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From Clouâtre-Ostermann-Ransford to Okubo-Ando

Let $\mathcal{A}$ be a unital operator algebra and let $\theta : \mathcal{A} \to B(\mathcal{H})$ be a continuous unital homomorphism. We prove that for all $n \in \mathbb{N}$ and all β in the dual space of $\mathcal{A}$,

$$\|\theta^{(n)}\| \leq \max(1, \|\theta^{(n)} + \beta^{(n)} I\|).$$

Here, $\theta^{(n)} : M_n(\mathcal{A}) \to B(\mathcal{H}^n)$ denotes the n-th matrix ampliation of θ. This extends a result of Clouâtre, Ostermann and Ransford (the case $n = 1$), who were motivated by Crouzeix's conjecture. Our result allows us to control the completely bounded norm of θ, which in turn has dilation theoretic consequences.

As an application, we obtain a new proof of the similarity theorem of Okubo and Ando, which says that if $T \in B(\mathcal{H})$ is an operator of class C_ρ, where $\rho \geq 1$, then there exists an invertible operator S such that $\|S^{-1}TS\| \leq 1$ and such that $\|S\|\|S^{-1}\| \leq \rho$. The conclusion of the Okubo–Ando theorem implies that the inequality

$$\|f(T)\| \leq \rho \|f\|_{\mathbb{D}}$$

holds for all polynomials f. A direct proof of this inequality was recently given by Clouâtre, Ostermann and Ransford. Our main result shows that this inequality in fact holds for matrix-valued polynomials, so that the existence of the similarity S follows from Paulsen's similarity theorem, which says that an operator T is completely polynomially bounded with constant ρ if and only if there exists an invertible operator S such that $\|S^{-1}TS\| \leq 1$ and such that $\|S\|\|S^{-1}\| \leq \rho$.

The key to proving our main result is to adapt a variational argument of Caldwell, Greenbaum and Li to matrix-valued holomorphic functions. Rather than working with biholomorphic automorphisms of the disc, we

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work with Potapov–Möbius transforms m_A, where A belongs to the unit ball of $M_n(\mathbb{C})$. A slightly simplified version of our key lemma, which is shown using this technique, then reads as follows: If $R \in M_n(B(\mathcal{H}))$ satisfies $\|R\| > 1$ and $\|m_{A\otimes I}(R)\| \le \|R\|$ for all A close to 0 and if $x \in \mathcal{H}^n$ with $\|x\| = 1$ and $\|Rx\| = \|R\|$, then

$$\langle Rx, (A \otimes I)x \rangle = 0$$

for all $A \in M_n(\mathcal{C})$.

Keywords: **completely bounded norm of homomorphism; Okubo–Ando theorem.**

2020 Mathematics Subject Classification: 47L30; 46L07.

1. Introduction

The Crouzeix conjecture is that the numerical range of an operator on a Hilbert space is always a 2-spectral set for the operator [5]. The sharpest result proved to date is by Crouzeix and Palencia [6], who proved that the numerical range is a $(1+\sqrt{2})$-spectral set. Examining the proofs in [6, 10], Clouâtre, Ostermann and Ransford were led to make the following conjecture, which if true would imply the Crouzeix conjecture [4].

Conjecture 1.1. [4] *Let $\mathcal{A}$ be a uniform algebra, and $\alpha : \mathcal{A} \to \mathcal{A}$ a unital anti-linear contraction. Let $\theta : \mathcal{A} \to \mathcal{M}_n(\mathbb{C})$ be a continuous unital algebra homomorphism.*

(i) If the linear map

$$\Lambda : \mathcal{A} \to M_n(\mathbb{C}), \quad f \mapsto \theta(f) + \theta(\alpha(f))^*,$$

has norm at most 2, then $\|\theta\| \le 2$.

(ii) If α is a complete contraction, θ is completely bounded, and $\|\Lambda\|_{cb} \le 2$, then $\|\theta\|_{cb} \le 2$.

They proved that both parts of their conjecture hold with the weaker conclusion $\|\theta\| \le 1+\sqrt{2}$ (resp. $\|\theta\|_{cb} \le 1+\sqrt{2}$). They also proved that (i) holds with constant 2 if the range of α is the scalar multiples of the identity, even without the assumption that $\alpha(1) = 1$. It was left open if (ii) also holds under this hypothesis. The object of this note is to prove that it does, which is an immediate consequence of the following result.

