T)$, $T = T(\rho, s)$ and the dependence $H(\rho, \chi_i)$ is directly determined by the law of magnetization.

As equations of state of the effective medium depend on the time explicitly, such medium is non-stationary. This kind of medium has recently been studied in electrodynamics [12]. It should be noted that equations (13)-(15) are essentially nonlinear even in the case of an ideal medium because $p_e = p_e(\rho, s, t)$. They are quasi-linear only in the case $\chi_i = const$.

3. Excitation of acoustic vibrations in oscillating magnetic field

At non-stationary parameters $\chi_i = \chi_i(t)$ the equation (13) allows stationary homogeneous solution:

$$\rho \equiv \rho_0, \quad v = v_0 \equiv 0, \quad s \equiv s_0 = const.$$

In this case, the energy enters to effective medium according to the equation

$$\frac{\partial u_e}{\partial t} = q(t).$$

If magnetocaloric effect is neglected, the temperature of the medium will be constant: $T = T_0$. But when this effect is taken into account the temperature of the homogeneous state depends on the time: $T = T(t)$, so that the condition of adiabaticity is performed ($s = s_0 = const$). Furthermore, the magnetic field is homogeneous: $\vec{H} = \vec{H}(t)$. Depending on the type of source $q(t)$ in the magnetizable medium new effects, that have not previously been studied, become possible.

The solution of system (13) for not heat-conducting medium ($\kappa = 0$) is sought in the form

$$\rho = \rho_0 + \rho'(x,t), \quad v = v'(x,t),$$

where the prime denotes the perturbation of parameters.

By linearizing of equations (13) relative to homogeneous state, we obtain:

$$\begin{gathered}\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial v'}{\partial x} = 0\,; \\ \rho_0 \frac{\partial v'}{\partial t} + a^2 \frac{\partial \rho'}{\partial x} + \left(\zeta + \frac{4}{3}\eta\right) \frac{\partial^2 v'}{\partial x^2} = 0; \\ s \equiv s_0 = const.\end{gathered} \tag{16}$$

Here $a^2 = (\frac{\partial p_e}{\partial \rho})_{s,\chi_i(t)}$, i.e. derivative of the effective pressure is calculated at constant entropy s and given functions $\chi_i(t)$. Therefore

$$a^2 = a^2(\rho_0, s_0, \chi_1(t), \chi_2(t), \chi_3(t)) = a^2(t).$$

As shown in [10], a^2 is equal to the square of the velocity of sound propagation in magnetizable medium and given by the following expression [4]:

$$\begin{aligned}
&a^2(\rho, t) = L_0 - L_1 (1 + L_2)^{-1}; \\
&L_0 = \rho x_{31} + x_{23} x_{32}; \\
&L_1 = 4\pi\rho\mu^3 m^2 \left[\chi_1^2(t) + \mu^{-2}(\chi_2^2(t) + \chi_3^2(t))\right] \times \\
&\times(\rho\mu_\rho + N\mu_T x_{23}) \left[\rho(\mu_\rho + \mu_T T_\rho) + \mu_T T_{s^0} x_{23}\right]; \\
&L_2 = (\mu^2 \mu_T^2 T_{s^0} N m \left[\chi_1^2(t) + \mu^{-2}(\chi_2^2(t) + \chi_3^2(t))\right] - \\
&- \mu^2 \mu_H B^{-1}(\chi_2^2(t) + \chi_3^2(t)))(\mu^2 + \mu_H B)^{-1}; \\
&m^{-1} = 4\pi\rho\mu(\mu^2 + \mu_H B); \\
&N^{-1} = 1 + T_{s^0}(s_T^m - \mu_T m B^2); \\
&x_{23} = \rho N \left[m\mu_T B^2(\mu_\rho + \mu_T T_\rho) - s_\rho^m - s_T^m T_\rho\right]; \\
&x_{31} = (p_\rho^0 + \psi_\rho + \psi_T T_\rho)/\rho + \rho\mu_\rho m B^2(\mu_\rho + \mu_T T_\rho); \\
&x_{32} = (p_{s^0}^0 + \psi_T T_{s^0})/\rho + \rho T_{s^0}\mu_\rho \mu_T m B^2; \\
&T_{s^0} = \left(\frac{\partial T}{\partial s^0}\right)_\rho;\ p_{s^0}^0 = \left(\frac{\partial p^0}{\partial s^0}\right)_\rho.
\end{aligned} \tag{17}$$

