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Вісник Харківського національного
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M. O. Bebiya
Ph.D in Physical and Mathematical Sciences
Associate Professor, Department of Applied Mathematics,
School of Mathematics and Computer Sciences
V. N. Karazin Kharkiv National University
Svobody Sq., 4, Kharkiv, Ukraine, 61022
m.bebiya@karazin.ua *http://orcid.org/0000-0003-3241-5879*

Y. M. Barska
MS in applied mathematics
V. N. Karazin Kharkiv National University
Svobody Sq., 4, Kharkiv, Ukraine, 61022
j.barskaya2@gmail.com *http://orcid.org/0009-0008-8444-2960*

On stabilization of canonical nonlinear systems in a special case

The paper addresses the stabilization problem for canonical nonlinear systems, which in the triangular case correspond to systems in p–normal form. These systems are inherently nonlinear because they are not feedback linearizable, and their linear approximation is degenerate and does not characterize the stability of the equilibrium at the origin. In the triangular case, backstepping is a standard feedback design tool due to its recursive nature. Backstepping provides a powerful framework for stabilizing a wide class of nonlinear systems in strict-feedback form. However, this recursive approach typically leads to feedback controllers of increasing complexity as the system dimension grows. This motivates the investigation of simpler control design methods. Several notable results on non-recursive stabilization have been obtained in recent years. The classes of polynomial control laws have been constructed for the case of strictly decreasing exponents of power terms in the right-hand side of the system. In particular, it has been shown that the system can be asymptotically stabilized by a linear feedback control law in the strictly decreasing case. Moreover, control coefficients can be chosen as arbitrary positive numbers. It is also established that it is possible to achieve stabilization in a certain special case of non-strictly decreasing exponents under additional conditions on the control coefficients. In this work, we solve the linear stabilization problem in a previously unexplored case of non-strictly decreasing exponents for a three-dimensional system. We propose constructive method of finding conditions for the control coefficients to guarantee asymptotic stabilization. Our approach is based on the Lyapunov function method. We also use stabilization through nonlinear approximation to generalize our result. The effectiveness of the proposed approach is demonstrated and confirmed by numerical examples.

Keywords: **stabilization; nonlinear systems; Lyapunov function method; critical case; feedback control; linear stabilization.**

2020 Mathematics Subject Classification: 93D05; 93D15; 93D30; 34H05; 34H15.

1. Introduction

Stabilization of nonlinear systems is an important and actively investigated direction in control theory [1, 3, 4, 5, 6, 7, 9]. Particular attention has been given to nonlinear systems whose behavior can not be adequately captured by linear approximation [1, 2, 3, 4, 5, 6, 8, 9]. This situation corresponds to the so-called critical case and requires nonlinear techniques beyond standard linearization, such as backstepping or its numerous modifications [5, 6, 7, 8]. While highly effective, the recursive structure of backstepping often leads to complex control laws whose implementation becomes challenging as the system dimension increases. This motivates the development of non-recursive approaches with reduced complexity, resulting in control laws of simpler form [1, 3, 4, 9].

In this paper, we focus on canonical nonlinear systems, for which linearization-based methods are not directly applicable. Specifically, we consider systems of the following form:

$$\begin{cases} \dot{x}_i = x_{i+1}^{p_i} + \varphi_i(x_1, x_2, \ldots, x_n), \quad i = 1, \ldots, n-1, \\ \dot{x}_n = u^{p_n}, \end{cases} \tag{1}$$

where $p_i \geq 1$ are ratios of positive odd integers, and $\varphi_i(x_1, \ldots, x_n)$ are continuous real-valued functions of higher order than $x_{i+1}^{p_i}$ satisfying $\varphi_i(0, \ldots, 0) = 0$ for all $i = 1, \ldots, n-1$. We address the problem of finding simple stabilizing controls for this system.

Classes of easily implementable stabilizing control laws for system (1) have been proposed in [3, 4, 9] using non-recursive Lyapunov design. In [9], it is shown that system (1) can be linearly stabilized under the assumption of strictly decreasing exponents, i.e., $p_1 > p_2 > \cdots > p_n \geq 1$. Moreover, under this assumption, the control coefficients can be selected as arbitrary positive numbers $k_i > 0$, $i = 1, \ldots, n$.

Throughout the paper we adopt the following convention. For the linear control law $u(x) = -k_1 x_1 - k_2 x_2 + \cdots - k_n x_n$, the parameters $k_1, k_2, \ldots, k_n$ are called control coefficients or control gains.

In what follows, we restrict our attention to the three-dimensional case and consider the linear stabilization problem. In a related study [1], the local asymptotic stability of system (1) for the case $n = 3$ was investigated. A specific case of non-strictly decreasing exponents was studied in [1], namely

$p_1 > p_2 = p_3 = 1$. It was proved that such a system can be stabilized by a linear control law of the form

$$u(x) = -k_1x_1 - k_2x_2 - k_3x_3,$$

where $k_i > 0$ for $i = 1, 2, 3$, provided that k_i satisfy the following constraint:

$$k_3^2 > \frac{k_2(3 + 2\sqrt{3})}{3}.$$

However, the question of the stabilizability of system (1) by linear feedback remains open in the general case. In this work, we investigate a previously unexplored case of non-strictly decreasing exponents in which $p_1 = p_2 > p_3 = 1$. We constructively demonstrate that, under this condition, system (1) is locally asymptotically stabilizable by a linear feedback control law.

Our primary objective is the synthesis of a linear state-feedback control law $u(x)$ that guarantees the local asymptotic stability of system (1) at the origin. To achieve this, we employ the Lyapunov function method by constructing a Lyapunov function obtained as a modification of the Lyapunov functions used in [1, 9]. This allows us to derive explicit sufficient conditions for the control gains and establish the asymptotic stability of the equilibrium at the origin of the corresponding closed-loop system.

2. Problem formulation and Lyapunov design

We study the nonlinear system

$$\begin{cases} \dot{x}_1 = x_2^p, \\ \dot{x}_2 = x_3^p, \\ \dot{x}_3 = u, \end{cases} \tag{2}$$

where $u \in \mathbb{R}$ is a control, $p > 1$ is a ratio of two positive odd integers. Here $p_1 = p_2 = p$.

We consider a linear state-feedback control law of the form

$$u(x) = -k_1x_1 - k_2x_2 - k_3x_3, \tag{3}$$

where $k_i > 0$ are real positive numbers for $i = 1, 2, 3$. Our objective is to establish conditions on the coefficients k_i under which the linear control $u(x)$ ensures asymptotic stability of the equilibrium at the origin of system (2).

Let us introduce the following Lyapunov function:

$$V(x) = \frac{1}{2}\left(g_1(p)\frac{k_2^p}{k_1}(k_1x_1)^2 + g_2(p)\frac{k_3^p}{k_2}(k_1x_1 + k_2x_2)^2 + \frac{1}{k_3}(k_1x_1 + k_2x_2 + k_3x_3)^2\right),$$

where $g_1(p)$, $g_2(p)$ are strictly positive functions that depend on p. It should be noted that in previous works [1] and [9], the weighting functions $g_1(p)$, $g_2(p)$ were set equal to 1.

