Вісник Харківського національного університету імені В.Н. Каразіна
Серія "Математика, прикладна математика і механіка"
УДК 517.982.22 № 1133, 2014, с.103–113

Stability of unconditional Schauder decompositions in Hilbert spaces

V. A. Marchenko
V.N. Karazin Kharkiv National University,
Svobody Sq., 4, 61022, Kharkiv, Ukraine
vitalii.marchenko@karazin.ua

We obtain the stability theorem for unconditional Schauder decompositions in Hilbert spaces. This result is a generalization of the classical theorem of T. Kato on similarity for sequences of projections in Hilbert spaces to the case of unconditional Schauder decompositions. Also we sharpen one theorem of V.N. Vizitei on the stability of Schauder decompositions in the case of unconditional Schauder decompositions.
Keywords: unconditional Schauder decomposition, projection, isomorphism.

Марченко В. А., **Стійкість безумовних розкладів Шаудера у гільбертових просторах.** Отримано теорему стійкості для безумовних розкладів Шаудера у гільбертових просторах. Цей результат є узагальненням класичної теореми Т. Като про подібність послідовностей проекторів у гільбертових просторах на випадок безумовних розкладів Шаудера. Також ми уточнюємо одну теорему В. Н. Візітея про стійкість розкладів Шаудера у випадку безумовних розкладів Шаудера.
Ключові слова: безумовний розклад Шаудера, проектор, ізоморфізм.

Марченко В. А., **Устойчивость безусловных разложений Шаудера в гильбертовых пространствах.** Получена теорема устойчивости для безусловных разложений Шаудера в гильбертовых пространствах. Этот результат является обобщением классической теоремы Т. Като о подобии последовательностей проекторов в гильбертовых пространствах на случай безусловных разложений Шаудера. Также мы уточняем одну теорему В.Н. Визитея об устойчивости разложений Шаудера.
Ключевые слова: безусловное разложение Шаудера, проектор, изоморфизм.

2000 Mathematics Subject Classification: 47A46, 46B15, 47B40.

1. Introduction

In 1940 came the publication of the work [14], dedicated to the well known S. Banach's problem – basis problem. In this paper an abstract theorem of stability of arbitrary bases in Banach spaces was first obtained. One of the main consequences of this Krein-Milman-Rutman theorem says that, in any Banach space with a basis, a basis may be formed from arbitrary dense set. In the present time this theorem has many generalizations, analogs and applications, see, e.g., [15, 23]. In 1951 N.K. Bari [3] opened the topic of the stability of bases, introduced the term and studied the properties of Riesz basis, and showed, inter alia, that any minimal system, quadratically close to the Riesz basis, is itself a Riesz basis.

The concept of Schauder decomposition (or basis of subspaces) is a natural generalization of the Schauder basis concept and was first introduced in 1950 by M.M. Grinblyum in [10]. In the same year, independently, M.K. Fage in [7, 8] proposed and studied this concept in Hilbert spaces. In 1960 A.S. Marcus generalize some results of N.K. Bari to the case of unconditional Schauder decompositions and, using the results obtained, establish certain conditions under which a dissipative operator has Bari basis of root subspaces, and the union of orthonormal bases from these subspaces forms Riesz basis or Bari basis, see [17].

Nowadays, Schauder decompositions together with Schauder bases are powerful tools of functional analysis and infinite dimensional linear systems theory, see [6, 20, 21, 22, 27]. About Schauder decompositions see, e.g., [15, 24, 4, 9].

Throughout what follows H will denote a Hilbert space with norm $\|\cdot\|$ and a scalar product $\langle\cdot,\cdot\rangle$, and $\mathbb{Z}_+$ will denote a set of nonnegative integers. In 1967 T. Kato published the following result.