Theorem 1. *Let $\mathcal{A}$ be a unital operator algebra and let $\theta : \mathcal{A} \to B(\mathcal{H})$ be a continuous unital homomorphism. Then for all $n \in \mathbb{N}$ and all β in the dual space of $\mathcal{A}$,*

$$\|\theta^{(n)}\| \le \max(1, \|\theta^{(n)} + \beta^{(n)} I\|).$$

Here, $(\theta + \beta I)(a) = \theta(a) + \beta(a) I$, and the notation $\theta^{(n)}$ means the n-th matrix ampliation; see Definition 1. The case $n = 1$ was proved in [4, Thm. 4.1]. (They stated the theorem for uniform algebras, but their proof extends to all operator algebras). The chief reason to consider ampliations of θ is that control of the completely bounded norm $\|\theta\|_{cb} = \sup_n \|\theta^{(n)}\|$ is crucial from the point of view of dilation theory. We discuss one such application.

The class C_ρ of operators with a unitary ρ-dilation (see Definition 2) was introduced by Sz.-Nagy and Foiaş. Operators of class C_ρ can be characterized intrinsically; see [14, Section I.11]. In particular, the class C_1 is the class of contractions (operators of norm at most one); this is the Sz-Nagy dilation theorem [12]. C_2 consists of all operators whose numerical range lies in the closed unit disk; this is Berger's strange dilation theorem [2].

Every operator in C_ρ was shown to be similar to a contraction in [13]. Okubo and Ando obtained the sharp bound on the condition number [8].

Theorem 2 (Okubo–Ando)**.** *Let $\rho \geq 1$ and let $T \in B(\mathcal{H})$ be of class C_ρ. Then there exists an invertible operator $S \in B(\mathcal{H})$ such that $\|S^{-1}TS\| \leq 1$ and such that $\|S\|\|S^{-1}\| \leq \rho$.*

We give another proof of the Okubo–Ando theorem in Section 3.

2. Definitions

We recall a few definitions. By an operator algebra, we mean a subalgebra $\mathcal{A}$ of $B(\mathcal{K})$ for some Hilbert space $\mathcal{K}$; we do not assume that the algebra is self-adjoint. Matrices with entries from $\mathcal{A}$ come equipped with natural norms. Indeed, if $\mathcal{A} \subset B(\mathcal{K})$, then $\mathcal{M}_n(\mathcal{A})$, the n-by-n matrices with entries from $\mathcal{A}$, is a subalgebra of $\mathcal{M}_n(B(\mathcal{K}))$, and this can be canonically identified with $B(\mathcal{K}^n)$. The norm on $\mathcal{M}_n(\mathcal{A})$ is its norm as a subalgebra of $B(\mathcal{K}^n)$.

Definition 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be operator algebras, and $\theta : \mathcal{A} \to \mathcal{B}$ a mapping. We write $\theta^{(n)} : \mathcal{M}_n(\mathcal{A}) \to \mathcal{M}_n(\mathcal{B})$ for the n-th ampliation of the map θ, defined by applying θ entry-wise. We define*

$$\|\theta\|_{cb} = \sup_{n\in\mathbb{N}} \|\theta^{(n)}\|.$$

We say θ is completely bounded if $\|\theta\|_{cb} < \infty$, and say θ is a complete contraction if $\|\theta\|_{cb} \leq 1$.

Since the work of Stinespring [11] and Arveson [1], it has been known that studying the ampliations can be very helpful in understanding the original map.

Definition 2. *Let $\rho > 0$. An operator $T \in B(\mathcal{H})$ is said to be of class C_ρ if there exists a unitary operator U on a Hilbert space containing $\mathcal{H}$ such that*

$$T^n = \rho P_{\mathcal{H}} U^n\big|_{\mathcal{H}} \quad \text{for all } n \geq 1.$$

In this case, U is called a unitary ρ-dilation of T. Equivalently,

$$f(T) + (\rho - 1)f(0) = \rho P_{\mathcal{H}} f(U)\big|_{\mathcal{H}} \quad \text{for all } f \in \mathbb{C}[z]. \tag{1}$$

3. Proof of the Okubo-Ando theorem

The conclusion of the Okubo–Ando theorem (Theorem 2) implies that if T is of class C_ρ, then the inequality

$$\|f(T)\| \le \rho \|f\|_{\overline{\mathbb{D}}} \tag{2}$$

holds for all $f \in \mathbb{C}[z]$, an improvement from the obvious bound

$$\|f(T)\| \le \rho \|f\|_{\overline{\mathbb{D}}} + |\rho - 1| |f(0)|$$

from (1).

