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Attractor for a composite system of nonlinear wave and thermoelastic plate equations

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We prove the existence of a compact finite dimensional global attractor for a coupled PDE system comprising a nonlinearly damped semilinear wave equation and a thermoelastic Mindlin-Timoshenko plate system with nonlinear viscous damping. We show the upper semi-continuity of the attractor with respect to the parameters related to the coupling terms and the shear modulus of the plate.
Keywords: acoustic model, attractor, upper semi-continuity.

Фастовская Т. Б., **Глобальный аттрактор нелинейной системы для волнового уравнения и термоупругой системы колебания пластин.** Доказывается существование конечномерного компактного глобального аттрактора системы, состоящей из нелинейного волнового уравнения с нелинейным демпингом и системы Миндлина-Тимошенко, описывающей акустическую камеру с упругой стенкой. Доказана верхняя полунепрерывность аттрактора по параметрам задачи.
Ключевые слова: модель акустики, аттрактор, верхняя полунепрерывность.

Фастовська Т. Б., **Глобальний атрактор нелінійної системи для хвильового рівняння та термопружної системи коливання пластин.** Доведено існування скінченномірного компактного глобального атрактора системи, що складається з нелінійного хвильового рівняння з нелінійним демпінгом та системи Міндліна-Тимошенка, що описує акустичну камеру з пружною стінкою. Доведено верхню напівнеперервність атрактора за параметрами задачі.
Ключові слова: модель акустики, атрактор, верхня напівнеперервність.

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Introduction

The mathematical model considered consists of a semilinear wave equation defined on a bounded domain, which is strongly coupled with thermoelastic Mindlin-Timoshenko plate equation on a part of the boundary. The model includes a weak structural damping and a thermal damping. This kind of models referred to as structural acoustic interactions, arise in the context of modelling gas pressure in an acoustic chamber which is surrounded by a combination of rigid and flexible walls (see, e.g. [13, 22]). The pressure in the chamber is described by the solution to a wave equation, while vibrations of the flexible wall are described by the solution to a plate equation. The Mindlin-Timoshenko model describes dynamics of a plate in view of transverse shear effects (see, e.g., [15, 24] and references therein).

More precisely, let $\Omega \in \mathbb{R}^3$ be a smooth bounded open domain with the boundary $\partial\Omega =: \Gamma = \overline{\Gamma_0 \cup \Gamma_1}$ consisting of two open (in the induced topology) connected disjoint parts Γ_0 and Γ_1 of positive measure. Γ_0 is flat and is referred to as the elastic wall. The dynamics of the acoustic medium in the chamber Ω is described by a interactive system of a semilinear wave equation and a Mindlin-Timoshenko system of thermoelasticity:

$$z_{tt} + g(z_t) - \Delta z + f(z) = 0,\ x \in \Omega, t > 0, \tag{1}$$

$$\frac{\partial z}{\partial n} = 0,\ x \in \Gamma_1,\ \frac{\partial z}{\partial n} = \kappa w_t,\ x \in \Gamma_0 \tag{2}$$

$$v_{tt} - \mathcal{A}v + \mu(v + \nabla w) + \beta\nabla\theta + b(v_t) + v[h(|v|^2) + \gamma w] = 0\ x \in \Gamma_0, t > 0, \tag{3}$$

$$w_{tt} - \mu \mathrm{div}(v + \nabla w) + b_0(w_t) + h_0(w) + \kappa z_t = 0, \tag{4}$$

$$\theta_t - \Delta\theta + \beta \mathrm{div} v_t = 0 \tag{5}$$

$$v = w = \theta = 0\ \ \partial\Gamma_0 \tag{6}$$

supplemented with initial conditions:

$$\begin{array}{c} z(0,\cdot) = z_0,\ \ z_t(0,\cdot) = z_1, \\ v(0,\cdot) = v_0,\ \ v_t(0,\cdot) = v_1, \\ w(0,\cdot) = w_0,\ \ w_t(0,\cdot) = w_1,\ \ \theta(0,\cdot) = \theta_0. \end{array} \tag{7}$$

The variable z describes the dynamics in the acoustic medium, while v denotes the angles of deflection of the filaments, w - the transverse displacement of the middle surface, and θ - the temperature variation averaged with respect to the thickness of the plate. The operator $\mathcal{A}$ is defined as follows

$$\mathcal{A} = \begin{pmatrix} \partial^2_{x_1} + \frac{1-\nu}{2}\partial^2_{x_2} & \frac{1+\nu}{2}\partial_{x_1 x_2} \\ \frac{1+\nu}{2}\partial_{x_1 x_2} & \frac{1-\nu}{2}\partial^2_{x_1} + \partial^2_{x_2} \end{pmatrix} = \nabla \mathrm{div} - \frac{1-\nu}{2}\mathrm{rotrot},$$

where $0 < \nu < 1$ is the Poisson ratio.

The non-decreasing functions $b(s)$, $b_0(s)$, and $g(s)$ describe the dissipation effects in the model, the terms $f(z)$, $h(v)$, $h_0(w)$, $vw \cdot v$ represent nonlinear forces acting on the wave and on the plate components respectively. The boundary term $\kappa z_t|_{\Gamma_0}$ represents the pressure exercised by the acoustic medium on the wall.

The parameter $0 \le \kappa \le 1$ has been introduced to cover the case of non-interacting wave and plate equations ($\kappa = 0$), while the parameter $0 \le \beta \le 1$ - the case of decoupled plate and heat conduction equations. The parameter $\mu > 0$ describes the shear modulus of the plate.

Due to broad engineering applications in aerospace industry, structural acoustic models have recently attracted an ample attention. A very large literature devoted to this model in the context of the control theory, (see e.g. the monograph [16] and references therein). The investigation of the uniform stability of structural acoustic models with thermoelastic wall in the case of a single equilibrium can be found in [17, 18, 19, 21]. The nonlinear structural acoustic model with thermal effects and without mechanical dissipation in the plate component comprising wave and thermoelastic Berger's equations has been studied in [2] in that the existence of a compact global attractor and it's properties were investigated. The same results were obtained for the wave/ Berger's system with mechanical damping without thermal effects [3]. Long-time behavior of a nonlinear structural acoustic model comprising wave and thermoelastic von Karman plate equations has been studied in [9]. We also refer to the paper [23] devoted to the problem of dynamics of a clamped von Karman plate in a gas flow in the presence of thermal effects. The existence and upper semicontinuity of attractors of the elastic and thermoelastic Mindlin-Timoshenko plate system were studied in [5, 10].

We consider the nonlinear acoustic model comprising wave and Mindlin-Timoshenko equations with thermal effects with and without non-conservative nonlinearity in the plate part.
The paper is organized as follows. Section 1 is devoted to the conservative system with monotone energy. We begin with the abstract formulation of the problem and its well-posedness. Our first main result, Theorem 3 states the existence of global attractors for problem (1)-(7) under rather general conditions on the nonlinearities. Since the dynamical system generated by the system without non-conservative nonlinearity is gradient, the main issue to be explored is the asymptotic compactness of the semi-flow. To show this property we use the idea due to Khanmamedov [14] in the form suggested in [8]. In comparison to the acoustic interaction with the Berger's and von Karman plate [3, 9] the existence of the compact global attractor requires the additional condition on the nonlinear damping referred to the elastic component (see Statement 3).

The next main results, Theorem 5 concerns the finite dimensionality of the attractors.

The main result of Section 2, Theorem 9, concerning problem (1)-(7) is the upper semicontinuity of the attractors with respect to the shear modulus and the coupling parameters. In contrast to the system considered in [2] the attractor is upper-semicontinuous not only with respect to the parameter decoupling wave and plate components but also with respect to the parameter decoupling plate and thermal components.

In Section 3 we establish the same results for the system with non-conservative nonlinearity. Due to the lost of monotonicity of the energy the existence of an absorbing ball is proved supplementary.

System with conservative forces ($\gamma = 0$).

In this section we consider the conservative model (the case $\gamma = 0$), which implies the monotonicity of the energy.

Basic assumptions. We impose the following basic assumptions on the nonlinearities of the problem. Note that the listed assumptions on the nonlinearities f, g and b_i, i=0,1,2 were first formulated in [9, Section 6.3, 12.3].

Statement 1

- *$g \in C(\mathbb{R})$ is a non-decreasing function, $g(0) = 0$, and there exists a constant $C > 0$ such that*

$$|g(s)| \leq C(1 + |s|^p), s \in \mathbb{R}, \tag{8}$$

where $1 \leq p \leq 5$.

- *$f \in Lip_{loc}(\mathbb{R})$ and there exists a positive constant M such that*

$$|f(s_1) - f(s_2)| \leq M(1 + |s_1|^q + |s_2|^q)|s_1 - s_2|, \quad s_1, s_2 \in \mathbb{R}, \tag{9}$$

where $q \leq 2$. Moreover,

$$\lambda = \frac{1}{2} \lim_{|s|\to\infty} \inf \frac{f(s)}{s} > 0 \tag{10}$$

.

- *$h \in Lip_{loc}(\mathbb{R}_+), h_0 \in Lip_{loc}(\mathbb{R})$ and there exists a positive constant M_1 such that*

$$|h(s_1) - h(s_2)| \leq M_1(1 + s_1^{q_1} + s_2^{q_1})|s_1 - s_2|, \quad s_1, s_2 \in \mathbb{R}_+, \tag{11}$$

and

$$|h_0(s_1) - h_0(s_2)| \leq M(1 + |s_1|^{q_2} + |s_2|^{q_2})|s_1 - s_2|, \quad s_1, s_2 \in \mathbb{R}, \tag{12}$$

where $q_1, q_2 \geq 0$. and

$$h^* = \lim_{s\to\infty} \inf \frac{h(s)}{s} > 0, \quad h_0^* = \lim_{|s|\to\infty} \inf \frac{h_0(s)}{s} > 0. \tag{13}$$

- *$b \in C(\mathbb{R}^2)$, $b_0 \in C(\mathbb{R})$ are non-decreasing functions such that $b(0) = 0$, $b_0(0) = 0$.*

Statement 2 *For any $\varepsilon > 0$ there exists c_ε such that $s \in \mathbb{R}$*

- $$s^2 \le \varepsilon + c_\varepsilon s g(s), \quad s \in \mathbb{R} \tag{14}$$

- $$s^2 \le \varepsilon + c_\varepsilon s b_0(s), \quad s \in \mathbb{R}, \qquad |s|^2 \le \varepsilon + c_\varepsilon s b(s), \quad s \in \mathbb{R}^2 \tag{15}$$

Statement 3 • *There exist $C > 0$ and $1 \le p, p_0 < \infty$ such that*

$$|b(s)| \le C(1 + |s|^p), s \in \mathbb{R}^2, \qquad |b_0(s)| \le C(1 + |s|^{p_0}), s \in \mathbb{R}. \tag{16}$$