Theorem 1 (T. Kato [13]) *Suppose that $\{P_n\}_{n=0}^{\infty}$ is a sequence of nonzero selfadjoint projections in H satisfying $\sum\limits_{n=0}^{\infty} P_n = I$, $P_nP_m = \delta_n^m P_n$ for $n, m \in \mathbb{Z}_+$, and let $\{J_n\}_{n=0}^{\infty}$ be a sequence of nonzero projections in H, such that $J_nJ_m = \delta_n^m J_n$ for $n, m \in \mathbb{Z}_+$. Also assume that*

$$\dim P_0 = \dim J_0 = m < \infty, \tag{1}$$

$$\sum_{n=1}^{\infty} \|P_n(J_n - P_n)x\|^2 \leq c^2\|x\|^2 \quad \text{for all } x \in H, \tag{2}$$

where c is a constant satisfying $0 \leq c < 1$. Then $\{J_n\}_{n=0}^{\infty}$ is similar to $\{P_n\}_{n=0}^{\infty}$, that is, there exists an isomorphism S, such that $J_n = S^{-1}P_nS$ for $n \in \mathbb{Z}_+$.

This result gave a new impetus to the development of the spectral theory. It is an effective tool for the analysis of spectral properties of various perturbations of operators in H and even 45 years later retains its relevance. In 1968 C. Clark [5] applied Theorem 1 to the study of spectral properties of relatively bounded perturbations of ordinary differential operators. In 1972 E. Hughes [11] used Theorem 1 in the proof of some perturbation theorems for relative spectral problems. T. Kato in [12] considered the problem of completeness of eigenprojections for slightly nonselfadjoint operators as a perturbation problem for selfadjoint operators and based the solution of this problem on his Theorem 1.

In 2012 J. Adduci and B. Mityagin applied Theorem 1 to the study of eigenfunction expansions of the perturbed harmonic oscillator $L = -\frac{d^2}{dx^2} + x^2 + B$, $B = b(x)$, with dense domain in $L_2(\mathbb{R})$ [1], and to the analysis of the perturbation $A = T + B$ of a selfadjoint operator T in a Hilbert space H with discrete spectrum [2]. Just recently, Theorem 1 was applied by B. Mityagin and P. Siegl to the study of the root system of singular perturbations of the harmonic oscillator type operators [18].

The purpose of the present paper is the study of stability of unconditional Schauder decompositions in Hilbert spaces. More precisely, the aim is to generalize Theorem 1, considering unconditional Schauder decompositions instead of orthogonal Schauder decompositions. It was found that the sequence of subspaces, corresponding to mutually disjoint projections, which are close in a certain sense to projections of unconditional Schauder decomposition of given structure, is itself an unconditional Schauder decomposition. As a direct consequence of this result we obtain one stability theorem for Riesz bases of sufficiently general structure in H. Also we sharpen one theorem of V.N. Vizitei on the stability of Schauder decompositions, which was published in [25], in the case of unconditional Schauder decompositions in H.

2. One lemma on unconditional Schauder decompositions in H

Throughout the paper we will use the following definitions.

Definition 1 ([24]) *A sequence $\{\mathfrak{M}_n\}_{n=0}^{\infty}$ of closed nonzero linear subspaces of H is called a Schauder decomposition of H provided each $x \in H$ has a unique, norm convergent expansion $x = \sum\limits_{n=0}^{\infty} x_n$, where $x_n \in \mathfrak{M}_n$ for $n \in \mathbb{Z}_+$.*

Definition 2 ([24]) *A Schauder decomposition $\{\mathfrak{M}_n\}_{n=0}^{\infty}$ of H is called 2-Besselian provided the convergence of $\sum\limits_{n=0}^{\infty} x_n$ in H, where $x_n \in \mathfrak{M}_n$, $n \in \mathbb{Z}_+$, implies the convergence of $\sum\limits_{n=0}^{\infty} \|x_n\|^2$.*

Definition 3 ([24]) *A pair of sequences $(\{\mathfrak{M}_n\}_{n=0}^{\infty}, \{P_n\}_{n=0}^{\infty})$, where $\{\mathfrak{M}_n\}_{n=0}^{\infty}$ is a sequence of closed nonzero linear subspaces of H and $\{P_n\}_{n=0}^{\infty}$ is a sequence of bounded linear projections satisfying $P_n H = \mathfrak{M}_n$ for all n, will be called a generalized biorthogonal system provided it satisfies $P_i P_j = \delta_i^j P_i$ for $i, j \in \mathbb{Z}_+$. The generalized biorthogonal system $(\{\mathfrak{M}_n\}_{n=0}^{\infty}, \{P_n\}_{n=0}^{\infty})$ is said to be H-complete, if $\overline{Lin}\{\mathfrak{M}_n\}_{n=0}^{\infty} = H$.*