Clearly, $V(x)$ is positive definite for any choice of positive feedback gains $k_1, k_2, k_3 > 0$. To simplify the subsequent analysis, we use the following linear coordinate transformation:

$$e_1 = k_1 x_1, \quad e_2 = k_1 x_1 + k_2 x_2, \quad e_3 = k_1 x_1 + k_2 x_2 + k_3 x_3. \tag{4}$$

Expressing the Lyapunov function in these new coordinates yields

$$V(e) = \frac{1}{2}\left(g_1(p)\frac{k_2^p}{k_1} e_1^2 + g_2(p)\frac{k_3^p}{k_2} e_2^2 + \frac{1}{k_3} e_3^2 \right). \tag{5}$$

The corresponding inverse coordinate transformation is given by

$$x_1 = k_1^{-1} e_1, \quad x_2 = k_2^{-1}(e_2 - e_1), \quad x_3 = k_3^{-1}(e_3 - e_2).$$

Differentiating (4) along the trajectories of system (2) under the state-feedback control law $u = u(x)$, we obtain

$$\begin{aligned}
\dot e_1 &= k_1 \dot x_1 = k_1 x_2^p = \frac{k_1}{k_2^p}(e_2 - e_1)^p,\\
\dot e_2 &= k_1 \dot x_1 + k_2 \dot x_2 = k_1 x_2^p + k_2 x_3^p = \frac{k_1}{k_2^p}(e_2 - e_1)^p + \frac{k_2}{k_3^p}(e_3 - e_2)^p,\\
\dot e_3 &= k_1 \dot x_1 + k_2 \dot x_2 + k_3 \dot x_3 = \frac{k_1}{k_2^p}(e_2 - e_1)^p + \frac{k_2}{k_3^p}(e_3 - e_2)^p + k_3 u(x).
\end{aligned}$$

Substituting the linear control law $u(x) = -(k_1 x_1 + k_2 x_2 + k_3 x_3) = -e_3$ into the third equation, we arrive at the system

$$\begin{cases}
\dot e_1 = \frac{k_1}{k_2^p}(e_2 - e_1)^p,\\
\dot e_2 = \frac{k_1}{k_2^p}(e_2 - e_1)^p + \frac{k_2}{k_3^p}(e_3 - e_2)^p,\\
\dot e_3 = \frac{k_1}{k_2^p}(e_2 - e_1)^p + \frac{k_2}{k_3^p}(e_3 - e_2)^p - k_3 e_3.
\end{cases} \tag{6}$$

We now proceed to compute the time derivative of the Lyapunov function $V(e)$, defined by (5), along the trajectories of the closed-loop system (6):

$$\dot V(e) = \frac{\partial V}{\partial e_1}\dot e_1 + \frac{\partial V}{\partial e_2}\dot e_2 + \frac{\partial V}{\partial e_3}\dot e_3.$$

Now we evaluate each term individually, which yields

$$\begin{aligned}
\frac{\partial V}{\partial e_1}\dot e_1 &= \left(g_1(p)\frac{k_2^p}{k_1} e_1 \right)\left(\frac{k_1}{k_2^p}(e_2 - e_1)^p \right) = g_1(p) e_1 (e_2 - e_1)^p,\\
\frac{\partial V}{\partial e_2}\dot e_2 &= \left(g_2(p)\frac{k_3^p}{k_2} e_2 \right)\left(\frac{k_1}{k_2^p}(e_2 - e_1)^p + \frac{k_2}{k_3^p}(e_3 - e_2)^p \right)\\
&= g_2(p)\frac{k_1 k_3^p}{k_2^{p+1}} e_2 (e_2 - e_1)^p + g_2(p) e_2 (e_3 - e_2)^p,\\
\frac{\partial V}{\partial e_3}\dot e_3 &= \left(\frac{1}{k_3} e_3 \right)\left(\frac{k_1}{k_2^p}(e_2 - e_1)^p + \frac{k_2}{k_3^p}(e_3 - e_2)^p - k_3 e_3 \right)\\
&= \frac{k_1}{k_2^p k_3} e_3 (e_2 - e_1)^p + \frac{k_2}{k_3^{p+1}} e_3 (e_3 - e_2)^p - e_3^2.
\end{aligned}$$

By summing the above expressions, we obtain the total time derivative

$$\begin{aligned}\dot{V}(e) &= g_1(p)e_1(e_2-e_1)^p + g_2(p)\frac{k_1k_3^p}{k_2^{p+1}}e_2(e_2-e_1)^p + g_2(p)e_2(e_3-e_2)^p \\ &+ \frac{k_1}{k_2^pk_3}e_3(e_2-e_1)^p + \frac{k_2}{k_3^{p+1}}e_3(e_3-e_2)^p - e_3^2.\end{aligned}$$

Recall that p is defined as a ratio of positive odd integers, which implies that $(e_i - e_j)^p = -(e_j - e_i)^p$ for any $i, j = 1, 2, 3$. Utilizing this property and rearranging the terms, we can express $\dot{V}(e)$ in the following form:

$$\begin{aligned}\dot{V}(e) &= -g_1(p)e_1(e_1-e_2)^p - g_2(p)e_2(e_2-e_3)^p - e_3^2 \\ &+ g_2(p)\frac{k_1k_3^p}{k_2^{p+1}}e_2(e_2-e_1)^p + \frac{k_1}{k_2^pk_3}e_3(e_2-e_1)^p + \frac{k_2}{k_3^{p+1}}e_3(e_3-e_2)^p.\end{aligned} \tag{7}$$

Before proceeding with the estimation of $\dot{V}(e)$, we recall several fundamental algebraic inequalities. These auxiliary results are well-established in the literature [1, 6, 9].

Lemma 1. *For any real number $p \geq 1$ and arbitrary scalars $x_i \in \mathbb{R}$ for $i = 1, \ldots, n$, the following inequality holds:*

$$|x_1 + x_2 + \cdots + x_n|^p \leq n^{p-1}\left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right).$$

Lemma 2. *Let $p \geq 1$ be a ratio of positive odd integers. Then, for all $x, a \in \mathbb{R}$, the following inequality holds:*

$$x(x+a)^p \geq 2^{1-p}x^{p+1} + xa^p.$$

Lemma 3 (Parameterized Young's Inequality)**.** *Let $m > 0$ and $n > 0$ be real constants. Then, for any strictly positive parameter $\gamma > 0$, the following inequality holds for all $x, y \in \mathbb{R}$:*

$$|x|^m|y|^n \leq \frac{m}{m+n}\gamma|x|^{m+n} + \frac{n}{m+n}\gamma^{-\frac{m}{n}}|y|^{m+n}.$$

We now derive several inequalities that will be required for the subsequent analysis. By applying Lemmas 1 and 2 to the corresponding individual terms of the derivative, given by (7), we establish the following upper bounds:

$$\begin{aligned}-g_1(p)e_1(e_1-e_2)^p &\leq g_1(p)\left(-2^{1-p}e_1^{p+1} + |e_1||e_2|^p\right), \\ -g_2(p)e_2(e_2-e_3)^p &\leq g_2(p)\left(-2^{1-p}e_2^{p+1} + |e_2||e_3|^p\right), \\ \frac{k_1k_3^p}{k_2^{p+1}}g_2(p)e_2(e_2-e_1)^p &\leq \frac{k_1k_3^p}{k_2^{p+1}}2^{p-1}g_2(p)\left(e_2^{p+1} + |e_1|^p|e_2|\right), \\ \frac{k_1}{k_3k_2^p}e_3(e_2-e_1)^p &\leq \frac{k_1}{k_3k_2^p}2^{p-1}\left(|e_2|^p|e_3| + |e_1|^p|e_3|\right), \\ \frac{k_2}{k_3^{p+1}}e_3(e_3-e_2)^p &\leq \frac{k_2}{k_3^{p+1}}2^{p-1}\left(e_3^{p+1} + |e_2|^p|e_3|\right).\end{aligned} \tag{8}$$

By Young's inequality, for any real numbers $a, b \geq 0$ and $n, m > 1$ such that $\frac{1}{n} + \frac{1}{m} = 1$, we have

$$ab \leq \frac{a^n}{n} + \frac{b^m}{m}$$

.