Due to the potentiality of one-dimensional motions $v' = \partial\varphi/\partial x$, where $\varphi = \varphi(x,t)$ is the velocity potential. Then from the second equation of (16) the equation for density perturbations is obtained

$$\rho' = -\frac{\rho_0}{a^2}\left(\frac{\partial \varphi}{\partial t} + \nu_0 \frac{\partial^2 \varphi}{\partial x^2}\right), \quad \nu_0 = \frac{1}{\rho_0}\left(\zeta + \frac{4}{3}\eta\right). \tag{18}$$

This allows to get from the first equation of (16) the following equation for velocity potential

$$\frac{\partial^2 \varphi}{\partial t^2} - a^2 \frac{\partial^2 \varphi}{\partial x^2} + \nu_0 \frac{\partial^3 \varphi}{\partial x^2 \partial t} - \left[\frac{\partial \varphi}{\partial t} + \nu_0 \frac{\partial^2 \varphi}{\partial x^2}\right] \frac{d}{dt}(\ln a^2) = 0. \tag{19}$$

Trivial solution $\varphi = const$ of this equation corresponds to the equilibrium state of magnetic fluid $\rho = const; v = 0$. The stability analysis of this equilibrium state is performed below.

The solution of equation (19) is sought in the form

$$\varphi(x,t) = \varphi(t)e^{ikx}.$$

For the amplitude of the perturbation $\varphi(t)$ we get

$$\ddot{\varphi} + \left[k^2\nu_0 - \frac{d}{dt}\ln(a^2)\right]\dot{\varphi} + k^2\left[a^2 - \nu_0\frac{d}{dt}\ln(a^2)\right]\varphi = 0. \tag{20}$$

For further study of equation (20) it is necessary to specify the explicit form of $a^2(t)$, given by the expression (17). In the case of general isotropic law of magnetization, equations for equilibrium state of effective medium can be obtained only by using numerical methods. For the study of qualitative characteristics of excited acoustic oscillations in magnetic fluids, the most important case of isotropic magnetization is considered.

For an ideal paramagnet the magnetization is determined by the Langevin equation [11]:

$$M = mnL(\xi), \quad \xi = \frac{mH}{kT}, \quad L = cth\xi - \xi^{-1},$$

where: m is the magnetic momentum of ferromagnetic particle, n is the volume concentration, k is the Boltzmann constant.

Then in weak fields ($\xi << 1$) we obtain

$$\mu = 1 + \alpha\rho; \, \alpha = \frac{4\pi c_1 m^2}{3\mathcal{M}kT},$$

where c_1 is mass concentration of magnetic particles, $\mathcal{M}$ is the mass of a single ferromagnetic particle. If the temperature changes are neglected: $\alpha = const$. Then

$$a^2 = a_0^2 + \frac{(\mu-1)^2}{8\pi\mu^3}\chi_1^2; \tag{21}$$

$$p_e = p^0(\rho, s^0) + \tfrac{1}{8\pi}(\chi_2^2 + \chi_3^2) - \tfrac{2\mu-1}{8\pi\mu^2}\chi_1^2; \quad s^0 = s_0 = s; \quad a_0^2 = \tfrac{\partial p^0(\rho,s^0)}{\partial\rho}.$$

Here a_0^2 is the square of sound velocity in the medium in the absence of a magnetic field.