Definition 4 ([24]) *A sequence of nonzero subspaces of H is said to be ω-linearly independent, if the relations $\sum_{n=0}^{\infty} x_n = 0$, $x_n \in \mathfrak{M}_n$, $n \in \mathbb{Z}_+$, imply $x_n = 0$, $n \in \mathbb{Z}_+$.*

Definition 5 *A Schauder decomposition $\{\mathfrak{M}_n\}_{n=0}^{\infty}$ will be called unconditional with constant M provided there exists $M \geq 1$ such that*

$$\left\| \sum_{i=0}^{n} \delta_i y_i \right\| \leq M \left\| \sum_{i=0}^{n} y_i \right\| \text{ for all } n \in \mathbb{Z}_+,\ y_n \in \mathfrak{M}_n,\ \{\delta_i\}_{i=0}^{n} \in \{0,1\}.$$

For example, every orthogonal Schauder decomposition in H is unconditional with constant $M = 1$. The following lemma provides some properties of unconditional Schauder decompositions in H and will be used further.

Lemma 1 *Assume that $\{\mathfrak{M}_n\}_{n=0}^{\infty}$ is an unconditional Schauder decomposition in H with constant M and corresponding sequence of projections $\{P_n\}_{n=0}^{\infty}$. Then for every $x \in H$ we have*

$$\frac{1}{2M}\left(\sum_{n=0}^{\infty} \|P_n x\|^2\right)^{\frac{1}{2}} \leq \|x\| \leq 2M\left(\sum_{n=0}^{\infty} \|P_n x\|^2\right)^{\frac{1}{2}}. \tag{3}$$

Proof. We note that, by the parallelogram identity, for each $x \in H$ and for every finite set of elements $\{P_j x\}_{j=0}^{n} \subset H$ there exists a set of numbers $\{\underline{\varepsilon}_j\}_{j=0}^{n} \subset \{-1,1\}$ such that

$$\left\| \sum_{j=0}^{n} \underline{\varepsilon}_j P_j x \right\|^2 = \min_{\varepsilon_j = \pm 1} \left\| \sum_{j=0}^{n} \varepsilon_j P_j x \right\|^2 \leq \frac{1}{2^{n+1}} \sum_{\varepsilon_j = \pm 1} \left\| \sum_{j=0}^{n} \varepsilon_j P_j x \right\|^2 = \sum_{j=0}^{n} \|P_j x\|^2. \tag{4}$$

Construct the following operators: $P_n^+ = \sum_{j:\underline{\varepsilon}_j = 1} P_j$, $P_n^- = \sum_{j:\underline{\varepsilon}_j = -1} P_j$. Further, applying (4), we obtain that

$$\|x\|^2 = \left(\lim_{n\to\infty} \left\| \sum_{j=0}^{n} P_j x \right\|\right)^2 = \left(\lim_{n\to\infty} \left\|\left(P_n^+ + P_n^-\right) x\right\|\right)^2 = \lim_{n\to\infty} \left\|(P_n^+ - P_n^-)^2 x\right\|^2$$

$$\leq 4M^2 \lim_{n\to\infty} \left\|(P_n^+ - P_n^-)x\right\|^2 = 4M^2 \lim_{n\to\infty} \left\| \sum_{j=0}^{n} \underline{\varepsilon}_j P_j x \right\|^2 \leq 4M^2 \sum_{j=0}^{\infty} \|P_j x\|^2.$$

Hence, a right-hand side of the inequality (3) is proved.

To prove a left-hand side of the inequality (3) we observe that, by the parallelogram identity, for each $x \in H$ and for every finite set of elements $\{P_j x\}_{j=0}^{n} \subset H$ there exists a set of numbers $\{\overline{\varepsilon}_j\}_{j=0}^{n} \subset \{-1,1\}$ such that

$$\left\| \sum_{j=0}^{n} \overline{\varepsilon}_j P_j x \right\|^2 = \max_{\varepsilon_j = \pm 1} \left\| \sum_{j=0}^{n} \varepsilon_j P_j x \right\|^2 \geq \frac{1}{2^{n+1}} \sum_{\varepsilon_j = \pm 1} \left\| \sum_{j=0}^{n} \varepsilon_j P_j x \right\|^2 = \sum_{j=0}^{n} \|P_j x\|^2. \tag{5}$$