Statement 4 • *There exist positive constants $m > 0$, $M > 0$ such that*

$$m \le \frac{g(s_1) - g(s_2)}{s_1 - s_2} \le M(1 + s_1 g(s_1) + s_2 g(s_2))^{2/3}, \quad s_1, s_2 \in \mathbb{R}, s_1 \ne s_2. \tag{17}$$

- *There exist $m_i > 0$, $M_i > 0$, $i = 1, 2$ such that*

$$m_1 |s_1 - s_2|^2 \le (b(s_1) - b(s_2))(s_1 - s_2), \tag{18}$$

$$\frac{b_j(s_1) - b_j(s_2)}{s_1 - s_2} \le M_1(1 + s_1 b_j(s_1) + s_2 b_j(s_2)), \quad s_1, s_2 \in \mathbb{R}, s_1 \ne s_2, \tag{19}$$

where $j = 1, 2$, $b = (b_1, b_2)$.

$$m_2 \le \frac{b_0(s_1) - b_0(s_2)}{s_1 - s_2} \le M_2(1 + s_1 b_0(s_1) + s_2 b_0(s_2)), \quad s_1, s_2 \in \mathbb{R}, s_1 \ne s_2. \tag{20}$$

- $f \in C^2(\mathbb{R})$,

$$|f''(s)| \le C(1 + |s|), \quad s \in \mathbb{R}. \tag{21}$$

- $h_0 \in C^2(\mathbb{R})$, $h \in C^2(\mathbb{R}_+)$ *and there exists a constant $c > 0$ and $1 \le p_2 < \infty$, $1 \le p_3 < \infty$ such that*

$$|h''(s)| \le c(1 + s^{p_2}), \; s \in \mathbb{R}_+ \tag{22}$$

and

$$|h_0''(s)| \le c(1 + |s|^{p_3}), \; s \in \mathbb{R}. \tag{23}$$

Abstract formulation. We represent the system (1)-(7) as an abstract evolution equation in an appropriate Hilbert space. For this purpose we introduce the following spaces and operators.Denote $u = (v, w) = (v_1, v_2, w)$.

Let $A : \mathcal{D}(A) \subset [L_2(\Gamma_0)]^3 \to [L_2(\Gamma_0)]^3$ be the positive self-adjoint operator on $\mathcal{D}(A) = [H^2 \cap H_0^1(\Gamma_0)]^3$ defined by

$$A = \begin{pmatrix} -\mathcal{A} + \mu I & \mu \nabla \\ -\mu div & -\mu \Delta \end{pmatrix}$$

Define also a positive self-adjoint operator $L : \mathcal{D}(L) \in L_2(\Omega) \to L_2(\Omega)$ by the formula

$$L = -\Delta + \lambda I,$$

with

$$\mathcal{D}(L) = \{H^2(\Omega) : \frac{\partial}{\partial n}|_\Gamma = 0\}$$

and λ is given by (9). Next, let N_0 be the Neumann map from $L_2(\Gamma_0)$ to $L_2(\Omega)$ defined by

$$\psi = N_0\phi \Leftrightarrow \begin{cases} (-\Delta + \lambda)\psi = 0 \\ \frac{\partial \psi}{\partial n}|_{\Gamma_0} = \phi, \frac{\partial \psi}{\partial n}|_{\Gamma_1} = 0 \end{cases}$$

It is well-known [20] that N_0 is continuous from $L_2(\Gamma_0)$ to $H^{3/2}(\Omega) \subset \mathcal{D}(A^{3/4-\epsilon})$, for any $\epsilon > 0$, and the following trace result takes place

$$N_0^* Lh = h|_{\Gamma_0}, \quad h \in \mathcal{D}(A^{1/2}). \tag{24}$$

We also introduce the operators $R_1 : H_0^1(\Gamma_0) \to [L_2]^3(\Gamma_0)$ and $R_2 : [H_0^1]^2(\Gamma_0) \to L_2(\Gamma_0)$ defined by the formulas

$$R_1\theta = \beta(\partial_1\theta, \partial_2\theta, 0)$$

and

$$R_2 = \beta\partial_1 v_1 + \beta\partial_2 v_2 = \beta divv.$$

Now we are at the point to give the abstract formulation of problem (1)-(7). With the above dynamic operators initial-value problem (1)-(7) can be rewritten as follows

$$z_{tt} + G(z_t) + Lz + F_1(z) - \kappa L N_0 u_t = 0, \; x \in \Omega, t > 0, \tag{25}$$

$$Du_{tt} + Au + R_1\theta + B(u_t) + F_2(u) + \kappa N_0^* L z_t = 0 \tag{26}$$

$$\gamma_1\theta_t - \Delta\theta + R_2 u_t = 0 \tag{27}$$

$$z(0) = z_0, \;\; z_t(0) = z_1, \;\; u(0) = u_0, \;\; u_t(0) = u_1, \;\; \theta(0) = \theta_0. \tag{28}$$

where the nonlinear terms are given by the following operators

$$G(h) = g(h),$$

$$B(u) = (b(v), b_0(w)),$$

here $u = (v, w)$. Denote

$$\Pi(z) = \int_\Omega \int_0^z (f(\xi) - \lambda\xi) d\xi dx. \tag{29}$$

Then

$$F_1(z) = \Pi'(z). \tag{30}$$

The term $F_2(u)$ is represented as follows

$$F_2(u) = (v_1 h(|v|^2), v_2 h(|v|^2), h_0(w)). \tag{31}$$

Denote

$$\Pi_0(u) = \frac{1}{2}\int\limits_{\Omega}\int\limits_{0}^{|v|^2} h(s)dsdx + \int\limits_{\Omega}\int\limits_{0}^{w} h_0(s)ds, \tag{32}$$

It follows from (10) and (13) that

$$\Pi(z) \geq -M_f \tag{33}$$

$$\Pi_0(u) \geq -M_h \tag{34}$$

for some nonnegative constants M_f and M_h. The natural energy functions associated with the solutions to the uncoupled wave and plate models are given respectively by

$$\mathcal{E}_z(z(t), z_t(t)) = E_z^0(z, z_t) + \Pi(z) \tag{35}$$

and

$$\mathcal{E}_{u,\theta}(u(t), u_t(t)) = E_u^0(u, u_t) + E_\theta^0(\theta) + \Pi_0(u). \tag{36}$$

Here we have set

$$E_z^0(z, z_t) = \frac{1}{2}(\|L^{1/2}z\|_\Omega^2 + \|z_t\|_\Omega^2), \tag{37}$$

$$E_u^0(u, u_t) = \frac{1}{2}(\|Au\|_{\Gamma_0}^2 + \|u_t\|_{\Gamma_0}^2), \tag{38}$$

and

$$E_\theta^0(\theta) = \frac{1}{2}\|\theta\|_{\Gamma_0}^2. \tag{39}$$

Denote also

$$E_z(z, z_t) = E_z^0(z, z_t) + \Pi(z) + M_f, \tag{40}$$

$$E_{u,\theta}(u, u_t, \theta) = E_u^0(u, u_t) + E_\theta^0(\theta) + \Pi_0(u) + M_h, \tag{41}$$

Finally we introduce the total energy $\mathcal{E}(t) = \mathcal{E}(z(t), z_t(t), u(t), u_t(t), \theta(t))$ of the system

$$\mathcal{E}(t) = \mathcal{E}_z(z, z_t) + \mathcal{E}_{u,\theta}(u, u_t, \theta), \tag{42}$$

where $\mathcal{E}_z(z, z_t)$ and $\mathcal{E}_{u,\theta}(u, u_t, \theta)$ are given by (35) and (36) respectively. Denote also

$$E^0(t) = E(z, z_t, u, u_t, \theta) = E_z^0(z, z_t) + E_u^0(u, u_t) + E_\theta^0(\theta). \tag{43}$$

The positive part of the total energy is given by

$$E(t) = E(z, z_t, u, u_t, \theta) = E_z(z, z_t) + E_{u,\theta}(u, u_t, \theta), \tag{44}$$

where $E_z(z, z_t)$ and $E_{u,\theta}(u, u_t, \theta)$ are given by (40) and (41) respectively.

It follows from (33) and (34) that there exist positive constants c, C, M_0 such that

$$cE(t) - M_0 \le \mathcal{E}(t) \le CE(t) + M_0 \qquad (45)$$

The phase spaces Y_1 for the acoustic component $[z, z_t]$ and Y_2 for the plate component $[u, u_t, \theta]$ of system are given by

$$Y_1 = \mathcal{D}(L^{1/2}) \times L_2(\Omega) = H_1(\Omega) \times L_2(\Omega)$$

and

$$Y_2 = \mathcal{D}(A^{1/2}) \times [L_2(\Gamma_0)]^3 \times L_2(\Gamma_0) = [H_0^1(\Gamma_0)]^3 \times [L_2(\Gamma_0)]^3 \times L_2(\Gamma_0)$$

with the norms

$$\|(z_1, z_2)\|^2_{Y_1} = \|L^{1/2} z_1\|^2_\Omega + \|z_2\|^2_\Omega$$

and

$$\|(u_1, u_2, \theta)\|^2_{Y_2} = \|A^{1/2} u_1\|^2_{\Gamma_0} + \|D^{1/2} u_2\|^2_{\Gamma_0} + \|\theta\|^2_{\Gamma_0}$$

respectively. The phase space for the problem (25)-(28) is defined as

$$H = Y_1 \times Y_2 \qquad (46)$$

with the norm

$$\|y\|^2_H = \|(z_1, z_2)\|^2_{Y_1} + \|(u_1, u_2, \theta)\|^2_{Y_2}$$

for $y = (z_1, z_2, u_1, u_2, \theta)$ and the corresponding inner product.

Well-posedness.