In this case it is obtained, that the magnetic field components, perpendicular to the direction of wave propagation, do not affect on the velocity of sound propagation. Moreover, the velocity of sound propagation along magnetic field direction is greater than in the absence of the field.

Suppose that the parameter χ_{10} is time-dependent according to harmonic law

$$\chi_1 = \chi_{10} + \beta\cos 2\omega t. \tag{22}$$

Then for the sound velocity in the medium we have

$$a^2(t) = a_0^2 + \frac{(\mu-1)^2}{4\pi\rho\mu^3}\left(\chi_{10}^2 + \frac{\beta^2}{2} + 2\chi_{10}\beta\cos 2\omega t + \frac{\beta^2}{2}\cos 4\omega t\right), \qquad (23)$$

where $a_0 = a_0(\rho_0, s_0)$, $\mu = \mu(\rho_0)$ are constant parameters, determined at equilibrium state. By substituting (23) in (20), the following equation is obtained

$$\begin{gathered}\frac{d^2\varphi}{d\tau^2} + [\psi_0 + 2\psi_{2s}\sin 2\tau + 2\psi_{4s}\sin 4\tau]\frac{d\varphi}{d\tau} + \\ +[\theta_0 + 2\theta_{2s}\sin 2\tau + 2\theta_{4s}\sin 4\tau + 2\theta_{2c}\cos 2\tau + 2\theta_{4c}\cos 4\tau]\,\varphi = 0,\end{gathered} \qquad (24)$$

where

$$\psi_0 = \frac{k^2\nu_0}{\omega},\quad \psi_{2s} = \frac{(\mu-1)^2\chi_{10}}{2\pi\rho\mu^3A^2}\beta,\quad \psi_{4s} = \frac{(\mu-1)^2}{4\pi\rho\mu^3A^2}\beta^2,$$

$$\theta_0 = \frac{k^2}{\omega^2}A^2,\quad \theta_{2s} = \frac{(\mu-1)^2k^2\nu_0\chi_{10}}{2\pi\rho\mu^3\omega A^2}\beta,\quad \theta_{2c} = \frac{(\mu-1)^2k^2\chi_{10}}{4\pi\rho\mu^3\omega^2}\beta,$$

$$\theta_{4s} = \frac{(\mu-1)^2k^2\nu_0}{4\pi\rho\mu^3\omega A^2}\beta^2,\quad \theta_{4c} = \frac{(\mu-1)^2k^2}{16\pi\rho\mu^3\omega^2}\beta^2,\quad A^2 = a_0^2 + \frac{(\mu-1)^2}{4\pi\rho\mu^3}\chi_{10}^2,$$

$\tau = \omega t$ is dimensionless time.

The equation (24) has periodic solutions, corresponding to acoustic waves.

4. Asymptotic solution

The equation (20) by substitution

$$\varphi(\tau) = a\,Z\exp\left(-\frac{k^2\nu_0\tau}{2\omega}\right) \qquad (25)$$

is reduced to the form

$$\frac{d^2Z}{d\tau^2} + \left[\frac{k^2a^2}{\omega^2} - \frac{2\nu_0k^2}{\omega a}\frac{da}{d\tau} - \left(\frac{k^2\nu_0}{16\omega} - \frac{1}{8a}\frac{da}{d\tau}\right)^2 + \frac{d}{d\tau}\left(\frac{1}{a}\frac{da}{d\tau}\right)\right]Z = 0.$$

In the case of time-dependent sound velocity in the form (23), by neglecting of terms of order β^2, the Hill equation for the function Z is obtained

$$\frac{d^2Z}{d\tau^2} + [\theta_0 - \psi_0^2 + 2(\theta_{1c} - \psi_{1s})\cos 2\tau + \theta_{1s}\sin 2\tau)]\,Z = 0. \qquad (26)$$

In the first approximation by the small parameter β marginal stability curves of the first unstable region is given by

$$\theta_0 = 1 + \psi_0^2 \pm ((\theta_{1c} - \psi_{1s})^2 + \theta_{1s}^2)^{1/2}.$$