Further, for every set of numbers $\{\overline{\varepsilon}_j\}_{j=0}^n \subset \{-1,1\}$ there exist two sets of numbers $\{\delta_j^+\}_{j=0}^n \subset \{0,1\}$ and $\{\delta_j^-\}_{j=0}^n \subset \{0,1\}$ such that

$$\left\|\sum_{j=0}^n \overline{\varepsilon}_j P_j x\right\| = \left\|\sum_{j=0}^n \delta_j^+ P_j x - \sum_{j=0}^n \delta_j^- P_j x\right\| \le 2M\|x\|.$$

Taking into account (5), we obtain

$$\sum_{j=0}^{\infty} \|P_j x\|^2 = \lim_{n\to\infty} \sum_{j=0}^n \|P_j x\|^2 \le 4M^2\|x\|^2,$$

which completes the proof of left-hand side of (3).

The lemma just proved is a slight variation of one lemma from [16, 26]. Note that Lemma 1 without specification of the constants in (3) follows from one lemma, which was obtained by W. Orlicz in [19]. Lemma 1 leads to the following remark of geometric nature.

Corollary 1 *Let Schauder decomposition $\{\mathfrak{M}_n\}_{n=0}^{\infty}$ in H is unconditional with constant M and corresponding sequence of projections $\{P_n\}_{n=0}^{\infty}$. Then every $x \in H$ is contained outside the open ball $B\left(0, \frac{1}{2M}\left(\sum\limits_{n=0}^{\infty}\|P_n x\|^2\right)^{\frac{1}{2}}\right)$ and inside the closed ball $B\left[0, 2M\left(\sum\limits_{n=0}^{\infty}\|P_n x\|^2\right)^{\frac{1}{2}}\right)$ of the space H, i.e. in the closed ring.*

3. Theorem of V.N. Vizitei and unconditional decompositions in H

Lemma 1, together with Theorem 15.17 from [24], which was obtained by V.N. Vizitei in 1965 [25], allow us to obtain the following stability theorem, which is valid for every unconditional Schauder decomposition in H. Thereby, we sharpen a theorem of V.N. Vizitei in a following way.

Theorem 2 *Assume that $\{\mathfrak{M}_n\}_{n=0}^{\infty}$ is an unconditional Schauder decomposition in H with corresponding sequence of projections $\{P_n\}_{n=0}^{\infty}$. Then the following statements hold.*

(i) There exists a constant $\lambda \in (0,1)$, such that every sequence of subspaces $\{\mathfrak{N}_n\}_{n=0}^{\infty}$ in H satisfying

$$\left(\sum_{n=0}^{\infty} \theta\left(\mathfrak{M}_n, \mathfrak{N}_n\right)^2\right)^{\frac{1}{2}} \le \lambda, \qquad (6)$$

where $\theta\left(\mathfrak{M}, \mathfrak{N}\right) = \max\left\{\sup\limits_{x\in\mathfrak{M}, \|x\|=1} dist(x, \mathfrak{N}), \sup\limits_{y\in\mathfrak{N}, \|y\|=1} dist(y, \mathfrak{M})\right\}$ is the opening of the subspaces $\mathfrak{M}$, $\mathfrak{N}$, is a Schauder decomposition in H, isomorphic to

$\{\mathfrak{M}_n\}_{n=0}^{\infty}$. *Moreover, a constant* λ *may be chosen as*

$$\lambda = \left(4 \sup_{0\le n<\infty} \left\| \sum_{j=0}^{n} P_j \right\| \left(1 + \sup_{0\le n<\infty} \|P_n\|\right)^2\right)^{-1}.$$

(ii) Every sequence of subspaces $\{\mathfrak{N}_n\}_{n=0}^{\infty}$ *in* H*, satisfying*

$$\sum_{n=0}^{\infty} \theta\left(\mathfrak{M}_n, \mathfrak{N}_n\right)^2 < \infty, \tag{7}$$

and admitting a sequence of projections $\{J_n\}_{n=0}^{\infty}$*, such that* $(\{\mathfrak{N}_n\}_{n=0}^{\infty}, \{J_n\}_{n=0}^{\infty})$ *is an* H*-complete generalized biorthogonal system, is* 2*-Besselian Schauder decomposition of* H*. If, additionally,* $\dim \mathfrak{M}_n < \infty$*,* $n \in \mathbb{Z}_+$*, then the same conclusion holds for every* ω*-linearly independent sequence of subspaces* $\{\mathfrak{N}_n\}_{n=0}^{\infty}$ *satisfying (7).*