Definition 1 *A triplet of functions $(z(t), u(t), \theta(t))$ which satisfy initial conditions (28) and such that*

$$(z(t), u(t)) \in C([0, T]; \mathcal{D}(L^{1/2}) \times \mathcal{D}(A^{1/2})) \cap C^1([0, T]; L_2(\Omega) \times [L_2(\Gamma_0)]^3)$$

and

$$\theta(t) \in C([0, T]; L_2(\Gamma_0))$$

is said to be

(S) *a strong solution to problem (25)-(28) on the interval $[0, T]$, iff*

- *for any $0 < a < b < T$*

$$(z_t, u_t) \in L_1([a, b], \mathcal{D}(L^{1/2}) \times \mathcal{D}(A^{1/2})), \quad \theta_t \in L_1([a, b], L_2(\Gamma_0))$$

  and

$$(z_{tt}, u_{tt}) \in L_1([a, b], L_2(\Omega) \times [L_2(\Gamma_0)]^3)$$

- *$L[z(t) - \alpha\kappa N_0 u_t] + G(z_t(t)) \in L^2(\Omega)$, $u(t) \in D(A)$, $\theta \in H^2 \cap H_0^1(\Gamma_0)$ for almost all $t \in [0, T]$*

- *equations (25)-(27) are satisfied in* $L_2(\Omega) \times L_2(\Gamma_0) \times L_2(\Gamma_0)$ *for almost all* $t \in [0, T]$

(G) *a generalized solution to problem (25)-(28) on the interval* $[0, T]$*, iff there exists a sequence* $\{(z_n(t), u_n(t), \theta_n(t))\}$ *of strong solutions to (25)-(28) with initial data* $(z_n^0, z_n^1, u_n^0, u_n^1, \theta_n^0)$ *such that*

$$\lim_{n\to\infty} \max_{t\in[0,T]} \{\|\partial_t z(t) - \partial_t z_n(t)\|_\Omega + \|L^{1/2}(z(t) - z_n(t))\|_\Omega\} = 0$$

$$\lim_{n\to\infty} \max_{t\in[0,T]} \{\|D^{1/2}(\partial_t u(t) - \partial_t u_n(t))\|_{\Gamma_0} + \|A^{1/2}(u(t) - u_n(t))\|_{\Gamma_0}\} = 0$$

and

$$\lim_{n\to\infty} \max_{t\in[0,T]} \{\|\theta(t) - \theta_n(t)\|_{\Gamma_0}\} = 0$$

Theorem 1 *Under Assumptions 1, 3 for any initial conditions*

$$y_0 = (z^0, z^1, u^0, u^1, \theta^0) \in H$$

there exists a unique generalized solution $y(t) = (z(t), z_t(t), u(t), u_t(t), \theta(t))$ *to the PDE system (25)-(28), which depends continuously on initial data. This solution satisfies the energy inequality*

$$\mathcal{E}(t) + \int_s^t (G(z_t), z_t)_\Omega d\tau + \int_s^t (B(u_t), u_t)_{\Gamma_0} d\tau + \int_s^t \|\nabla\theta\|_{\Gamma_0}^2 d\tau \le \mathcal{E}(s), \quad 0 \le s \le t, \quad (47)$$

with the total energy $\mathcal{E}(t)$ *given by (42). Moreover, the generalized solution to problem (25)-(28) is also weak, i.e. it satisfies the following system of variational equations:*

$$\frac{d}{dt}(z_t, \phi)_\Omega + (L^{1/2}z, L^{1/2}\phi)_\Omega + (g(z_t), \phi)_\Omega - \kappa(u_t, N_0^*\phi)_{\Gamma_0} + (f(z), \phi)_\Omega = 0 \quad (48)$$

$$\frac{d}{dt}(u_t + \kappa z, \psi)_{\Gamma_0} + (A^{1/2}u, A^{1/2}\psi)_{\Gamma_0} + (B(u_t), \psi)_{\Gamma_0} + (F_2(u), \psi)_{\Gamma_0} + (R_1\theta, \psi)_{\Gamma_0} = 0 \quad (49)$$

$$\frac{d}{dt}(\theta, \chi)_{\gamma_0} + (\nabla\theta, \nabla\chi)_{\Gamma_0} + (R_2 u_t, \chi)_{\Gamma_0} = 0 \quad (50)$$

for any $\phi \in H^1(\Omega)$*,* $\psi \in [H_0^1]^3(\Gamma_0)$*, and* $\chi \in H_0^1(\Gamma_0)$ *in the sense of distributions. If initial data are such that*

$$z^0, z^1 \in \mathcal{D}(L^{1/2}), \quad u^0 \in \mathcal{D}(A), \quad u^1 \in \mathcal{D}(A^{1/2}), \quad \theta^0 \in (H^2 \cap H_0^1)(\Gamma_0),$$

and

$$L[z^0 - \kappa N_0 u^1] + G(z^1) \in L_2(\Omega)$$

then there exists a unique strong solution $y(t)$ satisfying the energy identity:

$$\mathcal{E}(t) + \int_s^t (G(z_t), z_t)_\Omega d\tau + \int_s^t (B(u_t), u_t)_{\Gamma_0} d\tau + \int_s^t \|\nabla\theta\|^2_{\Gamma_0} d\tau = \mathcal{E}(s), \quad 0 \le s \le t,$$

Both strong and generalized solutions satisfy the inequalities

$$\mathcal{E}(t) \le \mathcal{E}(s), \quad t \ge s, \tag{51}$$

and

$$E(z(t), z_t(t), u(t), u_t(t), \theta(t)) \le C(1 + E(z^0, z^1, u^0, u^1, \theta^0)), \tag{52}$$

where E is given by (44) and C does not depend on κ, μ, and β.

Proposition 1 *Theorem 1 enables us to define the dynamical system (H, S_t) with the phase space H given by (46) and with the evolution operator $S_t : H \to H$ defined by the formula*

$$S_t y_0 = (z(t), z_t(t), u(t), u_t(t), \theta(t)), \quad y_0 = (z^0, z^1, u^0, u^1, \theta^0)$$

where $(z(t), u(t), \theta(t))$ is a generalized solutions to problem (25)-(28). Moreover, the monotonicity of the damping operators G and B, the Lipschitz conditions on F_1 and F_2 and the energy bound in (52) implies that the semigroup S_t is locally Lipschitz on H. Namely, there exist $a > 0$ and $b(\rho) > 0$ such that

$$\|S_t y_1 - S_t y_2\|_H \le a e^{b(\rho)t} \|y_1 - y_2\|_H, \quad \|y_i\|_H \le \rho, \ t \ge 0. \tag{53}$$

Stationary points. It follows from (45) that the energy $\mathcal{E}(z_0, z_1, u_0, u_1, \theta_0)$ is bounded from below on H and $\mathcal{E}(z_0, z_1, u_0, u_1, \theta_0) \to +\infty$ when $\|(z_0, z_1, u_0, u_1, \theta_0)\|_H \to +\infty$. This implies that there exists $R_* > 0$ such that the set

$$W_R = \{y = (z_0, z_1, u_0, u_1, \theta_0) \in H : \mathcal{E}(z_0, z_1, u_0, u_1, \theta_0) \le R\}$$

is a non-empty bounded set in H for all $R \ge R_*$. Moreover, any bounded set $B \in H$ is contained in W_R for some R and it follows from (51) that the set is forward invariant with respect to the semi-flow S_t , i.e. $S_t W_R \subset W_R$ for all $t > 0$. Thus, we can consider the restriction (W_R, S_t) of the dynamical system (H, S_t) on W_R, $R \ge R_*$.

We introduce the set of stationary points of S_t denoted by $\mathcal{N}$,

$$\mathcal{N} = \{V \in H : S_t V = V,\ t \geq 0\}$$

Every stationary point has the form $V = (z, 0, u, 0, 0)$, where $z \in H^1(\Omega)$ and $u \in H_0^1(\Omega)$ are weak solutions to the problems

$$-\Delta z + f(z) = 0 \ \text{ in } \ \Omega, \quad \frac{\partial z}{\partial n} = 0 \ \text{ on } \Gamma,$$

and

$$\begin{gathered} -\mathcal{A}v + \mu(v + \nabla w) + h(|v|^2)v = 0 \ x \in \Gamma_0, t > 0, \\ -\mu \mathrm{div}(v + \nabla w) + h_0(w) = 0, \\ v = w = \theta = 0 \ \ \partial\Gamma_0. \end{gathered}$$

It is clear that the set of stationary points does not depend on κ and μ. Therefore, one can easily prove the following assertion.

Lemma 1 *Under Assumption 1 the set $\mathcal{N}$ of stationary points for the semi-group S_t generated by problem (25)-(28) is a closed bounded set in H, and hence there exists $R_{**} \geq R_*$ (independent of κ, β, and μ) such that $\mathcal{N} \subset W_R$ for every $R \geq R_{**}$.*

Later we will also need the notion of unstable manifold $M^u(\mathcal{N})$ emanating from the set of stationary points.

Definition 2 *The unstable manifold $M^u(\mathcal{N})$ emanating from the set of stationary points $\mathcal{N}$ is a set of all $V \in H$ such that there exists a full trajectory $\bar{\gamma} = \{V(t) : t \in \mathbb{R}\}$ with the properties*

$$V(0) = V \ \ \textit{and} \ \ \lim_{t \to -\infty} dist_H(V(t), \mathcal{N}) = 0.$$

Existence of attractors. The main aim of the paper is to show the existence of a global attractor for the dynamical system generated by problem (25)-(28), and to study its properties.

By definition (see, e.g. [1, 6, 26]) a global attractor is a bounded closed set $\mathfrak{A} \subset H$ such that $S_t\mathfrak{A} = \mathfrak{A}$ for all $t \geq 0$ and

$$\lim_{t \to +\infty} \sup_{y \in \mathcal{B}} \mathrm{dist}(S_t y, \mathfrak{A}) = 0$$

for any bounded set $\mathcal{B} \in H$.

The fractal dimension

$$\dim_f M = \limsup_{\varepsilon \to 0} \frac{\ln N(M, \varepsilon)}{\ln(1/\varepsilon)},$$

where $N(M, \varepsilon)$ is the minimal number of closed sets of diameter 2ε which cover the set M.

To prove the existence of the compact global attractor of the dynamical system (H, S_t) we need to show some preliminary results.

Lemma 2 *Let Assumptions 1 and 3 hold. Assume that* $y_1, y_2 \in H$*, such that* $\|y_i\|_H \le R, \quad i = 1,2$ *and denote*

$$S_t y_1 = (d(t), d_t(t), \nu(t), \nu_t(t), \psi(t))$$

and

$$S_t y_2 = (\zeta(t), \zeta_t(t), \omega(t), \omega_t(t), \xi(t)).$$

Let

$$z(t) = d(t) - \zeta(t), \;\; u(t) = \nu(t) - \omega(t), \;\; \theta(t) = \psi(t) - \xi(t) \tag{54}$$

There exist $T_0 > 0$ *and positive constants* C_i*,* $i = \overline{1,4}$ *and* $C_5(R)$ *independent of* T*,* κ*,* μ*, and* β *such that for every* $T \ge T_0$ *the following inequality holds:*

$$\begin{aligned} TE^0(T) + \int_0^T E^0(t)dt \le C_1[(\int_0^T \|z_t\|^2 + \|\nabla\theta\|^2 + \|u_t\|^2 dt) \\ + G_0^T(z) + R_0^T(u)] + C_2 H_0^T(z) + C_3 Q_0^T(u) + C_4 \Psi_T(z,u) \\ + C_5(R)\int_0^T (\|z\|^2 + \|u\|^2)dt, \end{aligned} \tag{55}$$

where $E^0(t)$ *is given by (43). We also introduce the notations*

$$G_s^t(z) = \int_s^t (G(\zeta_t + z_t) - G(\zeta_t), \zeta_t)_\Omega d\tau, \tag{56}$$

$$H_s^t(z) = \int_s^t |(G(\zeta_t + z_t) - G(\zeta_t), \zeta)_\Omega d\tau, \tag{57}$$

$$R_s^t(u) = \int_s^t (B(\nu_t + u_t) - B(\nu_t), \nu_t)_{\Gamma_0} d\tau, \tag{58}$$

$$Q_s^t(u) = \int_s^t |(B(\nu_t + u_t) - B(\nu_t), \nu)_{\Gamma_0} d\tau, \tag{59}$$

and

$$\begin{aligned} \Psi_T(z,u) = |\int_0^T (\mathcal{F}_1(z), z_t)dt| + |\int_0^T \int_t^T (\mathcal{F}_1(u), u_t)d\tau dt| + |\int_0^T (\mathcal{F}_2(z), z_t)dt| \\ + |\int_0^T \int_t^T (\mathcal{F}_2(u), u_t)d\tau dt| \end{aligned} \tag{60}$$

with

$$\mathcal{F}_1(z) = F_1(\zeta + z) - F_1(\zeta), \quad \text{and} \quad \mathcal{F}_2(u) = F_2(\omega + u) - F_2(\omega), \tag{61}$$

where F_1 and F_2 are the same as in (30), (31).