Therefore, by setting $n := 1 + s$ and $m := 1 + \frac{1}{s}$, where $s > 0$ is an arbitrary constant, we can estimate the term $|e_i^p||e_j|$ as follows:

$$|e_i^p||e_j| \leq \frac{1}{1+s}|e_i|^{p(1+s)} + \frac{s}{s+1}|e_j|^{1+\frac{1}{s}}.$$

Suppose that $\{C_i\}_{i=1}^5$ are positive real numbers. Letting $s := \frac{1}{p}$, we obtain

$$|e_1||e_2|^p = |C_1^p e_1| \left|\frac{1}{C_1}e_2\right|^p \leq \frac{C_1^{p(p+1)}}{p+1}e_1^{p+1} + \frac{p}{(p+1)C_1^{p+1}}e_2^{p+1},$$
$$|e_1|^p|e_2| = |C_2^p e_1|^p \left|\frac{1}{C_2^{p^2}}e_2\right| \leq \frac{pC_2^{p(p+1)}}{p+1}e_1^{p+1} + \frac{1}{(p+1)C_2^{p^2(p+1)}}e_2^{p+1}.$$

To establish the following estimates, we choose any number $s_1 \in \left(\frac{1}{p}; 1\right)$ and apply Young's inequality with $n = 1 + s_1$:

$$|e_2|^p|e_3| = \left|\frac{1}{C_3}e_2\right|^p |C_3^p e_3| \leq \frac{1}{(1+s_1)C_3^{p(s_1+1)}}|e_2|^{p(s_1+1)} + \frac{s_1 C_3^{p(1+\frac{1}{s_1})}}{s_1+1}|e_3|^{1+\frac{1}{s_1}},$$
$$|e_1|^p|e_3| = |C_4^p e_1|^p \left|\frac{1}{C_4^{p^2}}e_3\right| \leq \frac{C_4^{p(1+s_1)}}{s_1+1}|e_1|^{p(1+s_1)} + \frac{s_1}{(1+s_1)C_4^{p^2(1+\frac{1}{s_1})}}|e_3|^{1+\frac{1}{s_1}},$$

Next, we fix $s_2 \in \left(\frac{2}{p} - 1; \frac{1}{p}\right)$ for $1 < p < 2$ or any $s_2 \in \left(0, \frac{1}{p}\right)$ for $p > 2$:

$$|e_3|^p|e_2| = \left|\frac{1}{C_5}e_3\right|^p |C_5^p e_2| \leq \frac{1}{(1+s_2)C_5^{p(s_2+1)}}|e_3|^{p(s_2+1)} + \frac{s_2 C_5^{p(1+\frac{1}{s_2})}}{s_2+1}|e_2|^{1+\frac{1}{s_2}}.$$

To establish asymptotic stability, we will determine additional constraints on the parameters C_i to guarantee the negative definiteness of $\dot{V}(e)$ in some small punctured neighborhood of the origin.

Applying the previously established inequalities sequentially, we obtain

$$
\begin{aligned}
\dot{V}(e) \le & -g_1(p)2^{1-p}e_1^{p+1} + g_1(p)\left(\frac{p}{(p+1)C_1^{p+1}}e_2^{p+1} + \frac{C_1^{p(p+1)}}{p+1}e_1^{p+1}\right) \\
& + g_2(p)\left(\frac{1}{(1+s_2)C_5^{p(s_2+1)}}|e_3|^{p(s_2+1)} + \frac{s_2C_5^{p(1+\frac{1}{s_2})}}{s_2+1}|e_2|^{1+\frac{1}{s_2}}\right) \\
& - g_2(p)2^{1-p}e_2^{p+1} - e_3^2 + \frac{k_1k_3^p}{k_2^{p+1}}2^{p-1}g_2(p)e_2^{p+1} \\
& + \frac{k_1k_3^p}{k_2^{p+1}}2^{p-1}g_2(p)\left(\frac{pC_2^{p(p+1)}}{p+1}e_1^{p+1} + \frac{1}{(p+1)C_2^{p^2(p+1)}}e_2^{p+1}\right) \\
& + \frac{k_1}{k_3k_2^p}2^{p-1}\left(\frac{1}{(1+s_1)C_3^{p(s_1+1)}}|e_2|^{p(s_1+1)} + \frac{s_1C_3^{p(1+\frac{1}{s_1})}}{s_1+1}|e_3|^{1+\frac{1}{s_1}}\right) \\
& + \frac{k_1}{k_3k_2^p}2^{p-1}\left(\frac{C_4^{p(1+s_1)}}{s_1+1}|e_1|^{p(1+s_1)} + \frac{s_1}{(1+s_1)C_4^{p^2(1+\frac{1}{s_1})}}|e_3|^{1+\frac{1}{s_1}}\right) \\
& + \frac{k_2}{k_3^{p+1}}2^{p-1}\left(\frac{1}{(1+s_1)C_3^{p(s_1+1)}}|e_2|^{p(s_1+1)} + \frac{s_1C_3^{p(1+\frac{1}{s_1})}}{s_1+1}|e_3|^{1+\frac{1}{s_1}}\right) \\
& + \frac{k_2}{k_3^{p+1}}2^{p-1}e_3^{p+1}.
\end{aligned}
$$

Grouping the coefficients of the dominant stabilizing terms e_1^{p+1}, e_2^{p+1}, and e_3^2, we can rewrite the right-hand side as

$$
\begin{aligned}
\dot{V}(e) \le & \left[-g_1(p)2^{1-p} + g_1(p)\frac{C_1^{p(p+1)}}{p+1} + \frac{k_1k_3^p}{k_2^{p+1}}2^{p-1}g_2(p)\frac{pC_2^{p(p+1)}}{p+1}\right]e_1^{p+1} \\
& + \Bigg[-g_2(p)2^{1-p} + g_1(p)\frac{p}{(p+1)C_1^{p+1}} \\
& + \frac{k_1k_3^p}{k_2^{p+1}}2^{p-1}g_2(p)\left(1 + \frac{1}{(p+1)C_2^{p^2(p+1)}}\right)\Bigg]e_2^{p+1} - e_3^2 + h(e),
\end{aligned}
\tag{9}
$$

where the function of higher-order terms $h(e)$ is given by

$$
\begin{aligned}
h(e) = & \left[\frac{k_1}{k_3k_2^p}2^{p-1}\frac{C_4^{p(1+s_1)}}{s_1+1}\right]|e_1|^{p(1+s_1)} \\
& + 2^{p-1}\left[\frac{k_1}{k_3k_2^p}\frac{1}{(1+s_1)C_3^{p(s_1+1)}} + \frac{k_2}{k_3^{p+1}}\frac{1}{(1+s_1)C_3^{p(s_1+1)}}\right]|e_2|^{p(s_1+1)}
\end{aligned}
$$

$$+\frac{k_2}{k_3^{p+1}}2^{p-1}e_3^{p+1}+g_2(p)\left(\frac{1}{(1+s_2)C_5^{p(s_2+1)}}|e_3|^{p(s_2+1)}+\frac{s_2C_5^{p(1+\frac{1}{s_2})}}{s_2+1}|e_2|^{1+\frac{1}{s_2}}\right)$$
$$+2^{p-1}\left[\frac{k_1}{k_3k_2^p}\left(\frac{s_1C_3^{p(1+\frac{1}{s_1})}}{s_1+1}+\frac{s_1}{(1+s_1)C_4^{p^2(1+\frac{1}{s_1})}}\right)+\frac{k_2}{k_3^{p+1}}\frac{s_1C_3^{p(1+\frac{1}{s_1})}}{s_1+1}\right]|e_3|^{1+\frac{1}{s_1}}.$$

We note that, in a sufficiently small neighborhood of the origin, the sign of the right-hand side of inequality (9) for $\dot V(e)$ is determined by the corresponding terms with the smaller exponents in each variable. Next we demonstrate that $h(e)$ consists entirely of higher-order terms and, thus, does not affect the sign of $\dot V(e)$ for sufficiently small e.