Note that every sequence of subspaces $\{\mathfrak{N}_n\}_{n=0}^{\infty}$, isomorphic to unconditional Schauder decomposition $\{\mathfrak{M}_n\}_{n=0}^{\infty}$ with constant M, is itself an unconditional Schauder decomposition with constant $M\|S\|\|S^{-1}\|$, where $\mathfrak{N}_n = S\mathfrak{M}_n$, $n \in \mathbb{Z}_+$.

4. A generalization of a theorem of T. Kato

The main result of the paper is formulated as follows.

Theorem 3 *Let* $\{\mathfrak{N}_n\}_{n=0}^{\infty}$ *is an orthogonal Schauder decomposition in* H *with corresponding sequence of projections* $\{F_n\}_{n=0}^{\infty}$*, where* $\dim F_0 < \infty$*, and assume that* $\{\mathfrak{M}_n\}_{n=0}^{\infty}$ *is an unconditional Schauder decomposition in* H *with constant* M *and corresponding sequence of projections* $\{P_n\}_{n=0}^{\infty}$*, where* $P_0 = F_0$*. Also suppose that* $\{J_n\}_{n=0}^{\infty}$ *is a sequence of nonzero projections in* H *such that* $J_n J_m = \delta_n^m J_n$ *for* $n, m \in \mathbb{Z}_+$*. If the condition (1) holds and for all* $x \in H$ *we have*

$$\sum_{n=1}^{\infty} \|P_n(J_n - P_n)x\|^2 \le \varsigma^2 \|x\|^2, \tag{8}$$

where $\varsigma \in \left[0, \frac{1}{2M}\right)$*, then* $\{J_n H\}_{n=0}^{\infty}$ *is also an unconditional Schauder decomposition in* H*, isomorphic to* $\{\mathfrak{M}_n\}_{n=0}^{\infty}$*.*

Proof. To prove the theorem we use the method which was used in [13]. Consider the operator S defined on H by

$$S = \sum_{n=0}^{\infty} P_n J_n. \tag{9}$$

To prove the existence of S in the strong sense we show that

$$\sum_{n=0}^{\infty} (P_n - P_n J_n) = \sum_{n=0}^{\infty} P_n \left(P_n - J_n\right)$$

converges in the strong sense. Indeed, since $\{\mathfrak{M}_n\}_{n=0}^{\infty}$ is an unconditional Schauder decomposition in H with constant M, for each $x \in H$ and for every $N \in \mathbb{Z}_+$ we have by virtue of Lemma 1, using (8), that

$$\left\| \sum_{n=k}^{k+N} P_n (P_n - J_n) x \right\|^2 \leq (2M)^2 \sum_{j=0}^{\infty} \left\| P_j \left(\sum_{n=k}^{k+N} P_n (P_n - J_n) x \right) \right\|^2 =$$

$$= (2M)^2 \sum_{n=k}^{k+N} \| P_n (P_n - J_n) x \|^2 \to 0,$$

when $k \to \infty$. Hence, $\sum\limits_{n=0}^{\infty} P_n (P_n - J_n) x$ converges and, consequently, the series

$$\sum_{n=0}^{\infty} P_n J_n x = \sum_{n=0}^{\infty} P_n x - \sum_{n=0}^{\infty} P_n (P_n - J_n) x$$

also converges. Consider the operator

$$R = \sum_{n=1}^{\infty} P_n (P_n - J_n) = I - P_0 - \sum_{n=1}^{\infty} P_n J_n$$

and note that $\|R\| < 1$, since for every $x \in H$,

$$\|Rx\|^2 = \left\| \sum_{n=1}^{\infty} P_n (P_n - J_n) x \right\|^2 \leq (2M)^2 \sum_{j=0}^{\infty} \left\| P_j \left(\sum_{n=1}^{\infty} P_n (P_n - J_n) x \right) \right\|^2 =$$