Proof. **Step 1 (Energy identity)** Without loss of generality, we can assume that $(d(t), \omega(t), \psi(t))$ and $(\zeta(t), \nu(t), \xi(t))$ are strong solutions. By (45) there exists a constant $C_R > 0$, independent of κ, μ, and β, such that

$$E_d^0(d(t), d_t(t)) + E_\zeta^0(\zeta(t), \zeta_t(t)) + E_\nu^0(\nu(t), \nu_t(t)) + E_\omega^0(\omega(t), \omega_t(t)) + E_\psi^0(\psi(t)) + E_\xi^0(\xi(t)) \le C_R \tag{62}$$

for all $t \ge 0$. We establish first an energy type equality.

Lemma 3 *For any $T > 0$ and all $0 \le t \le T$ $E^0(t)$ satisfies*

$$E^0(T) + G_t^T(z) + R_t^T(u) + \int_t^T \|\nabla\theta\|^2 d\tau = E^0(t) - \int_t^T (\mathcal{F}_1(z), z_t) d\tau - \int_t^T (\mathcal{F}_2(u), u_t) d\tau, \tag{63}$$

where $G_t^T(z)$ and $R_t^T(u)$ are given by (56), (58) while $\mathcal{F}_1(z)$ and $\mathcal{F}_2(u)$ are defined by (61).

Proof. It is easy to see that the differences (54) satisfy the following system of coupled equations

$$z_{tt} + G(z_t + \zeta_t) - G(\zeta_t) + Lz + \mathcal{F}_1(z) - \kappa L N_0 u_t = 0, \; x \in \Omega, t > 0, \tag{64}$$

$$Du_{tt} + Au + R_1\theta + B(u_t + \omega_t) - B(\omega_t) + \mathcal{F}_2(u) + \kappa N_0^* L z_t = 0 \tag{65}$$

$$\theta_t - \Delta\theta + R_2 u_t = 0. \tag{66}$$

By standard energy methods, taking the inner products in (64)-(66) with z_t, u_t and θ respectively, we obtain

$$E_z^0(T) + G_t^T(z) = E_z^0(t) - \int_t^T (\mathcal{F}_1(z), z_t)_\Omega d\tau + \kappa \int_t^T (L N_0 u_t, z_t)_\Omega d\tau, \; x \in \Omega, t > 0, \tag{67}$$

$$E_u^0(T) + R_t^T(z) = E_u^0(t) + \int_t^T (R_1\theta, u_t)_{\Gamma_0} d\tau - \int_t^T (\mathcal{F}_2(u), u_t)_{\Gamma_0} d\tau + \kappa \int_t^T (N_0^* L z_t, u_t)_{\Gamma_0} d\tau = 0 \tag{68}$$

$$E_\theta^0(T) + \int_t^T \|\nabla\theta\|_{\Gamma_0}^2 d\tau = E_\theta^0(t) - \int_t^T (R_2 u_t, \theta)_{\Gamma_0} d\tau = 0. \quad (69)$$

Then, collecting (67)-(69) we readily obtain the statement of the lemma.

Step 2. Reconstruction of the energy integral Multiplying equation (25) by z and integrating between 0 and T we obtain

$$\int_0^T \|L^{1/2}z\|^2 \leq C(E_z^0(T) + E_z^0(0)) + \int_0^T \|z_t\|^2 dt + H_0^T(z) + \kappa \int_0^T |(u_t, N_0^* Lz)| dt + \int_0^T |(\mathcal{F}_1(z), z)| dt. \quad (70)$$

It follows from (9) that

$$|(\mathcal{F}_1(z), z)| \leq C_R \|L^{1/2}z\|_\Omega \|z\|_\Omega. \quad (71)$$

Besides, using well-known interpolation results we get for $0 < \delta < 1/4$

$$|(u_t, N_0^* Lz)| \leq \|u_t\|_{\Gamma_0} \|N_0^* L^{1/2+\delta}\| \|L^{1/2-\delta}z\|_\Omega \leq \varepsilon \|u_t\|_{\Gamma_0}^2 + \varepsilon_1 \|L^{1/2}z\|_\Omega^2 + C_{\varepsilon,\varepsilon_1} \|z\|^2,$$

for any $\varepsilon, \varepsilon_1 > 0$. Then, by appropriately choosing ε and ε_1 we obtain from (70) and (71) that

$$\int_0^T \|L^{1/2}z\|^2 dt \leq C(E_z^0(T) + E_z^0(0)) + \varepsilon \int_0^T \|u_t\|^2 + 2\int_0^T \|z_t\|^2 dt + C_1 H_0^T(z) + C_2(R, \varepsilon) \int_0^T \|z\|^2 dt \quad (72)$$

for any $\varepsilon > 0$.

After multiplication (26) by u and integration between 0 and T

$$\int_0^T \|A^{1/2}u\|^2 \leq C(E_u^0(T) + E_u^0(0)) + \int_0^T \|B^{1/2}u_t\|^2 dt + Q_0^T(u) + \int_0^T (\mathcal{F}_2(u), u) d + \int_0^T (R_1\theta, u) dt + \kappa \int_0^T (N_0^* L z_t, u) dt. \quad (73)$$

Multiplying equation (27) by $(-\Delta)^{-1}\theta$ and integrating between 0 and T we obtain

$$\int_0^T \|\theta\|^2 \le C(E_\theta^0(T)+E_\theta^0(0)) + C_3\int_0^T \|u_t\|^2 dt \tag{74}$$

Combining (73) and (74) we arrive at

$$\int_0^T \|A^{1/2}u\|^2 dt + \int_0^T \|\theta\|^2 dt \le C(E_u^0(T)+E_u^0(0)+E_\theta^0(T)+E_\theta^0(0)) \\ + C_1(\int_0^T \|\nabla\theta\|^2 dt + \int_0^T \|u_t\|^2 dt) + Q_0^T(u) + C(R)\int_0^T \|z\|^2 dt \\ + C(R)\int_0^T \|u\|^2 dt. \tag{75}$$

Collecting (72) and (75) we get

$$\int_0^T E^0(t)dt \le C(E^0(T)+E^0(0)) + C_1\int_0^T (\|z_t\|^2+\|u_t\|^2+\|\nabla\theta\|^2)dt + C_2 H_0^T(z) \\ + C_3 Q_0^T(u) + C_4(R)\int_0^T (\|z\|^2+\|v\|^2)dt, \tag{76}$$

where $H_0^T(z)$ and $Q_0^T(u)$ are defined in (57) and (59). It follows from energy relation (63) that

$$E^0(0) = E^0(T) + G_0^T(z) + R_0^T(u) + \int_0^T \|\nabla\theta\|^2 dt \\ + \int_0^T (\mathcal{F}_1(z), z_t)dt + \int_0^T (\mathcal{F}_2(u), u_t)dt \tag{77}$$

and

$$TE^0(T) \le \int_0^T E^0(t)dt - \int_0^T\int_t^T (\mathcal{F}_1(z), z_t)d\tau - \int_0^T\int_t^T (\mathcal{F}_2(u), u_t)d\tau \tag{78}$$

therefore, combining (77) and (78) with (76) we arrive at (55) .

To prove the existence of a compact global attractor of the dynamical system (H, S_t) we need to show that it is asymptotically smooth. We recall [11] that a dynamical system (H, S_t) is called asymptotically smooth iff for any bounded set $\mathcal{B}$ in H such that $S_t\mathcal{B} \subset \mathcal{B}$ for $t > 0$ there exists a compact set $\mathcal{K}$ in the closure $\overline{\mathcal{B}}$ of $\mathcal{B}$, such that

$$\lim_{t\to+\infty} \sup_{y\in\mathcal{B}} dist_X\{S_t y, \mathcal{K}\} = 0$$

In order to establish this property we apply the compactness criterion due to [14]. This result is recorded below in the abstract formulation given and used in [8].

Proposition 2 *Let (H, S_t) be a dynamical system on a complete metric space H endowed with a metric d. Assume that for any bounded positively invariant set $\mathcal{B}$ in H and for any $\epsilon > 0$ there exists $T = T(\epsilon, \mathcal{B})$ such that*

$$d(S_T y_1, S_T y_2) \le \epsilon + \Psi_{\epsilon,\mathcal{B},T}(y_1, y_2), \quad y_i \in \mathcal{B},$$

where $\Psi_{\epsilon,\mathcal{B},T}(y_1, y_2)$ is a nonnegative function defined on $\mathcal{B} \times \mathcal{B}$ such that

$$\liminf_{m\to\infty} \liminf_{n\to\infty} \Psi_{\epsilon,\mathcal{B},T}(y_n, y_m) = 0 \tag{79}$$

for every sequence $\{y_n\}$ in $\mathcal{B}$. Then the dynamical system (H, S_t) is asymptotically smooth.

Lemma 4 *Let Assumptions 1-3 hold. Then, for any $\epsilon > 0$ and $T > 1$ there exist constants $C_\epsilon(R)$ and $C(R, T)$ such that*

$$E(T) \le \epsilon + \frac{1}{T}[C_\epsilon(R) + \Psi_T(z, u)] + C(R, T)lot(z, u), \tag{80}$$

where

$$lot(z, u) = \sup_{[0,T]}[\|z(t)\|_\Omega + \|u(t)\|_{\Gamma_0}]$$

Proof. To establish (80) we return to inequality (55) and proceed with the estimate of its right hand side. Preliminary we recall inequalities which hold under Assumptions 1 and 3 only (see, e.g. [3]). There exists a constant $C_0 > 0$ and such that

$$|(G(\zeta + z) - G(\zeta), h)| \le C_0[(G(\zeta), \zeta) + (G(\zeta + z), \zeta + z)]\|L^{1/2}h\| + C_0\|h\| \tag{81}$$

for any $\zeta, z, h \in \mathcal{D}(L^{1/2})$ and

$$|(B(\omega + u) - B(\omega), l)| \le C_0[(B(\omega), \omega) + (B(\omega + u), \omega + u)]\|A^{1/2}l\| + C_0\|l\| \tag{82}$$

for any $\omega, u, l \in \mathcal{D}(A^{1/2})$.