The original proof of Okubo and Ando [8] is based on analyzing geometric properties of the unitary ρ-dilation of T. More recently, Caldwell, Greenbaum and Li [3] gave a simple direct proof of Inequality (2) for $\rho = 2$ (in the case of finite dimensional Hilbert spaces). Their proof is based on a variational argument, which they attribute to Crouzeix. This was extended to a proof of the general Inequality (2) by Clouâtre, Ostermann and Ransford [4].

On the other hand, Theorem 1 shows that Inequality (2) also holds for matrix-valued polynomials, so we shall show that Theorem 2 follows from Paulsen's similarity theorem, which says that the completely bounded norm of the map

$$\mathbb{C}[z] \to B(\mathcal{H}), \ f \mapsto f(T)$$

equals the infimum of the condition numbers $\|S\|\|S^{-1}\|$ over all S such that $\|S^{-1}TS\| \le 1$.

Proof of Theorem 2. Let $\mathcal{A} = \mathbb{C}[z]$, equipped with the supremum norm over $\overline{\mathbb{D}}$. Define

$$\beta : \mathbb{C}[z] \to \mathbb{C}, \quad \beta(f) = (\rho - 1) f(0),$$

and let θ be the polynomial functional calculus of T. Then (1) shows that $\|\theta + \beta I\|_{cb} \le \rho$, hence $\|\theta\|_{cb} \le \rho$ by Theorem 1. Paulsen's similarity theorem [9, Theorem 9.1] then yields the invertible operator S. □

4. Proof of main result

The key to proving Theorem 1 is to adapt the variational argument in [3] to matrix-valued polynomials. To this end, we will use biholomorphic automorphisms of the unit ball of $B(\mathcal{H})$. In general, if $\mathcal{H}$ is a Hilbert space and if $A \in B(\mathcal{H})$ with $\|A\| < 1$, the Potapov–Möbius transform on the unit ball of $B(\mathcal{H})$ is defined by

$$m_A(X) = (I - AA^*)^{-1/2}(X + A)(I + A^*X)^{-1}(I - A^*A)^{1/2}. \tag{3}$$

If $\|A\| < 1$ and $\|X\| \le 1$, then $\|m_A(X)\| \le 1$, i.e. m_A maps the closed unit ball of $B(\mathcal{H})$ into itself; see e.g. [7, Theorem 8.19]. Let us note that even if $\|X\| > 1$, formula (3) makes sense as long as $\|A\|\|X\| < 1$.

Of particular importance for us is the case when the Hilbert space is of the form $\mathcal{H}^n$, so $B(\mathcal{H}^n) = \mathcal{M}_n(B(\mathcal{H})) = \mathcal{M}_n(\mathbb{C}) \otimes B(\mathcal{H})$. Given a scalar matrix $A \in \mathcal{M}_n(\mathbb{C})$ with $\|A\| < 1$, we shall define

$$\tilde{A} = A \otimes I_{\mathcal{H}}$$

and thus define $m_{\tilde{A}}(T)$ for $T \in \mathcal{M}_n(B(\mathcal{H}))$ with $\|A\|\|T\| < 1$. In particular, we have $\|m_{\tilde{A}}(T)\| \leq 1$ whenever $\|T\| \leq 1$. We will also use the observation that if $T \in M_n(\mathcal{A})$ for some norm-closed unital operator subalgebra $\mathcal{A}$ of $B(\mathcal{H})$, then $m_{\tilde{A}}(T) \in M_n(\mathcal{A})$ for all $A \in M_n(\mathbb{C})$ with $\|A\| < 1$ and $\|A\|\|T\| < 1$.

The following lemma is a generalization of [3, Theorem 5.1] to matrix-valued functions, sequences of operators and infinite-dimensional Hilbert spaces.

Lemma 1. *Let (R_k) be a sequence in $\mathcal{M}_n(B(\mathcal{H}))$. Let $C \in (0,\infty) \setminus \{1\}$. Assume that whenever $A \in \mathcal{M}_n(\mathbb{C})$ satisfies $\|A\|\|R_k\| < 1$ then $\|m_{\tilde{A}}(R_k)\| \leq C$.*

Let (x_k) be a sequence of unit vectors in $\mathcal{H}^n$ with $\lim_{k\to\infty} \|R_k x_k\| = C$. Then for every bounded sequence (A_k) in $\mathcal{M}_n(\mathbb{C})$,

$$\lim_{k\to\infty} \langle R_k x_k, (A_k \otimes I_{\mathcal{H}}) x_k \rangle = 0.$$