As follows from (25), it is necessary to find an unstable solution of the equation (26). Using the method of Whittaker [13, 14], as a first approximation is taken

$$Z \;=\; e^{\gamma\tau}\,\sin(\tau-\sigma). \tag{27}$$

By substituting (27) in (26) and equating coefficients at $\sin\tau$ and $\cos\tau$, for the first unstable region is obtained

$$\begin{aligned} 2\gamma\; &= (\theta_{1c}-\psi_{1s})\sin 2\sigma - \frac{\theta_{1s}}{2}\cos 2\sigma,\\ \theta_0 &= 1+\psi_0^2-\gamma^2+\,(\theta_{1c}-\psi_{1s})\cos 2\sigma\,+\,\frac{\theta_{1s}}{2}\sin 2\sigma. \end{aligned} \tag{28}$$

From this

$$\begin{aligned} \gamma^2 &= \;-(1+\theta_0-\psi_0^2)\pm\Big(4(\theta_0-\psi_0^2)+(\theta_{1c}-\psi_{1s})^2+\tfrac{\theta_{1s}^2}{4}\Big)^{1/2},\\ tg\sigma &= \;\frac{(\theta_{1c}-\psi_{1s})\pm[(\theta_{1c}-\psi_{1s})^2+\frac{\theta_{1s}^2}{4}-4\gamma^2]^{1/2}}{2\gamma-\theta_{1s}/2}. \end{aligned} \tag{29}$$

Values $\gamma^2\geq 0$, $0\leq\sigma\leq\pi/2$ correspond to unstable solutions.

In the first approximation, in view of (25) and (27), the solution of the equation (20) is obtained

$$\varphi(t)\;\;=\;A\,\exp((\gamma\omega-k^2\nu_0/2)t)\,\sin(\omega t-\sigma).$$

This is periodic solution if the following condition

$$\gamma\;=\;\frac{k^2\nu_0}{2\omega}$$

is satisfied. Then the equation (19) for the velocity potential has periodic solution

$$\varphi(x,t)\;\;=\;A\,\exp(i(kx-\omega\,t+\sigma))\,,$$

which corresponds to the potential of small-amplitude waves, excited as a result of parametric instability, and propagating at the velocity ω/k. The frequency of excited waves is twice less then frequency of the parametric excitation.

Taking into account (29), the equation, that determines the magnitude of the wave vector depending on parametric excitation frequency, is obtained:

$$\begin{aligned} &\Big(1-\tfrac{k^2A^2}{\omega^2}\Big)^2+\tfrac{1}{16}\left(\tfrac{k^2\nu_0}{\omega}\right)^2\left[9\left(\tfrac{k^2\nu_0}{\omega}\right)^2+40\right]-\tfrac{3}{2}\tfrac{k^2A^2}{\omega^2}\,\tfrac{k^2\nu_0}{\omega}\;=\\ &=\left[\tfrac{\beta\chi_0(\mu-1)^2}{4\pi\rho\mu^3A^2}\right]^2\left[\Big(\tfrac{k^2A^2}{\omega^2}-2\Big)^2+\left(\tfrac{k^2\nu_0}{\omega}\right)^2\right]. \end{aligned} \tag{30}$$

Hence it follows that the excited waves are dispersive and the dispersion is a result of the viscosity of the medium.

In the case of an ideal medium the equation (30) has two solutions

$$\frac{\omega^2}{k^2} = A^2 (1 \pm \varepsilon), \quad \varepsilon = \frac{\beta \chi_0 (\mu - 1)^2}{4\pi\rho\mu^3 A^2}. \tag{31}$$

Thus in this case waves propagate without dispersion. As A is the wave velocity in a constant field, the oscillating part of the magnetic field can leads to either increase or decrease of their velocity.

Values (31) correspond to periodic solutions of the equation (26) and the value of parameters, which belong to the boundaries of stability regions. Therefore, at the same frequency of the magnetic field can be excited waves of different lengths.