We recall that $s_1>\frac{1}{p}$ and $s_2<\frac{1}{p}$. Therefore, the exponents of the factors $|e_1|^{p(1+s_1)}$, $|e_2|^{p(s_1+1)}$, and $|e_2|^{1+\frac{1}{s_2}}$ satisfy the inequalities

$$p(1+s_1)>p+1 \quad\text{and}\quad 1+\frac{1}{s_2}>p+1.$$

This confirms that the sum of the terms with factors $|e_1|^{p(1+s_1)}$, $|e_2|^{p(s_1+1)}$, and $|e_2|^{1+\frac{1}{s_2}}$ are locally dominated by the terms with the factors e_1^{p+1} and e_2^{p+1} in a sufficiently small neighborhood of the origin.

It is clear that the terms with the factors e_3^{p+1}, $|e_3|^{1+\frac{1}{s_1}}$, and $|e_3|^{p(s_2+1)}$ are dominated by the term with the factor e_3^2 for sufficiently small e_3. The conditions for the exponents $p+1$, $1+\frac{1}{s_1}$, and $p(s_2+1)$ are verified as follows:

$p+1>2$	is trivially satisfied since $p>1$;
$1+\dfrac{1}{s_1}>2$	is satisfied due to the restriction $s_1<1$;
$p(s_2+1)>2$	is satisfied since $s_2>\max\left\{0,\dfrac{2-p}{p}\right\}$.

Thus, $h(e)$ does not affect the local negative definiteness of $\dot V(e)$.

2. Conditions on the coefficients of stabilizing controller

Based on the Lyapunov function method, to guarantee the asymptotic stability of the equilibrium at the origin, it is sufficient to show that the derivative $\dot V(e)$ is negative definite in some neighborhood of the origin. In order to ensure the negative definiteness, the coefficients associated with the dominant factors e_1^{p+1} and e_2^{p+1} must be strictly negative.

To determine whether there exist scaling constants C_1 and C_2 such that the coefficients associated with e_1^{p+1} and e_2^{p+1} are strictly negative, we consider the following system of inequalities:

$$\begin{cases}-g_1(p)2^{1-p}+g_1(p)\frac{C_1^{p(p+1)}}{p+1}+\frac{k_1k_3^p}{k_2^{p+1}}2^{p-1}g_2(p)\frac{pC_2^{p(p+1)}}{p+1}<0,\\ -g_2(p)2^{1-p}+g_1(p)\frac{p}{(p+1)C_1^{p+1}}+\frac{k_1k_3^p}{k_2^{p+1}}2^{p-1}g_2(p)\left(1+\frac{1}{(p+1)C_2^{p^2(p+1)}}\right)<0,\end{cases}$$

which can be equivalently rewritten as

$$\begin{cases} \frac{k_1 k_3^p}{k_2^{p+1}} 2^{p-1} g_2(p) \frac{p C_2^{p(p+1)}}{p+1} < g_1(p)\left(2^{1-p} - \frac{C_1^{p(p+1)}}{p+1}\right), \\ \frac{k_1 k_3^p}{k_2^{p+1}} 2^{p-1} g_2(p)\left(1 + \frac{1}{(p+1) C_2^{p^2(p+1)}}\right) < g_2(p) 2^{1-p} - g_1(p) \frac{p}{(p+1) C_1^{p+1}}. \end{cases}$$

Consequently, we obtain a decoupled system where the left-hand sides of the inequalities depend exclusively on the control gains k_i, while the right-hand sides depend solely on the scaling parameters C_1 and C_2:

$$\begin{cases} \frac{k_1 k_3^p}{k_2^{p+1}} < \frac{2^{1-p}}{g_2(p)} g_1(p)\left(2^{1-p} - \frac{C_1^{p(p+1)}}{p+1}\right) \frac{p+1}{p C_2^{p(p+1)}}, \\ \frac{k_1 k_3^p}{k_2^{p+1}} < \frac{2^{1-p}}{g_2(p)}\left(g_2(p) 2^{1-p} - g_1(p) \frac{p}{(p+1) C_1^{p+1}}\right) \frac{(p+1) C_2^{p^2(p+1)}}{(p+1) C_2^{p^2(p+1)} + 1}. \end{cases} \tag{10}$$

Given that the control gains satisfy $k_1, k_2, k_3 > 0$, the system of inequalities (10) admits a solution if and only if the right-hand sides are strictly positive. Since $C_1 > 0$ and $C_2 > 0$, this condition simplifies to the requirement that the terms within the parentheses be strictly positive:

$$2^{1-p} - \frac{C_1^{p(p+1)}}{p+1} > 0,$$
$$g_2(p) 2^{1-p} - g_1(p) \frac{p}{(p+1) C_1^{p+1}} > 0.$$

Isolating C_1 from these inequalities results in both an upper and a lower bound:

$$C_1^{p(p+1)} < 2^{1-p}(p+1),$$
$$C_1^{p+1} > \frac{g_1(p) p}{g_2(p)(p+1)} 2^{p-1}.$$

An admissible scaling constant C_1 exists if and only if the strict lower bound is less than the strict upper bound. By raising the lower bound to the power of p (to match the exponent of the upper bound), we arrive at the following sequence of equivalent inequalities:

$$\left(\frac{g_1(p) p}{g_2(p)(p+1)} 2^{p-1}\right)^p < 2^{1-p}(p+1),$$
$$\left(\frac{g_1(p)}{g_2(p)}\right)^p \left(\frac{p}{p+1}\right)^p < 2^{1-p^2}(p+1),$$
$$\left(\frac{g_1(p)}{g_2(p)}\right)^p < 2^{1-p^2} \frac{(p+1)^{p+1}}{p^p}.$$

Consequently, the necessary and sufficient conditions for the solvability of system (10) are formalized as follows:

$$0 < g_1(p) < 2^{\frac{1}{p}-p} \frac{(p+1)^{1+\frac{1}{p}}}{p} g_2(p), \tag{11}$$

and

$$\left(\frac{g_1(p)p}{g_2(p)(p+1)}2^{p-1}\right)^p < C_1^{p(p+1)} < 2^{1-p}(p+1). \tag{12}$$

2.1. Optimization of the scaling parameters

Let us denote the right-hand sides of the inequalities in system (10) as $R_1(C_1, C_2)$ and $R_2(C_1, C_2)$:

$$R_1(C_1, C_2) := \frac{2^{1-p}}{g_2(p)} g_1(p) \left(2^{1-p} - \frac{C_1^{p(p+1)}}{p+1}\right) \frac{p+1}{pC_2^{p(p+1)}}, \tag{13}$$

$$R_2(C_1, C_2) := \frac{2^{1-p}}{g_2(p)} \left(g_2(p)2^{1-p} - g_1(p)\frac{p}{(p+1)C_1^{p+1}}\right) \frac{(p+1)C_2^{p^2(p+1)}}{(p+1)C_2^{p^2(p+1)}+1}. \tag{14}$$

Since the left-hand sides of the inequalities in system (10) are identical, system (10) is equivalent to

$$\frac{k_1 k_3^p}{k_2^{p+1}} < \min\{R_1(C_1, C_2), R_2(C_1, C_2)\}.$$

To obtain a sharper estimate, we maximize this minimum over the scaling parameters. In other words, we are led to the following max-min optimization problem:

$$\max_{C_1, C_2 > 0} \min\{R_1(C_1, C_2), R_2(C_1, C_2)\}.$$

Analytical determination of the global maximum for this max-min problem is non-trivial due to the complex nonlinear structure of R_i. However, considering the behavior of these functions (the opposite monotonicity of the corresponding factors), it is reasonable to suggest that a near-optimal or optimal value is reached when they are equal. While a rigorous investigation of global optimality is beyond the scope of this study, we equate these functions to obtain a constructive sufficient condition:

$$R_1(C_1^*, C_2^*) = R_2(C_1^*, C_2^*). \tag{15}$$

This constraint allows us to simplify the analysis and find a feasible candidate pair (C_1^*, C_2^*) that ensures the fulfillment of the stability conditions (10). Observe from (13) and (14) that the function $R_i(C_1, C_2)$ can be decoupled as a product of two independent functions multiplied by weighting coefficient:

$$R_i(C_1, C_2) = \frac{2^{1-p}}{g_2(p)} F_i(C_1) H_i(C_2), \quad i \in \{1, 2\}.$$