Proof. The assumption implies that $\|R_k\| \leq C$ for all k by taking $A = 0$. Let $A \in \mathcal{M}_n(\mathbb{C})$ with $C\|A\| < 1$ and let $y \in \mathcal{H}^n$ be a unit vector. Define $z_k = (I - \tilde{A}^*\tilde{A})^{-1/2}(I + \tilde{A}^* R_k)y$. Then by assumption,

$$\begin{aligned} \|(I - \tilde{A}\tilde{A}^*)^{-1/2}(R_k + \tilde{A})y\|^2 &= \|m_{\tilde{A}}(R_k) z_k\|^2 \\ &\leq C^2 \|z_k\|^2 \\ &= C^2 \|(I - \tilde{A}^*\tilde{A})^{-1/2}(I + \tilde{A}^* R_k)y\|^2. \end{aligned}$$

We now replace A with tA for a small complex number t and expand to first order. We have $(I - |t|^2\tilde{A}\tilde{A}^*)^{-1/2} = I + \mathcal{O}(|t|^2)$ and $(I - |t|^2\tilde{A}^*\tilde{A})^{-1/2} = I + \mathcal{O}(|t|^2)$. Therefore, the last inequality implies that

$$\|R_k y\|^2 + 2\Re\bar{t}\langle R_k y, \tilde{A}y\rangle \leq C^2(\|y\|^2 + 2\Re t\langle y, \tilde{A}^* R_k y\rangle) + \mathcal{O}(|t|^2),$$

where the implied constant in the $\mathcal{O}(|t|^2)$ term is independent of the unit vector y and the number k because $\|R_k\| \leq C$. Rearranging yields

$$2(1 - C^2)\Re\bar{t}\langle R_k y, \tilde{A}y\rangle \leq C^2\|y\|^2 - \|R_k y\|^2 + \mathcal{O}(|t|^2)$$

for all complex numbers t in a neighborhood of zero and all unit vectors y. By varying the argument of t, we conclude that

$$2|1 - C^2||t||\langle R_k y, \tilde{A}y\rangle| \leq C^2\|y\|^2 - \|R_k y\|^2 + \mathcal{O}(|t|^2).$$

We apply the last inequality to $y = x_k$, the sequence in the statement of the lemma. Since the estimate above is independent of the unit vector y and k, it follows that

$$2|1 - C^2||t| \limsup_{k\to\infty} |\langle R_k x_k, \tilde{A}x_k\rangle| \leq \mathcal{O}(|t|^2)$$

for all t in a neighborhood of zero. Since $C \neq 1$, we conclude that

$$\lim_{k\to\infty} \langle R_k x_k, \tilde{A} x_k \rangle = 0.$$

We have proved this identity for all $A \in \mathcal{M}_n(\mathbb{C})$ with $C\|A\| < 1$, but then by homogeneity, it holds for all $A \in \mathcal{M}_n(\mathbb{C})$. This shows that the linear functionals

$$\mathcal{M}_n(\mathbb{C}) \to \mathbb{C}, \quad A \mapsto \langle \tilde{A} x_k, R_k x_k \rangle,$$

converge to zero pointwise, and hence in norm since $\mathcal{M}_n(\mathbb{C})$ is finite dimensional. This completes the proof.

Proof of Theorem 1. Given Lemma 1, the proof proceeds along the same lines as that of [4, Theorem 4.1]. Since the details differ, we provide the argument. Let $\mathcal{A} \subset B(\mathcal{K})$. Since θ and β are continuous, we may replace $\mathcal{A}$ with its norm closure if necessary and hence assume that $\mathcal{A}$ is closed. Fix $n \in \mathbb{N}$ and let $C = \|\theta^{(n)}\|$. If $C \leq 1$, there is nothing to prove, so assume $C > 1$.