It follows readily from (81), (82) that

$$H_0^T(z) \le C_R + CTlot(z, u) \tag{83}$$

and

$$Q_0^T(z) \leq C_R + CTlot(z,u). \tag{84}$$

Next, using Assumption 2 we get

$$\int_0^T (\|z_t\|^2_\Omega + \|u_t\|^2_{\Gamma_0} + \|\nabla\theta\|^2_{\Gamma_0}) \leq \varepsilon T + C_\varepsilon(R) \tag{85}$$

for every $\varepsilon > 0$. On the other hand, taking $t = 0$ in (63) and using the fact that $E(0) \leq C_R$, we get

$$G_0^T(z) + R_0^T(u) + \int_0^T \|\nabla\theta\|^2 dt \leq$$

$$C_R + |\int_0^T (\mathcal{F}_1(z), z_t)d\tau| + |\int_0^T (\mathcal{F}_1(u), u_t)d\tau| \tag{86}$$

therefore, (80) follows from Lemma 2 and estimates (83)-(86).

Theorem 2 *Let Assumptions 1-3 hold. Then the dynamical system (H, S_t) generated by problem (25)-(28) is asymptotically smooth.*

Proof. It follows from Lemma 4 that given $\epsilon > 0$ there exists $T = T(\epsilon) > 1$ such that for initial data $y_1, y_2 \in \mathcal{B}$ we have

$$\|S_T y_1 - S_T y_2\|_H = \|(z(T), z_t(T), u(T), u_t(T), \theta(T))\|_H \leq$$

$$C|E(T)|^{1/2} \leq \epsilon + \Psi_{\epsilon,\mathcal{B},T}(y_1, y_2), \tag{87}$$

where

$$\Psi_{\epsilon,\mathcal{B},T}(y_1, y_2) = C_{\epsilon,\mathcal{B},T}\{\Psi_T(z,u) + lot(z,u)\}^{1/2}$$

where $\Psi_T(z,u)$ is given by (60) and satisfies (79) (see e.g. [3]). Then, by Proposition 1 (87) implies the statement of the theorem.

Our first main result provides the existence of a global attractor for problem.

Theorem 3 *Under Assumptions 1-3 the dynamical system (H, S_t) generated by problem (25)-(28) possesses a compact global attractor $\mathfrak{A}$ which coincides with the unstable manifold $M^u(\mathcal{N})$ emanating from the set $\mathcal{N}$ of stationary points for S_t.*

The proof is similar to that given in [3].

Stabilizability estimate. In this section we derive a stabilizability estimate which will play a crucial role in the proofs of both finite-dimensionality and regularity of attractors.

The following lemma can be found in [3].

Lemma 5 *Under Assumption 4 the following estimate holds true for some $\delta > 0$*

$$|\int_t^T (\mathcal{F}_1(z), z_t)d\tau| \le C_{R,T} \max_{[0,T]} \|z\|^2_{1-\delta} + \varepsilon \int_0^T \|L^{1/2}z\|^2 d\tau + C_\varepsilon(R) \int_0^T (\|d_t(t)\|^2 + \|\zeta_t(t)\|^2)\|L^{1/2}z\|^2 d\tau$$

for all $t \in [0,T]$, where $\epsilon > 0$ can be taken arbitrarily small. Here, $\mathcal{F}_1$ is given by (61).

Now we state the analogue of Lemma 4 for the plate component which follows immediately from Assumption 1.

Lemma 6 *Under Assumptions 1 and 4 the following estimate holds true for all $t \in [0,T]$*

$$|\int_t^T (\mathcal{F}_2(u), u_t)d\tau| \quad \le \quad C_R \max_{[0,T]} \|u\|^2 \;+\; \varepsilon \int_0^T (\|\mathcal{A}^{1/2}u\|^2 \;+\; \|u_t\|^2)d\tau, \quad (88)$$

where $\varepsilon > 0$ can be taken arbitrarily small. Here, $\mathcal{F}_2$ is given by (61).

Now we are in position to estimate $\Psi_T(z,u)$ defined in (60).

Lemma 7 *For any $\varepsilon > 0$ the following estimate holds true*

$$\Psi_T(z,u) \le \epsilon \int_0^T E^0(t)dt + C(T,R)\Sigma_T(z,u)$$

with $\Sigma_T(z,u)$ given by

$$\Sigma_T(z,u) = C \max_{[0,T]}(\|u\|^2_{1-\delta} + \|z\|^2_{1-\delta}) + \int_0^T G_{d,\zeta}(\tau)\|L^{1/2}z\|^2 d\tau + \int_0^T B_{\omega,\nu}(\tau)\|A^{1/2}u\|^2 d\tau, \quad (89)$$

here $G_{d,\zeta}$ is given by

$$G_{d,\zeta} = m^{-1}[(G(d(t)), d(t))_\Omega + (G(\zeta(t)), \zeta(t))_\Omega] \quad (90)$$

Proof. It follows by the lower bound in (17) that $ms^2 \le sg(s)$, where $i = 1, 2$ and thus

$$\|d_t(t)\|^2_\Omega + \|\zeta_t(t)\|^2_\Omega \le G_{d,\zeta}, \quad \|\omega_t(t)\|^2_{\Gamma_0} + \|\nu_t(t)\|^2_{\Gamma_0} \le B_{\omega,\nu}.$$

Therefore, using Lemma 5 and Lemma 6 and the elementary inequality $\|\xi\| \le \epsilon + (4\epsilon)^{-1}\|\xi\|^2$, valid for arbitrary small $\epsilon > 0$, we obtain the statement of the lemma.

To proceed we need the following assertion

Lemma 8 *For any $T \ge T_0 > 0$ the following estimate holds true:*

$$TE^0(T) + \int\limits_0^T E^0(t)dt \le C[G_0^T(z) + R_0^T(u) + \int\limits_0^T \|\nabla\theta\|^2 d\tau] + C_2(T,R)\Sigma_T(z,u), \quad (91)$$

where $\Sigma_T(z,u)$ is the same as in (89).

Proof. It follows from Assumption 4 [7] that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|G(\zeta+z) - G(\zeta), l| \le C_\epsilon(G(\zeta+z) - G(\zeta), z) + \epsilon(1 + (G(\zeta),\zeta) + (G(\zeta+z),\zeta+z))\|L^{1/2}l\|^2 \quad (92)$$

for any $\zeta, z, l \in \mathcal{D}(L^{1/2})$ and

$$|B(\omega+u) - B(\omega), l| \le C_\epsilon(B(\omega+u) - B(\omega), u) + \epsilon(1 + (B(\omega),\omega) + (B(\omega+u),\omega+u))\|A^{1/2}l\|^2 \quad (93)$$

for any $\zeta, z, l \in \mathcal{D}(A^{1/2})$.

Owing to estimates (92) and (93) it is immediately seen that

$$H_0^T(z) \le C_\epsilon G_0^T(z) + \epsilon \int\limits_0^T E^0(t)dt + \epsilon m \int\limits_0^T G_{d,\zeta}(\tau)\|L^{1/2}z\|^2 d\tau$$

and

$$Q_0^T(z) \le C_\epsilon R_0^T(z) + \epsilon \int\limits_0^T E^0(t)dt + \epsilon m \int\limits_0^T B_{\omega,\nu}(\tau)\|A^{1/2}u\|^2 d\tau,$$

where

$$B_{\omega,\nu} = \min\{m_1, m_2\}^{-1}[(B(\omega(t)),\omega(t))_{\Gamma_0} + (B(\nu(t)),\nu(t))_{\Gamma_0}]. \quad (94)$$

Consequently,

$$H_0^T(z) + Q_0^T(z) \le \epsilon \int\limits_0^T E^0(t)dt + C_\epsilon[R_0^T(z) + G_0^T(z) + \Sigma_T(z,u)]. \qquad (95)$$

Notice that by the lower bounds in (17), (18), (20) we have

$$\int\limits_0^T \|z_t\|^2 dt \le \frac{1}{m} G_0^T(z), \quad \int\limits_0^T \|u_t\|^2 dt \le \frac{1}{\min\{m_1, m_2\}} R_0^T(u). \qquad (96)$$

Now we apply estimates (95), (96) and Lemma 7 to the basic inequality in Lemma 2. Choosing ε sufficiently small we obtain the statement of the lemma.

Now we are in position to prove the stabilizability inequality for the dynamical system (H, S_t).

Theorem 4 *Let Assumptions 1-4 hold. Then there exist positive constants C_1, C_2 and ω depending on R such that for any $y_1, y_2 \in W_R$ the following estimate holds true for any $\delta < 1$ and independent of κ, β, μ:*

$$\|S_t y_1 - S_t y_2\|_H^2 \le C_1 e^{-\omega t}\|y_1 - y_2\|_H^2 + C_2 \max_{[0,t]}(\|z(\tau)\|_{1-\delta}^2 + \|u(\tau)\|_{1-\delta}^2) \quad (97)$$

Above we have used the notation

$$S_t y_1 = (d(t), d_t(t), \omega(t), \omega_t(t), \psi(t)), \quad S_t y_1 = (\zeta(t), \zeta_t(t), \nu(t), \nu_t(t), \phi(t)).$$

Proof. Using inequality (63) and Lemma 8 we obtain that

$$G_0^T(z) + R_0^T(z) + \int\limits_0^T \|\nabla\theta\|^2 d\tau \le E^0(0) - E^0(T) + \epsilon \int\limits_0^T E^0(\tau)d\tau + C(T,R)\Sigma_T(z,u)$$

for any $\epsilon > 0$. Combining this estimate with (91) we get that there exists $T > 1$ such that

$$E^0(T) \le qE^0(0) + C_{R,T}\Sigma_T(z,u), \quad 0 < q \equiv q(T,R) < 1. \qquad (98)$$

Applying the procedure described in [4] we get from (98) that there exists $\omega > 0$ such that

$$E^0(t) \le C_1 e^{-\omega t} E^0(0)$$
$$+ C_2[\int\limits_0^t e^{-\omega(t-\tau)}[D_{h,\zeta}(\tau) + B_{\omega,\nu}(\tau) + \|\nabla\theta\|^2]E^0(\tau)d\tau + lot_t(z,u)]$$

for all $t \geq 0$. Therefore, by the Gronwall's lemma we get

$$E^0(t) \leq [C_1 e^{-\omega t} E^0(0) + C_2 lot_t(z,u)] e^{\int_0^t e^{-\omega(t-\tau)}[D_{h,\zeta}(\tau)+B_{\omega,\nu}(\tau)+\|\nabla\theta\|^2]d\tau}$$
$$\leq C_1 e^{-\omega t} E^0(0) + C_2 lot_t(z,u).$$

The above estimate and (62) yield estimate (97).

Properties of attractor. In this Subsection we establish the properties of the attractor to problem (25)-(28), namely, the finite dimensionality, boundedness in the higher-order spaces and upper-semicontinuity with respect to the parameters μ, β, κ.