Consequently, a sufficient condition to satisfy equality (15) is to equate the corresponding decoupled terms:

$$F_1(C_1^*) = F_2(C_1^*), \quad \text{and} \quad H_1(C_2^*) = H_2(C_2^*).$$

Let us first evaluate the equality $F_1(C_1) = F_2(C_1)$:

$$g_1(p)\left(2^{1-p} - \frac{C_1^{p(p+1)}}{p+1}\right) = g_2(p)2^{1-p} - g_1(p)\frac{p}{(p+1)C_1^{p+1}},$$

$$(g_1(p) - g_2(p))2^{1-p} - g_1(p)\frac{C_1^{p(p+1)}}{p+1} + g_1(p)\frac{p}{(p+1)C_1^{p+1}} = 0.$$

In the sequel we consider the case of natural p. Multiplying the entire expression by $(p+1)C_1^{p+1}$ to eliminate the denominator, we derive the following polynomial equation with respect to C_1:

$$g_1(p)C_1^{(p+1)^2} + 2^{1-p}(g_2(p) - g_1(p))(p+1)C_1^{p+1} - g_1(p)p = 0. \tag{16}$$

Similarly, we evaluate the equality $H_1(C_2) = H_2(C_2)$:

$$\frac{p+1}{pC_2^{p(p+1)}} = \frac{(p+1)C_2^{p^2(p+1)}}{(p+1)C_2^{p^2(p+1)}+1},$$

$$(p+1)\left((p+1)C_2^{p^2(p+1)} + 1\right) = (p+1)pC_2^{p^2(p+1)+p(p+1)}.$$

Dividing by the common factor $(p+1)$ and rearranging the terms, we obtain a polynomial equation in C_2:

$$pC_2^{p(p+1)^2} - (p+1)C_2^{p^2(p+1)} - 1 = 0. \tag{17}$$

The existence of at least one positive root is guaranteed, since each polynomial is negative at zero and tends to infinity as C_1 or C_2 increases. According to Descartes' rule of signs, the number of positive real roots of a polynomial is equal to the number of sign changes in the sequence of its coefficients, or is less than this number by an even integer. Therefore, since $p > 1$, each of the polynomials, (16) and (17), has exactly one positive real root.

Let C_1^*, C_2^* be the positive real roots of the polynomials (16) and (17) respectively; then the equality (15) holds. Let us choose positive definite functions $g_1(p)$ and $g_2(p)$ that satisfy the inequality (11), for example:

$$g_1(p) := 1, \quad g_2(p) := 2^p. \tag{18}$$

In this case, the corresponding polynomial equations simplify to the forms:

$$C_1^{(p+1)^2} + 2^{1-p}(2^p - 1)(p+1)C_1^{p+1} - p = 0,$$
$$pC_2^{p(p+1)^2} - (p+1)C_2^{p^2(p+1)} - 1 = 0.$$

Furthermore, according to (12), the root C_1^* must lie in the corresponding interval:

$$\left(\frac{p}{2(p+1)}\right)^p < (C_1^*)^{p(p+1)} < 2^{1-p}(p+1). \tag{19}$$

For any specified odd integer $p > 1$, the exact values of C_1^* and C_2^* can be computed numerically. As illustrated in Fig. 1 the blue dots represent the calculated roots C_1^* of the polynomial (16) that successfully satisfy the bounding condition (19).

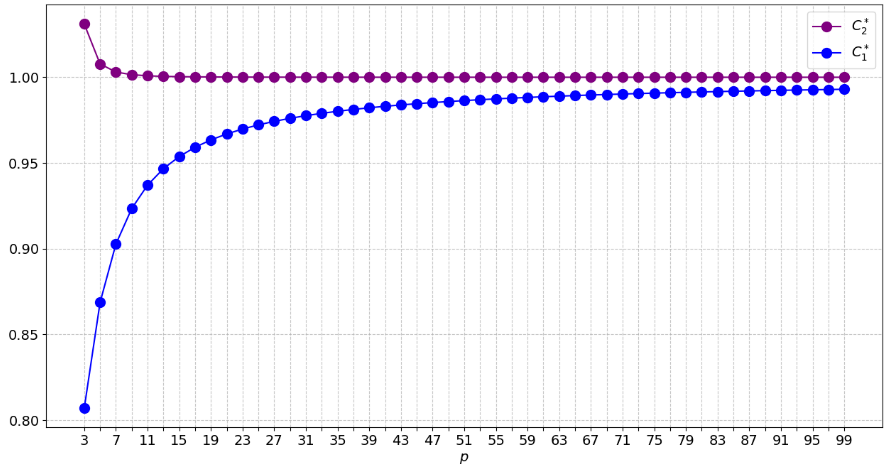


Fig. 1. Dependence of the roots C_1^* and C_2^* on the chosen exponent p.
Рис. 1. Залежність коренів C_1^* та C_2^* від вибраного показника p.

An analysis of the data depicted in Fig. 1 indicates that the identified positive root C_1^* consistently satisfies the necessary condition (12) for all odd integers $p > 1$. Thus, by numerically evaluating the roots C_1^* and C_2^*, we can derive the final stabilizing condition on the control coefficients from system (10):

$$\frac{k_1 k_3^p}{k_2^{p+1}} < R_1(C_1^*, C_2^*) = R_2(C_1^*, C_2^*). \tag{20}$$

Consequently, a strict upper bound for the control gains can be explicitly determined as a function of the exponent p. For instance, referencing the numerical results compiled in Table 1, stabilizing system (2) for $p = 3$ requires the following strict inequality:

$$\frac{k_1 k_3^3}{k_2^4} < 0.0066639059756,$$

where $k_i > 0$ for $i \in \{1, 2, 3\}$. When this condition is satisfied, the control law $u(x) = -k_1x_1 - k_2x_2 - k_3x_3$ guarantees the asymptotic stability of the equilibrium at the origin of the corresponding closed-loop system.

Remark 1. *The restriction to odd integers for the parameter p is imposed here only for convenience of exposition. In fact, the results remain valid for $p > 1$ being a ratio of odd integers with some minor modifications required in the proof.*

Remark 2. *It is important to note that while the system of inequalities* (10) *remains solvable for any odd integer $p > 1$, the computed scaling parameters*

C_1^* and C_2^* are not necessarily globally optimal. Furthermore, the numerical bounds derived using this methodology are highly dependent upon the selection of the functions $g_1(p)$ and $g_2(p)$. Due to the inherent complexity of the nonlinear analysis, predicting exactly how alternative choices for these functions might influence the final numerical results is mathematically nontrivial. Consequently, we subsequently propose an alternative approach to establish a strict bound for the control coefficients that relies exclusively on the system exponent p.

Table 1. Numerical solution depending on p.

p	C_1^*	C_2^*	$R_1(C_1^*, C_2^*) = R_2(C_1^*, C_2^*)$
3	0.8069196040	1.03109362	0.0066639059756
5	0.8686373318	1.00779697	0.00011151180503
7	0.9026865506	1.00305250	1.7902×10^{-6}
9	0.9234310693	1.00149759	2.837×10^{-8}
11	0.9370926072	1.00084302	4.472×10^{-10}
13	0.9466788767	1.00052068	7.03×10^{-12}
15	0.9537502144	1.00034384	1.103×10^{-13}
17	0.9591735749	1.00023883	1.729×10^{-15}
19	0.9634622911	1.00017258	2.71×10^{-17}

2.2. Alternative approach to conditions for control gains

Now we present an alternative approach that guarantees the solvability of the system of inequalities (10) for any odd integer $p > 1$. Numerical analysis demonstrates that this method provides a slightly less conservative estimate for the stabilizing control coefficients.