$$= (2M)^2 \sum_{n=1}^{\infty} \| P_n (P_n - J_n) x \|^2 \leq (2M)^2 \varsigma^2 \|x\|^2,$$

by virtue of Lemma 1 and applying (8). Further observe that, since

$$S = P_0 J_0 + I - P_0 - R, \qquad \|S\| < \|J_0\| + 3 < \infty.$$

Thus, a theorem will be proved if we show that S is continuously invertible. To this end we consider the operator

$$\widetilde{S} = \sum_{n=1}^{\infty} P_n J_n = I - P_0 - R. \tag{10}$$

Since $\dim P_0 = m < \infty$ by (1) we have that $(I - P_0)$ is Fredholm operator with

$$nul\,(I - P_0) = m, \quad ind\,(I - P_0) = 0, \quad \gamma\,(I - P_0) = 1,$$

where $nul\,T$ denotes the nullity, $ind\,T$ the index, and $\gamma\,(T)$ the reduced minimum modulus, of the operator T (for these notions see, e.g., [12], Chapter IV, §5.1). Indeed, first we note that $nul\,(I - P_0) = \dim P_0 = m$,

$$def\,(I - P_0) = \dim H|_{Im(I-P_0)} = \dim H|_{\overline{Im(I-P_0)}} =$$

$$= \dim co\ker (I - P_0) = \dim (Im\,(I - P_0))^{\perp} = m,$$

$ind\,(I - P_0) = nul\,(I - P_0) - def\,(I - P_0) = 0$, where $defT$ denotes the deficiency of T, see, e.g., [12, 4]. Second, since $\{F_n\}_{n=0}^{\infty}$ is a sequence of orthoprojections corresponding to orthogonal Schauder decomposition $\{\mathfrak{N}_n\}_{n=0}^{\infty}$, where $P_0 = F_0$, we have that

$$\inf_{v\in\ker(I-P_0)} \|x - v\| = \inf_{v\in ImF_0} \left(\sum_{n=0}^{\infty} \|F_n(x-v)\|^2 \right)^{\frac{1}{2}} =$$

$$= \left(\sum_{n=1}^{\infty} \|F_n(x - F_0 x)\|^2 \right)^{\frac{1}{2}} = \left(\sum_{n=0}^{\infty} \|F_n(x - F_0 x)\|^2 \right)^{\frac{1}{2}} = \|(I - P_0)x\|.$$

Consequently, $\gamma\,(I - P_0) =$

$$= \sup \left\{ \gamma :\ \|\,(I - P_0)\,x\| \geq \gamma \inf_{v\in\ker(I-P_0)} \|x - v\|,\ x \in D\,(I - P_0) = H \right\} = 1.$$

Furthermore, since $\|R\| < 1 = \gamma\,(I - P_0)$, $\widetilde{S} = (I - P_0) - R$ is also Fredholm with

$$nul\,\widetilde{S} \leq nul\,(I - P_0) = m, \quad ind\,\widetilde{S} = ind\,(I - P_0) = 0 \tag{11}$$

(see [12], Chapter IV, Theorem 5.22). Since $S = P_0 J_0 + \widetilde{S}$, where $P_0 J_0$ is compact, S is also Fredholm and $ind\,S = ind\,\widetilde{S} = 0$ (see [12], Chapter IV, Theorem 5.26). Therefore we obtain that $nul\ S = def\ S$, and S will be invertible if and only if $nul\ S = def\ S = 0$. Thus it is sufficient to show that $nul\ S = 0$. To this end we first show that

$$\ker \widetilde{S} = ImJ_0. \tag{12}$$

If $x \in ImJ_0$, i.e. $x = J_0 y$, then $\widetilde{S}x = \widetilde{S}J_0 y = \sum_{n=1}^{\infty} P_n J_n J_0 y = 0$ and, consequently, $x \in \ker \widetilde{S}$. On the other hand, $\ker \widetilde{S} \subset ImJ_0$, since $\ker \widetilde{S}$ and ImJ_0 are linear subspaces, $\dim ImJ_0 = m$ and $\dim \ker \widetilde{S} \leq m$ by (11). Assume now that $x \in \ker S$. Then,

$$0 = P_0 Sx = P_0 \sum_{n=0}^{\infty} P_n J_n x = P_0 J_0 x$$

and $\widetilde{S}x = Sx - P_0 J_0 x = 0$. Hence, $x \in \ker \widetilde{S}$, $x = J_0 y$ by (12) and, therefore,

$$P_0 x = P_0 J_0 y = P_0 \sum_{n=0}^{\infty} P_n J_n J_0 y = P_0 \sum_{n=0}^{\infty} P_n J_n x = 0.$$