Theorem 5 *Let Assumptions 1-4 hold. Then the attractor* $\mathfrak{A}$ *has a finite fractal dimension.*

The proof is similar to that given in [3].

Theorem 6 *The attractor* $\mathfrak{A}$ *is a bounded set in the space*

$$H_* = W^2_{6/p}(\Omega) \times \mathcal{D}(L^{1/2}) \times \mathcal{D}(A) \times \mathcal{D}(A^{1/2}) \times \mathcal{D}(-\Delta)$$

for $3 < p \leq 5$ *and in the space*

$$H_{**} = H^2(\Omega) \times \mathcal{D}(L^{1/2}) \times \mathcal{D}(A) \times \mathcal{D}(A^{1/2}) \times \mathcal{D}(-\Delta)$$

in the other cases. Moreover,

$$\sup_{t\in\mathbb{R}}\{\|z\|^2_{W^2_{6/p}(\Omega)} + \|z_t\|^2_{H^1(\Omega)} + \|z_{tt}\|^2\} \leq C \tag{99}$$

$$\sup_{t\in\mathbb{R}}\{\|v_{tt}\|^2 + \|w_{tt}\|^2 + \|v_t\|^2_{[H^1_0(\Omega)]^2} + \|\theta_t\|^2\} \leq C, \tag{100}$$

$$\sup_{t\in\mathbb{R}} \|w_t\|^2_{H^1_0(\Omega)} \leq C, \tag{101}$$

$$\sup_{t\in\mathbb{R}} \|\theta\|_{H^2\cap H^1_0(\Omega)} \leq C, \tag{102}$$

$$\sup_{t\in\mathbb{R}} \|w\|_{H^2\cap H^1_0(\Omega)} \leq C, \tag{103}$$

$$\sup_{t\in\mathbb{R}} \|v + \nabla w\| \leq \frac{1}{\sqrt{\mu}} C \tag{104}$$

$$\sup_{t\in\mathbb{R}} \|v_t + \nabla w_t\| \leq \frac{1}{\sqrt{\mu}} C, \tag{105}$$

where C *does not depend on* κ, μ, *and* β.

Proof. Estimate (104) follows readily from the uniform, with respect to κ and μ, boundedness of the attractor in H. Let $\{y(t) = (z(t), z_t(t), u(t), u_t(t), \theta(t))\} \in H$

be a full trajectory from the attractor $\mathfrak{A}$. Let $|\sigma| \le 1$. Applying Theorem 4 with $y_1 = y(s+\sigma)$, $y_2 = y(s)$ for the interval $[s,t]$ in place of $[0,t]$ we obtain

$$\|y(t+\sigma) - y(t)\|_H^2 \le C_1 e^{-\omega(t-s)}\|y(s+\sigma) - y(s)\|_H^2 + C_2 \max_{\tau\in[s,t]}(\|z(\tau+\sigma) - z(\tau)\|_{1-\delta}^2 + \|u(\tau+\sigma) - u(\tau)\|_{1-\delta}^2)$$

for any $t, s \in \mathbb{R}$ such that $s \le t$ and $|\sigma| \le 1$. Letting $s \to -\infty$ gives

$$\|y(t+\sigma) - y(t)\|_H^2 \le C_2 \max_{\tau\in[-\infty,t]}(\|z(\tau+\sigma) - z(\tau)\|_{1-\delta}^2 + \|u(\tau+\sigma) - u(\tau)\|_{1-\delta}^2) \quad (106)$$

By interpolation we get

$$\|z(\tau+\sigma) - z(\tau)\|_{1-\delta}^2 + \|u(\tau+\sigma) - u(\tau)\|_{1-\delta}^2 \le \varepsilon\|y(t+\sigma) - y(t)\|_H^2 + C_\varepsilon(\|z(\tau+\sigma) - z(\tau)\|^2 + \|u(\tau+\sigma) - u(\tau)\|^2) \quad (107)$$

for every $\varepsilon > 0$. Therefore we obtain from (106) and (107)

$$\max_{\tau\in[-\infty,t]} \|y(t+\sigma) - y(t)\|_H^2 \le C \max_{\tau\in[-\infty,t]}(\|z(\tau+\sigma) - z(\tau)\|^2 + \|u(\tau+\sigma) - u(\tau)\|^2) \quad (108)$$

for any $t \in \mathbb{R}$ and $|\sigma| < 1$. On the attractor we have

$$\frac{1}{\sigma}\|z(\tau+\sigma) - z(t)\| \le \frac{1}{\sigma}\int_0^\sigma \|z_t(\tau+t)\| d\tau \le C, \quad t \in \mathbb{R},$$

and

$$\frac{1}{\sigma}\|u(\tau+\sigma) - u(t)\| \le \frac{1}{\sigma}\int_0^\sigma \|u_t(\tau+t)\| d\tau \le C, \quad t \in \mathbb{R},$$

which gives with (108)

$$\max_{\tau\in\mathbb{R}} \left\| \frac{y(\tau+\sigma) - y(\tau)}{\sigma} \right\|_H^2 \le C \text{ for } |\sigma| < 1.$$

This implies

$$\|z_{tt}\|^2 + \|L^{1/2}z_t\|^2 + \|u_{tt}\|^2 + \|A^{1/2}u_t\|^2 + \|\theta_t\|^2 \le C \quad (109)$$

and (105).

It follows readily from (5) that

$$\|\Delta\theta(t)\| \le C(\|u_t\|_{H^1(\Gamma_0)} + \|\theta_t\|) \le C$$

and from (4) that

$$\|\Delta w\| \leq C(\frac{1}{\mu} + \|v\|_{H^1(\Gamma_0)}) \leq C,$$

which implies (102) and (103). From (3) and (4) we conclude

$$\|\mathcal{A}u\| \leq C(\mu). \tag{110}$$

In case $1 \leq p \leq 3$ we have for the wave component

$$\|g(z_t)\| \leq C(1 + \|z_t\|^2_{L_{2p}(\Omega)}) \leq C(1 + \|z_t\|_1^2)$$

Therefore $z(t)$ solves the problem

$$(-\Delta + \lambda)z = h_1(t) \ \text{ in } \Omega, \ \frac{\partial z}{\partial n} = h_2(t) \ \text{ on } \Gamma, \tag{111}$$

where $h_1(t) \in L_\infty(\mathbb{R}, L_2(\Omega))$ and $h_2(t) \in L_\infty(\mathbb{R}, H^s(\Omega))$ for any $s < 3/2$. By the elliptic regularity theory we conclude that $z(t)$ is a bounded function with values in $H^2(\Omega)$.

In case $3 < p \leq 5$ we have that $g(z_t)$ is bounded in $L_{6/p}(\Omega)$ and therefore, z solves (111) with $h_1(t) \in L_\infty(\mathbb{R}, L_{6/p}(\Omega))$. The elliptic regularity theory gives that $z(t)$ is a bounded function with values in $W^2_{6/p}(\Omega)$, which implies together with (109) estimate (99).

Estimate (110) gives the boundedness of the component v in $H^1 \cap H^1_0(\Gamma_0)$ on the attractor for every $\mu > 1$, but not uniformly.

The following result is a corollary of Theorems 3, 5, 6.

Theorem 7 *Let f and g satisfy the conditions in Assumptions 1 and 2. Then the dynamical system (H_1, S_t^1) generated by the problem*

$$\begin{aligned} z_{tt} + g(z_t) - \Delta z + f(z) = 0 \ \text{ in } \ \Omega \times (0,T) \\ \tfrac{\partial z}{\partial n} = 0 \ \text{ on } \ \Gamma \times (0,T) \end{aligned} \tag{112}$$

possesses a compact global attractor $\mathfrak{A}_1 \equiv M^u(\mathcal{N}_1)$, where $\mathcal{N}_1$ is the set of equilibria for (112). If f and g satisfy Assumption 4, then the attractor $\mathfrak{A}_1$ has a finite fractal dimension and $\mathfrak{A}_1$ is a bounded set in the space $W^2_{6/p}(\Omega) \times \mathcal{D}(L^{1/2})$ in case $3 < p \leq 5$, and in the space $\mathcal{D}(L) \times \mathcal{D}(L^{1/2})$ in other cases.

Arguing as in [10] one can obtain the following result on the existence of attractor.

Theorem 8 *Let b_i, $i = 1, 2$, h, and h_0 satisfy the conditions in Assumptions 1 - 3 and $H_2 = H_0^2(\Gamma_0) \times H_0^1(\Gamma_0)$. Then the dynamical system (H_2, S_t^2) generated by the problem*

$$\begin{aligned} (1 - \Delta)w_{tt} + div b(-\nabla w_t) + b_0(w_t) + \Delta^2 w - div[h(|\nabla w|^2)\nabla w] + h_0(w) = 0, \\ w(x,t) = 0, \quad \nabla w(x,t) = 0 \qquad x \in \partial\Gamma_0, \ \ t > 0 \end{aligned} \tag{113}$$

possesses a compact global attractor $\mathfrak{A}_2 \equiv M^u(\mathcal{N}_2)$, where $\mathcal{N}_2$ is the set of equilibria for (113). If f, h, h_0, b_i, $i = 1, 2$ satisfy additionally Assumption 4, then the attractor $\mathfrak{A}_2$ has a finite fractal dimension.

Our last main result consists in the upper-semicontinuity of the family of attractors of problem (25)-(28) with respect to the parameters μ, κ, β.

Theorem 9 *Let Assumptions 1-4 hold. Denote by $S_t^{\mu,\kappa,\beta}$ the evolution operator of problem (25)-(28) in the space*

$$H_\mu = H = (L^{1/2}) \times L^2(\Omega) \times \mathcal{D}(A^{1/2}) \times L^2(\Gamma_0) \times H^1(\Gamma_0).$$

Let $\mathfrak{A}^{\mu,\kappa,\beta}$ be a global attractor for the system $(S_t^{\mu,\kappa,\beta}, H_\mu)$. Then the family of the attractors $\mathfrak{A}^{\mu,\kappa,\beta}$ is upper semi-continuous on $\Lambda = [1,\infty) \times [0,1] \times [0,1]$. Namely, we have that

$$\lim_{(\mu,\kappa,\beta)\to(\infty,0,0)} \sup_{y\in\mathfrak{A}^{\mu,\kappa,\beta}} \{dist_{H^{\delta_1,\delta_2}}(y, \mathfrak{A}_1 \times \mathfrak{A}_2 \times 0)\} = 0, \tag{114}$$

where

$$H^{\delta_1,\delta_2} = (L^{1/2-\delta_1}) \times L^2(\Omega) \times [[H^{1-\delta_2}(\Gamma_0)]^2 \times H^1(\Gamma_0)] \times L^2(\Gamma_0) \times L^2(\Gamma_0).$$

Here $\delta_2 > 0$, $\delta_1 \geq 0$ in case $p < 5$ and $\delta_1 > 0$ in case $p = 1$.