By substituting the functions $g_1(p)$ and $g_2(p)$ defined in (18) into the original system (10), we obtain

$$\begin{cases} \frac{k_1 k_3^p}{k_2^{p+1}} < 2^{1-2p}\left(2^{1-p} - \frac{C_1^{p(p+1)}}{p+1}\right)\frac{p+1}{pC_2^{p(p+1)}}, \\ \frac{k_1 k_3^p}{k_2^{p+1}} < 2^{1-2p}\left(2 - \frac{p}{(p+1)C_1^{p+1}}\right)\frac{(p+1)C_2^{p^2(p+1)}}{(p+1)C_2^{p^2(p+1)}+1}. \end{cases} \tag{21}$$

To satisfy condition (12), we select a parameter value lying within the corresponding interval. To ensure this, we take the geometric mean of the lower and upper bounds of the interval as a candidate value. This leads to the following sequence of calculations:

$$C_1^{p(p+1)} = \sqrt{\left(\frac{p}{2(p+1)}\right)^p 2^{1-p}(p+1)}$$

$$C_1^{p(p+1)} = \sqrt{\frac{p^p}{(p+1)^{p-1}}2^{1-2p}}$$

$$C_1^{p(p+1)} = \frac{p^{\frac{p}{2}}}{(p+1)^{\frac{p-1}{2}}} 2^{\frac{1}{2}-p}.$$

Consequently, C_1^{p+1} can be explicitly calculated by taking the p-th root:

$$C_1^{p+1} = \left(C_1^{p(p+1)}\right)^{\frac{1}{p}} = \frac{\sqrt{p}}{(p+1)^{\frac{p-1}{2p}}} 2^{\frac{1}{2p}-1}.$$

Substituting the expression for C_1^{p+1} into the system of inequalities (21), we obtain

$$\begin{cases} \frac{k_1 k_3^p}{k_2^{p+1}} < 2^{1-2p} \left(2^{1-p} - \frac{\frac{p^{\frac{p}{2}}}{(p+1)^{\frac{p-1}{2}}} 2^{\frac{1}{2}-p}}{p+1} \right) \frac{p+1}{pC_2^{p(p+1)}}, \\ \frac{k_1 k_3^p}{k_2^{p+1}} < 2^{1-2p} \left(2 - \frac{p}{(p+1)\frac{\sqrt{p}}{(p+1)^{\frac{p-1}{2p}}} 2^{\frac{1}{2p}-1}} \right) \frac{(p+1)C_2^{p^2(p+1)}}{(p+1)C_2^{p^2(p+1)}+1}. \end{cases}$$

By algebraic manipulation, the system is reduced to a more compact form

$$\begin{cases} \frac{k_1 k_3^p}{k_2^{p+1}} < 2^{\frac{3}{2}-3p} \left(\sqrt{2} - \sqrt{p^p (p+1)^{-p-1}} \right) \frac{p+1}{pC_2^{p(p+1)}}, \\ \frac{k_1 k_3^p}{k_2^{p+1}} < 2^{2-2p} \left(1 - 2^{-\frac{1}{2p}} \sqrt{p (p+1)^{\frac{-p-1}{p}}} \right) \frac{(p+1)C_2^{p^2(p+1)}}{(p+1)C_2^{p^2(p+1)}+1}. \end{cases} \tag{22}$$

Let us introduce the auxiliary functions $\widetilde{F}_1(p)$ and $\widetilde{F}_2(p)$ to denote the constant terms

$$\widetilde{F}_1(p) := 2^{\frac{3}{2}-3p} \left(\sqrt{2} - \sqrt{p^p (p+1)^{-p-1}} \right),$$

$$\widetilde{F}_2(p) := 4^{1-p} \left(1 - 2^{-\frac{1}{2p}} \sqrt{p (p+1)^{\frac{-p-1}{p}}} \right).$$

Both functions are positive definite, monotonically decreasing, and asymptotically approach 0. Furthermore, they satisfy the inequality $\widetilde{F}_2(p) > \widetilde{F}_1(p)$ for all $p > 1$.

Thus, system (22) can be rewritten as follows:

$$\begin{cases} \frac{k_1 k_3^p}{k_2^{p+1}} < \widetilde{R}_1(p, C_2), \\ \frac{k_1 k_3^p}{k_2^{p+1}} < \widetilde{R}_2(p, C_2), \end{cases} \tag{23}$$

where the functions $\widetilde{R}_1$ and $\widetilde{R}_2$ are defined as

$$\widetilde{R}_1(p, C_2) := \widetilde{F}_1(p) \frac{p+1}{pC_2^{p(p+1)}},$$

$$\widetilde{R}_2(p, C_2) := \widetilde{F}_2(p) \frac{(p+1)C_2^{p^2(p+1)}}{(p+1)C_2^{p^2(p+1)} + 1}.$$

For a fixed p, the function $\widetilde{R}_1(p, C_2)$ is monotonically decreasing for positive C_2, while $\widetilde{R}_2(p, C_2)$ is monotonically increasing. Therefore, similarly to the previous section, we determine the optimal value for C_2 at the intersection point, where the right-hand sides of system (23) are equal:

$$\widetilde{F}_1(p)\frac{p+1}{pC_2^{p(p+1)}} = \widetilde{F}_2(p)\frac{(p+1)C_2^{p^2(p+1)}}{(p+1)C_2^{p^2(p+1)}+1},$$

$$\widetilde{F}_1(p)(p+1)^2C_2^{p^2(p+1)} + \widetilde{F}_1(p)(p+1) = \widetilde{F}_2(p)p(p+1)C_2^{p(p+1)^2}.$$

By grouping the terms, we obtain the following polynomial equation in C_2:

$$\widetilde{F}_2(p)pC_2^{p(p+1)^2} - \widetilde{F}_1(p)(p+1)C_2^{p^2(p+1)} - \widetilde{F}_1(p) = 0. \tag{24}$$

Let $\widetilde{C}_2$ denote the positive real root of this equation. By continuity, this root exists, since the corresponding polynomial is negative at zero and tends to infinity as C_2 increases. Its uniqueness is guaranteed by Descartes' rule of signs, since the functions $\widetilde{F}_1(p)$ and $\widetilde{F}_2(p)$ are strictly positive for all $p > 1$. Thus, we select $C_2 = \widetilde{C}_2$ as the optimal scaling value for solving system (23).

Numerical calculations, illustrated in Fig. 2, demonstrate that $\widetilde{C}_2$ asymptotically approaches 1 as the degree p increases. Since $\widetilde{C}_2$ is tightly bounded, we can analyze the behavior of the right-hand sides of system (23) at these boundary values. This allows us to derive a strict estimate for the control coefficients k_1, k_2, k_3 that depends exclusively on the exponent p, thereby eliminating the need for complex numerical root-finding algorithms.