As a result, $(I - R)x = (\widetilde{S} + P_0)x = 0$. Since $\|R\| < 1$, we obtain $x = 0$. Thus, $\ker S = \{0\}$, $nul\ S = 0$ and S is continuously invertible. Finally, we note that $J_n = S^{-1} P_n S$, $n \in \mathbb{Z}_+$, implies $\mathfrak{M}_n = SJ_n H$, $n \in \mathbb{Z}_+$, which completes the proof.

Definition 6 *We will say that* $\{\phi_n\}_{n=0}^{\infty}$ *is a Riesz basis in* H *with constant* M *provided the sequence of corresponding subspaces* $\{Lin\{\phi_n\}\}_{n=0}^{\infty}$ *forms an unconditional Schauder decomposition with constant* M.

In the case when all the subspaces $\mathfrak{M}_n$ are one dimensional, we deduce from Theorem 3 the following stability theorem for Riesz bases in H.

Theorem 4 *Let* $\{h_n\}_{n=0}^{\infty}$ *be an orthonormal basis of* H *and assume that* $\{\phi_n\}_{n=0}^{\infty}$ *is a Riesz basis in* H *with constant* M *and corresponding sequence of coordinate functionals* $\{\phi_n^*\}_{n=0}^{\infty}$, *where* $\phi_0 = \phi_0^* = h_0$. *Consider a biorthogonal sequence* $(\{\psi_n\}_{n=0}^{\infty}, \{\psi_n^*\}_{n=0}^{\infty})$ *in* H *such that* $0 < \inf_n \|\psi_n\| \leq \sup_n \|\psi_n\| < \infty$. *If for all* $x \in H$ *we have*

$$\sum_{n=1}^{\infty} |\langle \psi_n^*, x\rangle \langle \phi_n^*, \psi_n\rangle - \langle \phi_n^*, x\rangle|^2 \|\phi_n\|^2 \leq \varsigma^2 \|x\|^2,$$

where $\varsigma \in [0, (2M)^{-1})$, *then* $\{\psi_n\}_{n=0}^{\infty}$ *is also a Riesz basis of* H.

5. Conclusions

We obtain a stability theorem for unconditional Schauder decompositions in H, which is a generalization of the classical theorem of T. Kato [13]. More precisely, it is proved that the sequence of mutually disjoint projections, which is close in some sense to the sequence of projections corresponding to unconditional Schauder decomposition of given structure, itself generates an unconditional Schauder decomposition isomorphic to the original. As a direct consequence of this result, we obtain a stability theorem for Riesz bases. Also we sharpen one stability theorem of V.N. Vizitei in the case of unconditional Schauder decompositions.

In conclusion, we note the following. Just as Theorem 1 plays a special role in the study of spectral properties of nonselfadjoint and unbounded operators in H (see, e.g., [1, 2, 5, 11, 12, 18]), Theorem 2 and Theorem 3 may be very useful in the analysis of spectral properties of different type operators in H. It is enough to do the following. We should consider perturbations of nonselfadjoint operators generating unconditional spectral Schauder decompositions, instead of perturbations of selfadjoint operators generating an orthogonal spectral Schauder decompositions. And this, in turn, allows us to extend in qualitative manner the class of spectral problems which we can solve via known methods.

REFERENCES

1. Adduci J., Mityagin B. Eigensystem of an L^2-perturbed harmonic oscillator is an unconditional basis // Cent. Eur. J. Math., 2012. – **10**, no 2. – P. 569–589.

2. Adduci J., Mityagin B. Root system of a perturbation of a selfadjoint operator with discrete spectrum // Integral Equations Operator Theory, 2012. – **73**, no 2. – P. 153–175.