Proof. We base the proof on the idea presented in [12]. Assume that the statement of the theorem is not true. Then there exists a sequence $\{(\mu^n, \kappa^n, \beta^n\} \to (\infty, 0)$ such that $\mu^n \geq \mu_\infty$, $\kappa^n \leq \kappa_0$, $\beta^n \leq \beta_0$ and for any $n \in \mathbb{N}$ and a sequence $y^n \in \mathfrak{A}_{\mu^n,\kappa^n,\beta^n}$ such that

$$dist_{H^{\delta_1,\delta_2}}(y, \mathfrak{A}_1 \times \mathfrak{A}_2 \times 0) \geq \varepsilon, \quad n = 1, 2, ... \tag{115}$$

for some $\varepsilon > 0$. Let $y^n(t) = \{z^n(t), z_t^n(t), u^n(t), u_t^n(t), \theta^n(t)\}$ be a full trajectory in $\mathfrak{A}_{\mu^n,\kappa^n,\beta^n}$ passing through y^n $(y^n(0) = y^n)$. The functions y^n satisfy equations (25)-(28). It follows from (100), (101), (103) that the sequence $\{z^n(t), w^n(t), \theta^n(t)\}$ is uniformly with respect to n bounded in the space

$$\begin{aligned}\mathfrak{C}_1 = &\big(C_{bnd}(\mathbb{R}; W^2_{6/p}(\Omega)) \cap C^1_{bnd}(\mathbb{R}; \mathcal{D}(L^1/2)) \cap C^2_{bnd}(\mathbb{R}; L^2(\Omega))\big) \\ &\times \big(C_{bnd}(\mathbb{R}; (H^2 \cap H^1_0)(\Gamma_0)) \cap C^1_{bnd}(\mathbb{R}; H^1_0(\Gamma_0)) \cap C^2_{bnd}(\mathbb{R}; L^2(\Gamma_0))\big) \times \\ &\qquad \big(C_{bnd}(\mathbb{R}; H^2 \cap H^1_0(\Omega)) \cap C^1_{bnd}(\mathbb{R}; L^2(\Omega))\big).\end{aligned}$$

Hence, by Aubin's compactness theorem [25] $\{z^n(t), w^n(t), \theta^n(t)\}$ is a compact sequence in the space

$$\begin{aligned}\mathcal{W}_1 = &\Big(C([-T,T]; (L^{1/2-\delta_1})) \cap C^1([-T,T]; L^2(\Omega))\Big) \\ &\times \Big(C([-T,T]; H^1_0(\Gamma_0)) \cap C^1([-T,T]; L^2(\Gamma_0))\Big) \\ &\qquad \times C([-T,T]; H^1(\Gamma_0))\end{aligned}$$

for every $T > 0$. Estimate (100) yields that the sequence $\{v^n\}$ is uniformly with respect to n bounded in the space

$$\mathfrak{C}_2 = \big(C_{bnd}(\mathbb{R}; [H_0^1(\Omega)]^2) \cap C_{bnd}^1(\mathbb{R}; [H_0^1(\Omega)]^2) \cap C_{bnd}^2(\mathbb{R}; [L^2(\Omega)]^2\big).$$

Thus, we deduce that there exists a function $\{\mathbf{z}(t), \mathbf{w}(t), \Theta(t)\} \in \mathfrak{C}_1$ such that

$$\lim_{k\to\infty} \max_{[-T,T]} \{\|z^{n_k}(t) - \mathbf{z}(t)\|^2_{\mathfrak{D}(L^{1/2-\delta_1})} + \|z_t^{n_k}(t) - \mathbf{z}_t(t)\|^2_{L^2(\Omega)} + \|w^{n_k}(t) - \mathbf{w}(t)\|^2_{H_0^1(\Gamma_0)} + \|w_t^{n_k}(t) - \mathbf{w}_t(t)\|^2_{L^2(\Gamma_0)} + \|\theta^{n_k}(t) - \Theta(t)\|^2_{H_0^1(\Gamma_0)} = 0 \quad (116)$$

for any $\delta_1 > 0$ in case $p < 5$ and $\delta_1 \geq 0$. Analogously, the sequence $\{v^n\}$ is compact in the space $C([-T,T]; [H_0^{1-\delta_2}(\Omega)]^2) \cap C^1([-T,T]; [L^2(\Gamma_0)]^2)$. Moreover, by (104), (105) we get that

$$\lim_{k\to\infty} \max_{[-T,T]} \{\|v^{n_k} + \nabla \mathbf{w}\|_{[H_0^{1-\delta_2}(\Gamma_0)]^2} + \|v_t^{n_k} + \nabla \mathbf{w}_t\|_{[L^2(\Gamma_0)]^2}\} = 0 \quad (117)$$

for every $T > 0$. By the trace theorem we infer from (117) that

$$\lim_{k\to\infty} \|v^{n_k} + \nabla \mathbf{w}\|_{[L^2(\partial\Gamma_0)]^2} = 0,$$

therefore,

$$\nabla \mathbf{w}|_{\partial\Gamma_0} = 0.$$

We can choose functions ϕ, ψ and χ in (48)-(50) of the following form: $\psi(t) = (-\partial_{x_1} l, -\partial_{x_2} l, l)\cdot p(t)$ and $\chi(t) = \chi \cdot p(t)$, where $\phi \in (L^{1/2})$, $l \in H_0^2(\Omega)$, $\chi \in H_0^1(\Omega)$ and $p(t)$ is a scalar continuously differentiable function such that $p(T) = 0$. It is easy to see that

$$(\mathcal{A}u^{n_k}, \psi) = [-\nu(\mathrm{div} v^{n_k}, \Delta l) - (1-\nu)\int_\Omega [\partial_{x_1} v_1^{n_k} \cdot \partial_{x_1}^2 l + \partial_{x_2} v_2^{n_k} \cdot \partial_{x_2}^2 l + (\partial_{x_1} v_2^{n_k} + \partial_{x_2} v_1^{n_k})\partial_{x_1 x_2} l] dx] p(t). \quad (118)$$

Therefore, passing to the limit $k \to \infty$ we get

$$\lim_{k\to\infty} \int_0^T (\mathcal{A}u^{n_k}, \psi) dt = \int_0^T (\Delta \mathbf{w}, \Delta l) p(t) dt.$$

By Assumptions 1, 2, 3 we pass to the limit in the nonlinear terms. Observing (116) and (118) we get

$$-\int_0^T (\mathbf{z}_t, \phi'(t)) dt + \int_0^T (L^{1/2}\mathbf{z}, L^{1/2}\phi) dt + \int_0^T (g(\mathbf{z}_t), \phi) dt + \int_0^T (f(\mathbf{z}), \phi) dt = (z_1, \phi(0)) \quad (119)$$

$$-\int_0^T (\mathbf{w}_t, l)p'(t)dt - \int_0^T (\nabla \mathbf{w}_t, \nabla l)p'(t)dt + \int_0^T (K\mathbf{w}, Kh)p\,(t)dt$$
$$+ \int_0^T (divb(\nabla \mathbf{w}_t) + b_0(\mathbf{w}_t), l)p\,(t)dt$$
$$+ \int_0^T (div[l(|\nabla \mathbf{w}|^2)\nabla \mathbf{w}], l)p\,(t)dt = (w_1, l)p\,(0) + (\nabla w_1, \nabla l)p\,(0), \quad (120)$$

$$-\int_0^T (\Theta, \tau)p'(t)dt + \int_0^T (\nabla\Theta, \nabla\tau)p\,(t)dt = (\theta_0, \tau)p\,(0), \quad (121)$$

where $K : H_0^2(\Gamma_0) \to L^2(\Gamma_0)$ such that $K^2 = \Delta^2 : H^4 \cap H_0^2(\Gamma_0) \to L^2(\Gamma_0)$.

One can deduce from (119)-(121) that $\mathbf{z}(t)$, $\mathbf{w}(t)$ are weak solutions to problems (112) and (113) possessing the properties

$$\sup_{t\in\mathbb{R}}\{\|\mathbf{z}(t)\|^2_{\mathcal{D}(L^{1/2})} + \|\mathbf{z}_t(t)\|^2_{L^2(\Omega)}\} \le C$$

$$\sup_{t\in\mathbb{R}}\{\|\mathbf{w}(t)\|^2_{H^2\cap H_0^1(\Gamma_0)} + \|\mathbf{w}_t(t)\|^2_{H_0^1(\Gamma_0)} + \|\Theta(t)\|^2_{L^2(\Gamma_0)}\} \le C$$

and

$$\nabla \mathbf{w}|_{\partial\Gamma_0} = 0.$$

Consequently, $\{\mathbf{z}(t), \mathbf{z}_t(t)\}$ and $\{\mathbf{w}(t), \mathbf{w}_t(t)\}$ are full trajectories to (112) and (113) which belong to the attractor $\mathfrak{A}^1$ and $\mathfrak{A}^2$. The function $\Theta(t)$ is a full trajectory to the problem

$$\Theta_t + \Delta\Theta = 0, \quad x \in \Gamma_0,\ t > 0$$
$$\Theta = 0, \quad x \in \partial\Gamma_0,$$

which is exponentially stable. Consequently, $\Theta \equiv 0$. Thus, it follows from (116) and (117) that

$$\lim_{n_k\to 0}\{\|v^{n_k}(0) + \nabla\mathbf{w}(0)\|^2_{[H_0^{1-\delta_2}(\Gamma_0)]^2} + \|w^{n_k}(0) - \mathbf{w}(0)\|^2_{H_0^1(\Gamma_0)}$$
$$+\|v_t^{n_k}(0) + \nabla\mathbf{w}_t(0)\|^2_{[L^2(\Gamma_0)]^2} + \|w_t^{n_k}(0) - \mathbf{w}_t(0)\|^2_{L^2(\Gamma_0)}$$
$$+\|\theta^{n_k}(0)\|^2_{H_0^1(\Gamma_0)}\} = 0$$

and

$$\lim_{n_k\to 0}\{\|z^{n_k}(0) + \mathbf{z}(0)\|^2_{\mathcal{D}(L^{1/2-\delta_1})} + \|z_t^{n_k}(0) - \mathbf{z}_t(0)\|^2_{L^2(\Omega)}\} = 0$$

and we obtain a contradiction to (115). Consequently, (114) holds true.

System with non-conservative forces ($\gamma \neq 0$).

Consider now system (1)-(7) with $\gamma \neq 0$. This case corresponds to the non-conservative nonlinearity and non-monotone energy.