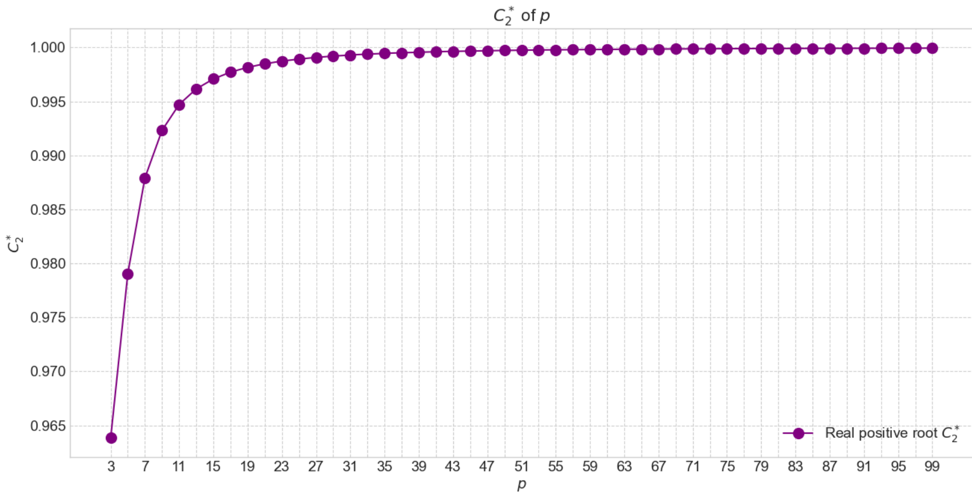
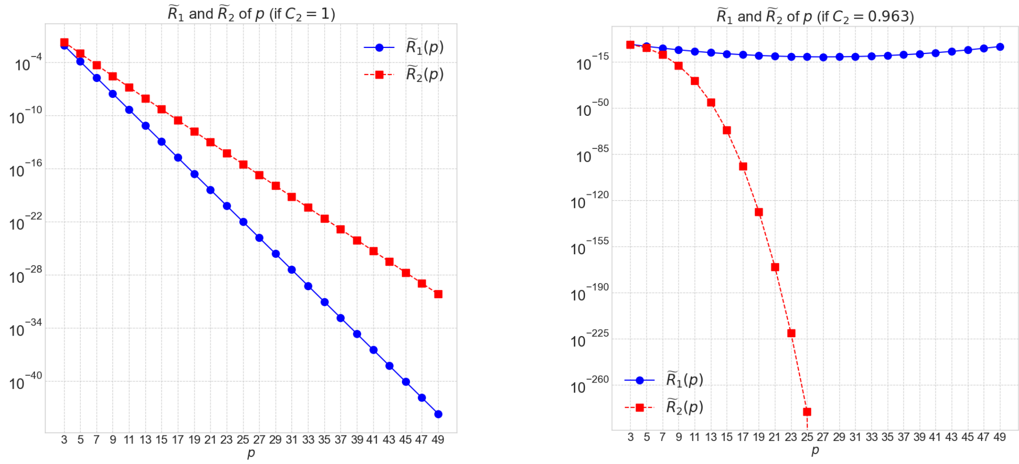


Fig. 2. Dependence of the optimal root $\widetilde{C}_2$ on the exponent p.
Рис. 2. Залежність оптимального кореня $\widetilde{C}_2$ від показника p.

To constructively estimate the maximum of the minimum of the right-hand sides of system (23) from below, it is sufficient to evaluate the functions at $C_2 = 1$.

This structural behavior is confirmed by the plots in Fig. 3, which show that $\widetilde{R}_1(p, 1) < \widetilde{R}_1(p, \widetilde{C}_2)$ for $p > 1$.



$C_2 = 1$ $\qquad\qquad$ $C_2 = 0.963$

Fig. 3. Behavior of $\widetilde{R}_1(p, C_2)$ and $\widetilde{R}_2(p, C_2)$ at boundary values of C_2.
Рис. 3. Поведінка $\widetilde{R}_1(p, C_2)$ та $\widetilde{R}_2(p, C_2)$ для граничних значень C_2.

Consequently, according to the Lyapunov function method, the following condition:

$$\frac{k_1 k_3^p}{k_2^{p+1}} < \widetilde{R}_1(p, 1) = 2^{\frac{3}{2} - 3p} \left(\sqrt{2} - \sqrt{(p+1)^{-p-1} p^p} \right) \frac{p+1}{p}, \tag{25}$$

guarantees the asymptotic stability of the equilibrium at the origin of system (2) under the linear feedback law $u = u(x)$, given by (3).

Table 2. Numerical values of the right-hand side of (25) for various p.

p	$\widetilde{R}_1(p, 1)$	p	$\widetilde{R}_1(p, 1)$	p	$\widetilde{R}_1(p, 1)$
3	0.0080245868397	13	6.9209×10^{-12}	23	6.4452×10^{-21}
5	0.00011967741513	15	1.0805×10^{-13}	25	1.0076×10^{-22}
7	1.8383×10^{-6}	17	1.688×10^{-15}	27	1.5753×10^{-24}
9	2.8504×10^{-8}	19	2.6378×10^{-17}	29	2.4629×10^{-26}
11	4.4373×10^{-10}	21	4.123×10^{-19}	31	3.8506×10^{-28}

3. Examples and generalizations

3.1. Example of three-dimensional canonical nonlinear system

Example 1. To demonstrate the applicability of the theoretical results established in the previous chapter, we first consider the stabilization problem for a canonical three-dimensional nonlinear system of the form

$$\begin{cases} \dot{x}_1 = x_2^{11}, \\ \dot{x}_2 = x_3^{11}, \\ \dot{x}_3 = u. \end{cases} \tag{26}$$

In this case $p = 11$.

The control gains k_1, k_2, and k_3 are selected to satisfy the sufficient condition (25). By choosing $k_1 = 5, k_2 = 15$, and $k_3 = 2$, we guarantee that

$$\frac{k_1 k_3^{11}}{k_2^{12}} \approx 0.789 \times 10^{-10} < 4.4373 \times 10^{-10},$$

which strictly satisfies the numerical bounds presented in Table 2 for $p = 11$.

Consequently, the stabilizing linear control law is defined as

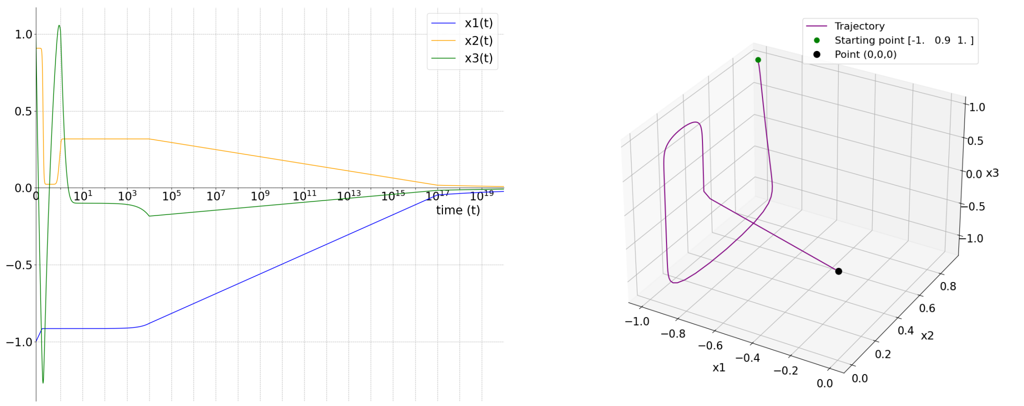
$$u(x) = -5x_1 - 15x_2 - 2x_3.$$

Applying this control input to system (26) leads to the closed-loop system

$$\begin{cases} \dot{x}_1 = x_2^{11}, \\ \dot{x}_2 = x_3^{11}, \\ \dot{x}_3 = -5x_1 - 15x_2 - 2x_3. \end{cases}$$

The dynamic behavior of the closed-loop system was simulated using the following initial state vector:

$$x_1(0) = -1, \quad x_2(0) = 0.9, \quad x_3(0) = 1.$$



Trajectories $x_1(t), x_2(t), x_3(t)$ Phase trajectory

Fig. 4. Trajectories of system (26) under the linear control law $u = u(x)$.

Рис. 4. Траєкторії системи (26) при лінійному законі керування $u = u(x)$.

As illustrated in Fig. 4, all solutions $x_i(t)$ converge asymptotically to the origin. The phase trajectory further demonstrates the stability of the equilibrium at the origin under the proposed linear feedback control law.

3.2. Generalized nonlinear system with higher-order terms

Following the methodology described in [1, 3], we extend our analysis to a generalized class of systems incorporating higher-order nonlinear perturbations:

$$\begin{cases} \dot{x}_i = x_{i+1}^{p_i} + f_i(x_1, \dots, x_n), \quad i = 1, \dots, n-1, \\ \dot{x}_n = u^{p_n}, \end{cases} \tag{27}$$

where the terms $f_i(x_1, \dots, x_n)$ represent continuous nonlinearities of a higher order relative to the principal terms $x_{i+1}^{p_i}$.