3. Bari N.K. Biorthogonal systems and bases in Hilbert space // Moskov. Gos. Univ. Učenye Zapiski. Matematika, 1951. – **148**, no 4. – P. 69—107 (in Russian)

4. Bilalov B.T., Veliev S.G. Some Questions of Bases, – Baku: Elm, 2010. – 304 p. (in Russian).

5. Clark C. On relatively bounded perturbations of ordinary differential operators // Pacific J. Math., 1968. – **25**, no 1. – P. 59–70

6. Curtain R.F., Zwart H.J. An Introduction to Infinite-Dimensional Linear Systems Theory, Texts in Applied Mathematics, Volume 21.– New-York: Springer-Verlag, 1995. – 698 p.

7. Fage M.K. Idempotent operators and their rectification // Dokl. Akad. Nauk, 1950. – **73**, – P. 895–897 (in Russian).

8. Fage M.K. The rectification of bases in Hilbert space // Dokl. Akad. Nauk, 1950. – **74**, – P. 1053–1056 (in Russian).

9. Gohberg I.C., Krein M.G. Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space, Transl. Math. Monogr., 18, – Providence, Rhode Island: American Mathematical Society, 1969. – 378 p.

10. Grinblyum M.M. On the representation of a space of type B in the form of a direct sum of subspaces // Dokl. Akad. Nauk, 1950. – **70**, – P. 749–752 (in Russian).

11. Hughes E. Perturbation theorems for relative spectral problems // Canad. J. Math., 1972. – **24**, no 1. – P. 72–81.

12. Kato T. Perturbation Theory for Linear Operators, 2nd ed. (reprint), Classics Math.,– Berlin: Springer-Verlag, 1995. – 619 p.

13. Kato T. Similarity for sequences of projections // Bull. Amer. Math. Soc., 1967. – **73**, no 6. – P. 904–905.

14. Krein M., Milman D., Rutman M. On a property of a basis in a Banach space // Comm. Inst. Sci. Math. Mec. Univ. Kharkoff [Zapiski Inst. Mat. Mech.], 1940. – **16**, no 4. – P. 106–110 (in Russian, with English summary).

15. Lindenstrauss J., Tzafriri L. Classical Banach Spaces I and II.– Reprint of the 1977, 1979 ed., Berlin: Springer-Verlag, 1996. – 188 p.

16. Mackey G.W. Commutative Banach algebras, Lecture notes (multigraphed),– Harvard University, edited by A. Blair, 1952. – 95 p.

17. Marcus A.S. A basis of root vectors of a dissipative operator // Dokl. Akad. Nauk, 1960. – **132**, no 3. – P. 524–527 (in Russian)

18. Mityagin B., Siegl P. Root system of singular perturbations of the harmonic oscillator type operators // preprint available at http://arxiv.org/abs/1307.6245.

19. Orlicz W. Über die Divergenz von allgemeinen Orthogonalreihen & Über unbedingte Convergenz in Funktionraümen, Studia Math., 1933. – **4**. – P. 27–37.

20. Rabah R., Sklyar G.M., Rezounenko A.V. Generalized Riesz basis property in the analysis of neutral type systems // C. R. Math. Acad. Sci. Paris, Ser. I, 2003. – **337**. – P. 19–24.

21. Rabah R., Sklyar G.M., Rezounenko A.V. Stability analysis of neutral type systems in Hilbert space // J. Differential Equations, 2005. – **214**. – P. 391–428.

22. Rabah R., Sklyar G.M. The analysis of exact controllability of neutral-type systems by the moment problem approach // SIAM J. Control Optim., 2007. – **46**, no 6. – P. 2148–2181.

23. Singer I. Bases in Banach Spaces I,– Berlin: Springer-Verlag, 1970. – 668 p.

24. Singer I. Bases in Banach Spaces II,– Berlin: Springer-Verlag, 1981. – 880 p.

25. Vizitei V.N. On the stability of bases of subspaces in a Banach space, Studies on Algebra and Mathematical Analysis, Moldov. Acad. Sci., – Chişinău: Kartja Moldovenjaska, 1965. – P. 32–44 (in Russian)

26. Wermer J. Commuting spectral measures on Hilbert space // Pacific J. Math., 1954. – **4**. – P. 355–361.

27. Zwart H. Riesz basis for strongly continuous groups // J. Differential Equations, 2010. – **249**. – P. 2397–2408.

Article history: Received: 14 April 2014; Final form: 25 October 2014;
Accepted: 28 October 2014.