Note that Assumption 1 with $h^* h_0^* > 2\gamma^2$ guarantees that there exist a positive constant C_0 such that

$$H(r) = C_0 + \frac{1}{2}\int_0^r h(\xi)d\xi \geq 0, \quad r \in \mathbb{R}_+, \quad H_0(s) = C_0 + \frac{1}{2}\int_0^s h_0(\xi)d\xi \geq 0, \quad s \in \mathbb{R}.$$

Moreover, there exist positive constants C, C_1 and C_2 such that

$$\gamma rs + H(r) + H_0(s) + C \geq 0, \quad r \in \mathbb{R}_+, \ s \in \mathbb{R}. \tag{122}$$

and

$$\gamma rs \leq C_1(\sigma^2 + H(r)) + C_2, \quad r \in \mathbb{R}_+, \ s \in \mathbb{R}. \tag{123}$$

The additional assumption for the non-conservative case is the following:

Statement 5

- *There exist positive constants c_1 and c_2 such that*

$$-rh(r) \leq -c_1 H(r) + c_2, \quad r \in \mathbb{R}_+ \tag{124}$$

and

$$-rh_0(r) \leq -c_1 H_0(r) + c_2, \quad s \in \mathbb{R} \tag{125}$$

- *For any $\varepsilon > 0$ there exists a positive constant C_ε such that*

$$-\gamma rs \leq \varepsilon[H(r) + H_0(s)] + C_\varepsilon, \quad r \in \mathbb{R}_+, s \in \mathbb{R} \tag{126}$$

and

$$\gamma r\sigma \leq \varepsilon[\sigma^2 + H(r)] + C_\varepsilon, \quad r \in \mathbb{R}_+, \sigma \in \mathbb{R}. \tag{127}$$

- *There exist positive constants c_1 and c_2 such that*

$$-rf(r) \leq -c_1 \Pi(r) + c_2, \quad r \in \mathbb{R} \tag{128}$$

The assumptions (124)-(127) were made to guarantee the existence of the global attractor for the Mindlin plate system in [4]. Now we are in position to give the abstract formulation of system (1)-(7). Denote

$$F^*(u) = (0, 0, \frac{\gamma}{2}|v|^2),$$

$$F_2(u) = (v_1[\gamma w + h(|v|^2)], v_2[\gamma w + h(|v|^2)], h_0(w)). \tag{129}$$

and

$$\Pi_1(u) = \frac{\gamma}{2}\int_\Omega w|v|^2 dx.$$

Let

$$\mathcal{E}_0(t) = E_z^0(z, z_t) + E_u^0(u, u_t) + E_\theta^0(\theta) + \Pi(z) + \Pi_0(u), \tag{130}$$

where E_z^0, E_z^0, E_z^0, Π, Π_0 are given by (37)-(39) and (29), (32) respectively. We define the total energy in the following way:

$$\mathcal{E}(t) = \mathcal{E}_0(t) + \Pi_1(u). \tag{131}$$

It is easy to see from (122) and (123) that

$$-\frac{1}{2}\Pi_0(u) - C_1 \le \Pi_1(u) \le C_2 \int\limits_\Omega [|w|^2 + H(|v|^2)]dx + C_3 \tag{132}$$

Applying the same arguments as in case $\gamma = 0$ we obtain the following theorem.

Theorem 10 *Under Assumptions 1 with $h^* h_0^* > 2\gamma^2$, 3 for any initial conditions*

$$y_0 = (z^0, z^1, u^0, u^1, \theta^0) \in H$$

there exists a unique generalized solution $y(t) = (z(t), z_t(t), u(t), u_t(t), \theta(t))$ to the PDE system (25)-(28) with F_2 defined by (129), which depends continuously on initial data. This solution satisfies the energy inequality

$$\mathcal{E}(t) + \int\limits_s^t (G(z_t), z_t)_\Omega d\tau + \int\limits_s^t (B(u_t), u_t)_{\Gamma_0} d\tau \\ + \int\limits_s^t \|\nabla\theta\|_{\Gamma_0}^2 d\tau \le \mathcal{E}(s) + \int\limits_s^t (F^*(u), u_t) d\tau, \quad 0 \le s \le t,$$

with the total energy $\mathcal{E}(t)$ given by (131). Moreover, if initial data are such that

$$z^0, z^1 \in (L^{1/2}), \ \ u^0 \in \mathcal{D}(A), \quad u^1 \in \mathcal{D}(A^{1/2}), \ \ \theta^0 \in \mathcal{D}(-\Delta)$$

and

$$L[z^0 - \kappa N_0 u^1] + G(z^1) \in L_2(\Omega)$$

then there exists a unique strong solution $y(t)$ satisfying the energy identity:

$$\mathcal{E}(t) + \int\limits_s^t (G(z_t), z_t)_\Omega d\tau + \int\limits_s^t (B(u_t), u_t)_{\Gamma_0} d\tau \\ + \int\limits_s^t \|\nabla\theta\|_{\Gamma_0}^2 d\tau = \mathcal{E}(s) + \int\limits_s^t (F^*(u), u_t) d\tau, \quad 0 \le s \le t. \tag{133}$$

In contrast to the conservative case, the non-conservative system is not gradient and the energy is not monotone, i.e. one cannot guarantee the existence of a bounded absorbing set without additional arguments. To prove the dissipativity of system (25)-(28) in case $\gamma \neq 0$ we resort to the Lapunov's method combined with the barriers method.

Theorem 11 *Let Assumptions 1-3, 5 hold. Then the dynamical system (H, S_t) generated by problem (25)-(28) possesses an absorbing ball $\mathcal{B}(R)$ of the radius R independent of β, κ, and μ.*

Proof. Consider the functional

$$V(z, z_t, u, u_t, \theta) = \mathcal{E}(t) + \delta[(z_t, z) + (u_t, u)],$$

where $\delta > 0$ will be chosen later. It follows from (132) that there exist positive constants C_i, $i = \overline{1, 4}$ such that

$$C_1 E^0(z, z_t, u, u_t, \theta) - C_2 \leq V(z, z_t, u, u_t, \theta) \leq C_3 E^0(z, z_t, u, u_t, \theta) + C_4.$$

After differentiating the Lyapunov function by t we obtain

$$\frac{d}{dt}V = (G(z_t), z_t) + (B(u_t), u_t) - (F^*(u), u_t) + \delta[\|z_t\|^2 + \|u_t\|^2 \\ - (G(z_t), z) - \|L^{1/2}z\|^2 - \kappa(LN_0u_t, z) - (F_1(z), z) - \|A^{1/2}u\|^2 - (R_1\theta, z) \\ - (B(u_t), u) - (F_2(u), u) - \kappa(N_0^* L z_t, u)].$$

Taking under consideration (24), (124)-(126), 128 we get

$$\frac{d}{dt}V \leq -(G(z_t), z_t) - (B(u_t), u_t) - (F^*(u), u_t) - \|\nabla\theta\|^2 + \delta[\|z_t\|^2 + \|u_t\|^2 \\ - (G(z_t), z) - \frac{1}{2}\|L^{1/2}z\|^2 - \frac{1}{2}\|A^{1/2}u\|^2 + \|\nabla\theta\|^2 \\ - (B(u_t), u) + C[\|z_t\|^2 + \|u_t\|^2] - c_1/2[\Pi_0(u) + \Pi(z)] + C]. \quad (134)$$

It follows from (127) that for any $\varepsilon > 0$

$$(F^*(u), u_t) = \frac{\gamma}{2}\int_\Omega |v|^2 w_t dx \leq \varepsilon \int_{\Gamma_0} [|w|^2 + H(|v|^2)]dx + C_2 \\ \leq \varepsilon[\|w_t\|^2 + \Pi_0(u)] + C. \quad (135)$$

Consider now the term $(B(u_t), u)$. Let $\Gamma_0^1 = \{x \in \Gamma_0 : |u_t(x)| \geq 1\}$ and $\Gamma_0^2 = \Gamma_0 \setminus \Gamma_0^1$. We obviously have that

$$|(B(u_t), u)| \leq \int_{\Gamma_0} |b(u_t)||u|dx \leq \int_{\Gamma_0^1} |b(u_t)||u|dx + C\int_{\Gamma_0^2} |u|dx \\ \leq ([\int_{\Gamma_0^1} |b(u_t)|^{\frac{p_1}{1+p_1}}dx]\|A^{1/2}u\| + C\|u\|^2) \\ \leq C(B(u_t), u_t)E^0(z, u, \theta) + \bar{C}\|u\|^2 \leq C(B(u_t), u_t)[V + 1]^{1/2} + \bar{C}\|u\|^2. \quad (136)$$

Analogously,

$$|(G(z_t), z)| \le C(G(z_t), z_t)[V+1]^{1/2} + \bar{C}\|z\|^2. \qquad (137)$$

Consequently, collecting Assumption 2, (134)-(137) and choosing $\delta = 4\varepsilon(1/2 + \bar{C}\max\{\lambda_z, \lambda_u\})$, where λ_z and λ_u are the first eigenvalues of L and A respectively, we get

$$\frac{d}{dt}V(t) + \varepsilon V(t) \le d_1(\varepsilon + C) + d_2(\varepsilon[1+V(t)]^{1/2} - d_4)[(G(z_t), z_t) + (B(u_t), u_t)]. \qquad (138)$$

Applying to (138) the barriers method described in [7, Th. 3.15] we obtain the statement of the theorem.

Applying the same arguments as in Section 2 we get the following theorem

Theorem 12 *Let Assumptions 1-5 hold. Denote by $S_t^{\mu,\kappa,\beta}$ the evolution operator of problem (25)-(28) in the space*

$$H_\mu = H = \mathcal{D}(L^{1/2}) \times L^2(\Omega) \times \mathcal{D}(A^{1/2}) \times L^2(\Gamma_0) \times H^1(\Gamma_0).$$

Let $\mathfrak{A}^{\mu,\kappa,\beta}$ be a global attractor for the system $(S_t^{\mu,\kappa,\beta}, H_\mu)$. Then the family of the attractors $\mathfrak{A}^{\mu,\kappa,\beta}$ is upper semi-continuous on $\Lambda = [1,\infty) \times [0,1] \times [0,1]$. Namely, we have that

$$\lim_{(\mu,\kappa,\beta)\to(\infty,0,0)} \sup_{y\in\mathfrak{A}^{\mu,\kappa,\beta}} \{dist_{H^{\delta_1,\delta_2}}(y, \mathfrak{A}_1 \times \mathfrak{A}_3 \times 0)\} = 0,$$

where

$$H^{\delta_1,\delta_2} = (L^{1/2-\delta_1}) \times L^2(\Omega) \times [[H^{1-\delta_2}(\Gamma_0)]^2 \times H^1(\Gamma_0)] \times L^2(\Gamma_0) \times L^2(\Gamma_0).$$

and $\mathfrak{A}_3$ is the attractor of the system

$$\begin{gathered}(1-\Delta)w_{tt} + divb(-\nabla w_t) + b_0(w_t) + \Delta^2 w - div[h(|\nabla w|^2)\nabla w] \\ + h_0(w) - \gamma/2\Delta[w^2] = 0, \\ w(x,t) = 0, \quad \nabla w(x,t) = 0 \qquad x \in \partial\Gamma_0, \quad t > 0\end{gathered}$$

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