5. Numerical solution

For the case of weak magnetic fields ($\xi << 1$) the equation (24) for the velocity potential was obtained. The equation (24) includes periodic functions of time, so the solution of this equation is sought in the Floquet form

$$\varphi(\tau) = e^{\gamma\tau} Y(\tau),$$

where $\gamma = s + i\alpha$ is the Floquet exponent; $Y(\tau)$ is a periodic function with period $\frac{\pi}{\omega}$, therefore it can be expanded in the Fourier series

$$Z(\tau) = \sum_{n=-\infty}^{\infty} \phi_{2n} e^{2n\tau i}.$$

Then

$$\varphi(\tau) = \sum_{n=-\infty}^{\infty} \phi_{2n} e^{q_{2n}\tau}, \quad q_{2n} = s + i(\alpha + 2n). \tag{32}$$

By substituting (32) in (24), we obtain

$$\begin{gathered}\sum_{n=-\infty}^{\infty} e^{q_{2n}\tau}[(q_{2n}^2 + q_{2n}\psi_0 + \theta_0)\phi_{2n} + \\ +(\theta_{2c} - i(\theta_{2s} + q_{2n}\psi_{2s}))\phi_{2n+1} + (\theta_{2c} + i(\theta_{2s} + q_{2n}\psi_{2s}))\phi_{2n-1} + \\ +(\theta_{4c} - i(\theta_{4s} + q_{2n}\psi_{4s}))\phi_{2n+2} + (\theta_{4c} + i(\theta_{4s} + q_{2n}\psi_{4s}))\phi_{2n-2}] = 0.\end{gathered} \tag{33}$$

In matrix form (33) can be written as

$$(C + \beta B + \beta^2 D)\phi = 0, \tag{34}$$

where C is diagonal matrix with complex coefficients, B and D are banded matrices with two and three subdiagonals:

$$C = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ & & & & \iddots \\ \dots & c_{-1,-1} & 0 & 0 & \dots \\ \dots & 0 & c_{0,0} & 0 & \dots \\ \dots & 0 & 0 & c_{1,1} & \dots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}; B = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \dots & 0 & b_{-1,0} & 0 & 0 & \dots \\ \dots & b_{0,-1} & 0 & b_{0,1} & 0 & \dots \\ \dots & 0 & b_{1,0} & 0 & b_{1,2} & \dots \\ \dots & 0 & 0 & b_{2,1} & 0 & \dots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix};$$

$$D = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \dots & d_{-1,-1} & 0 & d_{-1,1} & 0 & \dots \\ \dots & 0 & d_{0,0} & 0 & d_{0,2} & \dots \\ \dots & d_{1,-1} & 0 & d_{1,1} & 0 & \dots \\ \dots & 0 & d_{2,0} & 0 & d_{2,2} & \dots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix};$$

$$c_{n,n} = q_{2n}^2 + \frac{k^2\nu_0}{\omega}q_{2n} + \frac{k^2}{\omega^2}\left(a_0^2 + \frac{(\mu-1)^2}{4\pi\rho\mu^3}\chi_{10}^2\right); d_{n,n} = \frac{(\mu-1)^2}{8\pi\rho\mu^3}\frac{k^2}{\omega^2};$$

$$b_{n,n\pm1} = \frac{(\mu-1)^2}{2\pi\rho\mu^3}\left[\frac{k^2}{2\omega^2} \mp \frac{i(q_{2n} + \frac{\nu_0 k^2}{\omega})}{a_0^2 + (\mu-1)^2\chi_0^2/(4\pi\rho\mu^3)}\right]\chi_{10};$$

$$d_{n,n\pm2} = \frac{(\mu-1)^2}{4\pi\rho\mu^3}\left[\frac{k^2}{4\omega^2} \mp \frac{i(q_{2n} + \frac{\nu_0 k^2}{\omega})}{a_0^2 + (\mu-1)^2\chi_0^2/(4\pi\rho\mu^3)}\right].$$