Let (a_k) be a sequence in the unit ball of $\mathcal{M}_n(\mathcal{A})$ and let (x_k) be a sequence of unit vectors in $\mathcal{H}^n$ such that $\|\theta^{(n)}(a_k)x_k\| \to C$. Let $R_k = \theta^{(n)}(a_k) \in \mathcal{M}_n(B(\mathcal{H}))$. Then $\|R_k\| \geq 1$ for sufficiently large k; we may assume that this holds for all k. We claim that the R_k satisfy the assumption of Lemma 1. Indeed, if $A \in \mathcal{M}_n(\mathbb{C})$ with $\|A\|\|R_k\| < 1$, then $\|A\| < 1$, and since θ is a unital homomorphism,

$$m_{A\otimes I_{\mathcal{H}}}(R_k) = m_{A\otimes I_{\mathcal{H}}}(\theta^{(n)}(a_k)) = \theta^{(n)}(m_{A\otimes I_{\mathcal{K}}}(a_k)).$$

The discussion preceding Lemma 1 shows that $m_{A\otimes I_{\mathcal{K}}}(a_k)$ belongs to the closed unit ball of $\mathcal{M}_n(\mathcal{A})$, hence $\|m_{A\otimes I_{\mathcal{H}}}(R_k)\| \leq C$ for all k, as claimed. Now

$$\begin{aligned}\|\theta^{(n)} + \beta^{(n)} I\|^2 &\geq \|(\theta^{(n)} + \beta^{(n)} I)(a_k)x_k\|^2 \\ &\geq \|\theta^{(n)}(a_k)x_k\|^2 + 2\Re\langle R_k x_k, (\beta^{(n)}(a_k) \otimes I_{\mathcal{H}})x_k\rangle\end{aligned}$$

for all k. Taking the limit as $k \to \infty$, the second summand vanishes by Lemma 1, and the first summand converges to $C^2 = \|\theta^{(n)}\|^2$, so

$$\|\theta^{(n)}\| \leq \|\theta^{(n)} + \beta^{(n)} I\|. \qquad \square$$

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Від Клуатр-Остерманн-Рансфорд до Окубо-Андо

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Нехай $\mathcal{A}$ — унітальна операторна алгебра, а $\theta : \mathcal{A} \to B(\mathcal{H})$ — неперервний унітальний гомоморфізм. Ми доводимо, що для всіх $n \in \mathbb{N}$ та всіх β у дуальному просторі до $\mathcal{A}$,

$$\|\theta^{(n)}\| \leq \max(1, \|\theta^{(n)} + \beta^{(n)} I\|).$$

Тут $\theta^{(n)} : M_n(\mathcal{A}) \to B(\mathcal{H}^n)$ позначає n-ту матричну ампліацію відображення θ. Це розширює результат Клуатра, Остермана та Рансфорда (випадок $n = 1$), які були мотивовані гіпотезою Крузейкса. Наш результат дозволяє нам контролювати повністю обмежену норму θ, що, у свою чергу, має теоретико-дилатаційні наслідки.

Як застосування, ми отримуємо новий доказ теореми подібності Окубо та Андо, яка стверджує, що якщо $T \in B(\mathcal{H})$ є оператором класу C_ρ, де $\rho \geq 1$, тоді існує оборотний оператор S такий, що $\|S^{-1}TS\| \leq 1$ та такий, що $\|S\|\|S^{-1}\| \leq \rho$. Висновок теореми Окубо–Андо означає, що нерівність

$$\|f(T)\| \leq \rho \|f\|_{\overline{\mathbb{D}}}$$

виконується для всіх поліномів f. Прямий доказ цієї нерівності нещодавно надали Клуâтр, Остерманн та Рансфорд. Наш основний результат показує, що ця нерівність насправді виконується для матрично-значних поліномів, так що існування подібності S випливає з теореми подібності Паульсена, яка стверджує, що оператор T є повністю поліноміально обмеженим з константою ρ тоді і тільки тоді, коли існує оборотний оператор S такий, що $\|S^{-1}TS\| \leq 1$ та такий, що $\|S\|\|S^{-1}\| \leq \rho$.

Ключем до доведення нашого основного результату є адаптація варіаційного аргументу Колдуелла, Грінбаума та Лі до матрично-значних голоморфних функцій. Замість роботи з біголоморфними автоморфізмами диска, ми працюємо з перетвореннями Потапова–Мебіуса m_A, де A належить одиничній кулі $M_n(\mathbb{C})$. Дещо спрощена версія нашої ключової леми, яка показана за допомогою цієї техніки, тоді звучить

так: Якщо $R \in M_n(B(\mathcal{H}))$ задовольняє $\|R\| > 1$ та $\|m_{A\otimes I}(R)\| \leq \|R\|$ для всіх A, близьких до 0, та якщо $x \in \mathcal{H}^n$ з $\|x\| = 1$ та $\|Rx\| = \|R\|$, тоді

$$\langle Rx, (A \otimes I)x \rangle = 0$$

для всіх $A \in M_n(\mathcal{C})$.
Ключові слова: **цілком обмежена норма гомоморфізму; теорема Окубо–Андо.**

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