In the case of pure oscillating magnetic field $\chi_{10} = 0 : b_{n,n\pm1} = 0$. Then by inverting of the matrix C, from (34) follows the ordinary eigenvalue problem:

$$(C^{-1}D)\phi = \frac{1}{\beta^2}\phi\,. \tag{35}$$

At the stability analysis is usually used the following procedure [16]: the first step is to fix the wavenumber k and the amplitude β, as well as values of other hydrodynamic parameters of the system, and then the Floquet exponent $\gamma = s+i\alpha$ is calculated. Marginal stability curves in the plane (k, β) are curves on which $s(k, \beta) = 0$. This condition is satisfied by interpolation of β at fixed k between negative and positive values of s.

But in our calculations the method described in [17] is used: the Floquet exponent $\gamma = s + i\alpha$ is pre-fixed, then the eigenvalue problem (35) is solved at fixed value of k. The largest real positive eigenvalue $\frac{1}{\beta^2}$, corresponding to a minimum amplitude β, is sought by interpolation of k. To construct marginal stability curves in the plane (k, β) we have to set $s = 0$ and $\alpha = 0$ ($\alpha = 1$), which corresponds to the case of harmonic (subharmonic) oscillations. The above method for calculation of boundaries of instability regions is used to solve the problem (35). Matrices A and D are cut to size, providing the required accuracy of calculations. In all calculations the typical ferrofluid parameters were accepted

$$\nu = 0.1(\mathrm{P}),\ \mu = 2,\ \sigma = 30\left(\frac{\mathrm{erg}}{\mathrm{cm}^2}\right),\ \rho = 1.2\left(\frac{\mathrm{g}}{\mathrm{cm}^3}\right),\ a_0 = 1.5\cdot 10^5\left(\frac{\mathrm{cm}}{\mathrm{s}}\right).$$

Boundaries of the first two unstable regions (the Ince-Strutt diagram for a viscous fluid) is shown on Fig.1.a) and Fig.1.b). Marginal stability curves form narrow regions ("tongues"), the value of parameters outside (inside) of these regions corresponds to stability (instability). The absolute minimum of this curves

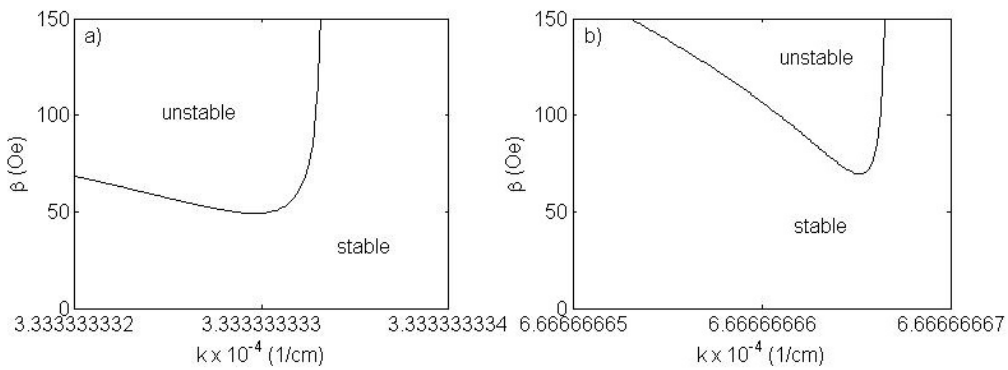

Fig. 1. a) The first and b) the second region of parametric instability at excitation frequency $\omega = 100$ (Hz) of magnetic field.

determines the critical wavenumber k_c and the critical amplitude β_c, at which instability occurs.

Fig.2.a) shows, that if magnetic field frequency increases, acoustic waves with less wavelength are excited. Moreover, at increasing of frequency for excitation of parametric instability must be applied the oscillating field of greater amplitude (see Fig.2.b))

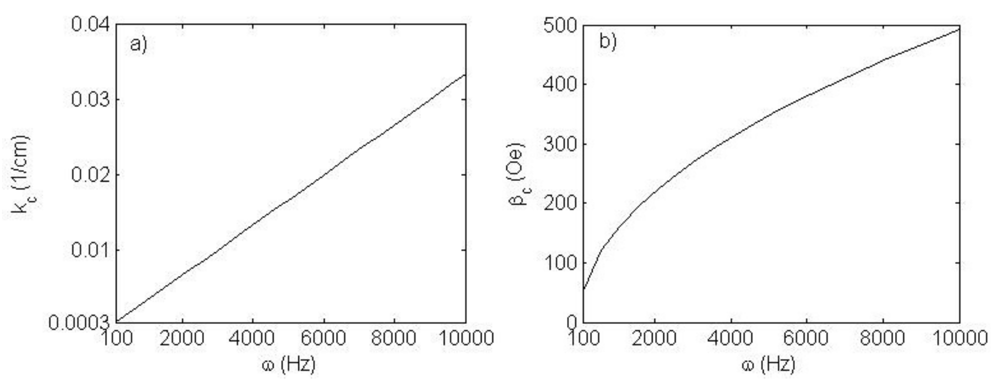

Fig. 2. The dependence of a) the critical wavenumber k_c and b) the critical amplitude β_c on the frequency ω of oscillating magnetic field.

In the case, when the magnetic field consist of constant and oscillating parts, the eigenvalue problem (34) must be solved. Using the column vector $\phi := \beta\xi$, the equation (34) reduces to the ordinary eigenvalue problem for matrix doubled in size

$$\begin{pmatrix} -D^{-1}B & -D^{-1}C \\ I & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \xi \end{pmatrix} = \beta \begin{pmatrix} \phi \\ \xi \end{pmatrix}, \tag{36}$$

where I is the identity matrix, which has the same size as B, C and D.

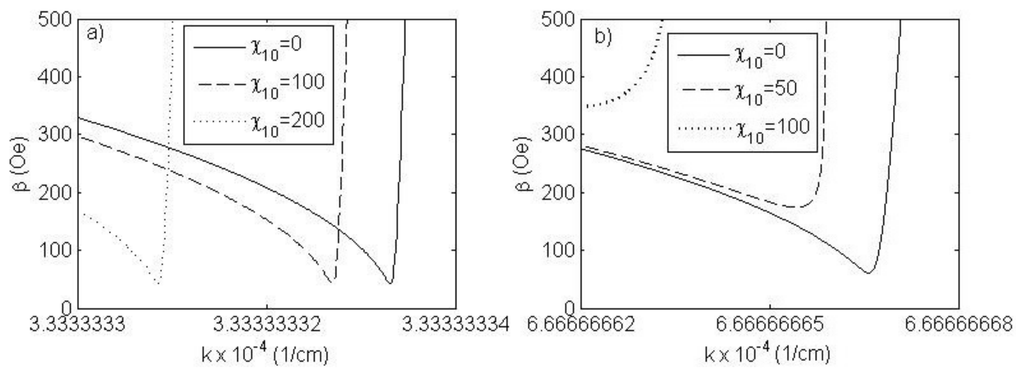

Fig. 3. a) The first and b) the second region of parametric instability for different values of stationary field χ_{10} and frequency $\omega = 100$ (Hz).

Similarly to the previous case, to construct regions of parametric instability in the plane of parameters (k, β) at fixed values of χ_{10}, the smallest real positive eigenvalue β of the problem (36) is sought. The calculation revealed that stationary component of the magnetic field has less (greater) impact on the structure of odd (even) instability regions. For the first unstable region at increasing of χ_{10} critical amplitude β_c remains almost unchanged, but exited sound waves have larger wavelength (see Fig.3.a)). Whereas for the second unstable region Fig.3.b) shows, that if χ_{10} increases, the critical amplitude β_c also increases, i.e. instability threshold shifts to higher values.

Conclusions

The parametric instability of ferrofluid volumes in weak homogeneous magnetic field, which consist of constant and oscillating parts, is considered. The appearance of unstable zones is studied. The problem was reduced to the Hill equation, which is studied using asymptotic and numerical methods. Marginal stability curves, that form narrow unstable regions corresponding to acoustical oscillations in ferrofluid, were obtained. The dependence of a structure of unstable tongues on the frequency ω and constant part χ_{10} of magnetic field is studied. It is shown, that increasing of ω leads to increasing of critical wavenumber k_c and critical amplitude β_c of magnetic field, required for the onset of instability. Also the increasing of χ_{10} causes to the appearance of shorter wavelength and can shifts a threshold of instability.

REFERENCES

1. Taketomi S., Tikadzumi S. Magnetic fluids. — M.: Mir, 1993. — 272 p. (in Rus.)

2. Polunin V. M. Acoustics of Nanodispersed Magnetic Fluids. — CRC Press, 2015. — 453 p.
3. Patsegon N. F., Popova L. N. Spatial structures in magnetizable fluids. / Journ. of Math. Sci., 2012. — V. 180, **2**. — P. 175–186.
4. Tarapov I. E. Simple waves in nonconducting magnetizable medium. / Journ. of Appl. Math. Mech. (PMM), 1973. — V. 37, **5**. — P. 813–821.
5. Patsegon N. F., Potseluiev S. I. Volumetric parametric resonance in magnetic fluids, the excitation of acoustic oscillations. / International Mathematical Conference "Differential Equations, Computational Mathematics, Theory of functions and mathematical methods of mechanics". — Kyiv, Taras Shevchenko National University, April 23-24, 2014. — 99 p. (in Rus.)
6. Potseluiev S. I., Patsegon N. F. Parametric instability of the free surface of nonlinear magnetizable fluid. / Contemporary problems of mathematics, mechanics and computing sciences, — Kharkov: "Apostrophe 2011. — P. 104–120.
7. Patsegon N. F., Potseluiev S. I. The stability of a free surface of viscous magnetizable fluid at multiparametric excitation. / Applied hydromechanics, 2014. — **16**(88). — No 3. — P. 36–51. (in Rus.)
8. De Groot S. R., Mazur P. Non-Equilibrium Thermodynamics. — Dover Pub., 2011. — 526 p.
9. Landau L. D., Lifshitz E. M. Electrodynamics of Continuous Media. – Pergamon Press, 2004.— 476 p.
10. Patsegon N. F. General properties of wave motion in nonconducting magnetized media. / Magnetohydrodynamics, 1990. — V. 26. — **3**. — P. 279–283.
11. Rosenzweig R. Ferrohydrodynamics. — Dover Pub., 2014. — 344 p.
12. Nerukh A., Sakhnenko N., Benson T., Sewell P. Non-stationary Electromagnetics. — Pan Stanford Pub., 2012. — 616 p.
13. Nayfeh A. Perturbation methods. – Wiley-VCH, 2000. — 437 p.
14. McLachlan N.W. Theory and application of Mathieu functions. – Oxford University Press, 1951. — 412 p.
15. Tarapov I. E. Continuum Mechanics. Part 2. — Kharkiv: Golden Pages, 2002. — 514 p. (in Rus.)
16. Frolov K. V. at al. Vibration in the technique. Directory in 6 volumes /Editor: Chelomei V. N. – M. : Mechanical Engineering, 1978.— **1**. — Vibrations of linear systems / Editor: Bolotin V. V. — 352 p. (in Rus.)
17. Kumar K., Tuckerman. L. S. Parametric instability of the interface between two fluids. / Journ. Fluid Mech., 1994. — **279**. — p. 49–68.

Article history: Received: April 17, 2015; Final form: 21 May 2015;
Accepted: 25 May 2015