To stabilize system (27), we apply the same linear control law $u(x)$:

$$u(x) = -k_1 x_1 - \dots - k_n x_n,$$

where $k_i \in \mathbb{R}$ and $k_i > 0$ for $i = 1, \dots, n$. This control law stabilizes the nonlinear approximation of the system, given by

$$\begin{cases} \dot{x}_i = x_{i+1}^{p_i}, \quad i = 1, \dots, n-1, \\ \dot{x}_n = u^{p_n}. \end{cases} \tag{28}$$

A key condition for the applicability of this approach is the requirement that the continuous functions f_i satisfy the following estimates in some neighborhood of the origin:

$$|f_i(x_1, \dots, x_n)| \le \rho_i(x_1, \dots, x_n) \left(|x_{i+1}|^{p_i+\delta_i} + |x_{i+2}|^{p_i+\delta_i} + \dots + |x_n|^{p_i+\delta_i} \right), \tag{29}$$

where $\rho_i(x_1, \dots, x_n) \ge 0$ are continuous functions and $\delta_i > 0$ are constants.

Since the order of the functions $f_i(x_1, \dots, x_n)$ is higher than the order of $x_{i+1}^{p_i}$, the control $u(x)$ also stabilizes the system (27). This allows us to retain the choice of the Lyapunov function and the change of variables applied to the system (28). Moreover, the additional higher-order terms arising due to f_i during the differentiation of the Lyapunov function will be absorbed into the function $h(x)$ as higher-order terms. Such terms do not affect the sign of the derivative in a sufficiently small neighborhood of the origin.

Let us focus on the specific case where $n = 3$, $p_1, p_2 := p > 1$, and $p_3 = 1$. System (27) then takes the form

$$\begin{cases} \dot{x}_1 = x_2^p + f_1(x_1, x_2, x_3), \\ \dot{x}_2 = x_3^p + f_2(x_1, x_2, x_3), \\ \dot{x}_3 = u. \end{cases}$$

Assuming the estimates (29) hold for f_1 and f_2, the linear control law $u(x) = -k_1 x_1 - k_2 x_2 - k_3 x_3$ stabilizes the equilibrium at the origin of the corresponding closed-loop system, provided that the sufficient condition (25) is satisfied.

Example 2. Consider the stabilization problem for the following nonlinear system:

$$\begin{cases} \dot{x}_1 = x_2^3 + x_2^4 \sin(x_1 + x_2), \\ \dot{x}_2 = x_3^3 + x_3^4 \cos(x_1), \\ \dot{x}_3 = u. \end{cases} \tag{30}$$

In this example, the system includes trigonometric perturbations

$$f_1(x) = x_2^4 \sin(x_1 + x_2) \quad \text{and} \quad f_2(x) = x_3^4 \cos(x_1).$$

These terms satisfy the growth condition (29) with parameters $\delta_1 = \delta_2 = 1$ and bounding functions $\rho_1 = \rho_2 = 1$.

Let us select the control gains as $k_1 = 1$, $k_2 = 4$, and $k_3 = 1$. Evaluating the sufficient stability condition gives

$$\frac{k_1 k_3^3}{k_2^4} = 0.00390625 < 0.008024,$$

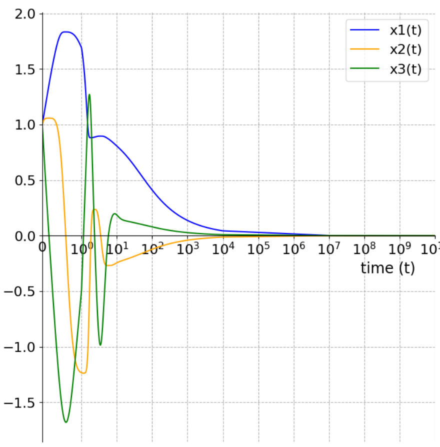
which strictly satisfies the numerical bound derived from (25) for $p = 3$.

Consequently, the stabilizing linear control law takes the form

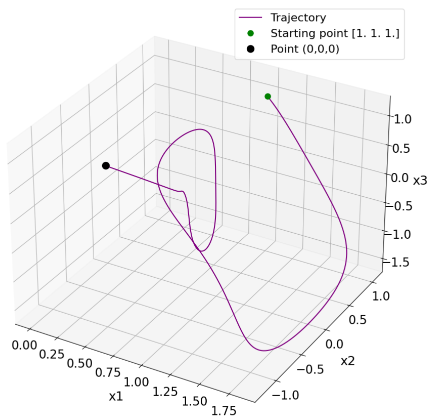
$$u(x) = -x_1 - 4x_2 - x_3.$$

The closed-loop system was simulated using the following initial state vector:

$$x_1(0) = 1, \quad x_2(0) = 1, \quad x_3(0) = 1.$$



Trajectories $x_1(t), x_2(t), x_3(t)$



Phase trajectory

Fig. 5. Trajectories of system (30) under the proposed linear control law.

Рис. 5. Траєкторії системи (30) при запропонованому лінійному керуванні.

The numerical simulation confirms that the proposed linear control law $u(x) = -x_1 - 4x_2 - x_3$ effectively compensates for the higher-order nonlinear perturbations. As predicted by the theoretical analysis, the trajectories exhibit asymptotic convergence to the origin for initial states lying in a sufficiently small neighborhood of the origin.

4. Conclusion

In this work, the linear stabilization problem has been solved for a class of three-dimensional systems with power nonlinearities in a previously uninvestigated special case. Compared to previous results [3, 4, 9], the condition of strictly decreasing powers has been relaxed to a non-strictly decreasing condition. It has been proven that for the investigated class of systems, the application of a linear control law is effective, the proof is based on the construction of a special Lyapunov function and the analysis of its derivative. Additional conditions on the linear control coefficients k_1, k_2, and k_3, which guarantee the local asymptotic stability of the equilibrium at the origin, were obtained in several ways. Note that in the case of strictly decreasing powers, there were no additional conditions on the coefficients. Moreover, a sufficient condition that is easy to verify in practice was obtained for systems with arbitrarily large powers.

The obtained theoretical results were numerically confirmed by examples, which demonstrated the convergence of the phase trajectories of the systems to the origin. The class of systems was extended by means of nonlinear approximation. The prospect for future research lies in considering systems of higher dimensions. The results of this work give reason to believe that by a proper choice of the Lyapunov function coefficients, the existence of a stabilizing linear control can be guaranteed for systems of arbitrary dimension.

Conflicts of Interest: The authors declare no conflict of interest.

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Про стабілізацію канонічних нелінійних систем у спеціальному випадку

М. О. Бебія, Ю. М. Барська
Харківський національний університет імені В. Н. Каразіна
майдан Свободи 4, 61022, Харків, Україна

У статті розглядається задача стабілізації для канонічних нелінійних систем, які у трикутному випадку відповідають p–нормальним системам. Ці системи є суттєво нелінійними, оскільки вони не підлягають фідбек-лінеаризації, а їхня лінійна апроксимація є виродженою і не характеризує стійкість точки спокою у початку координат. У трикутному випадку backstepping є стандартним методом синтезу керування зі зворотним зв'язком через свою рекурсивну природу. Backstepping є потужним інструментом стабілізації для широкого класу нелінійних систем, що мають strict-feedback форму. Однак цей рекурсивний підхід зазвичай призводить до керувань зі зворотним зв'язком, складність яких зростає зі збільшенням розмірності системи. Це

мотивує дослідження більш простих методів побудови керувань. Упродовж останніх років було отримано декілька вагомих результатів щодо нерекурсивної стабілізації. Було побудовано класи поліноміальних керувань для випадку строгого спадання степенів степеневих членів правої частини системи. Зокрема, було показано, що такі системи можуть бути асимптотично стабілізовані лінійним керуванням зі зворотним зв'язком у випадку строго спадаючих степенів. Більше того, коефіцієнти керування може бути обрано як довільні додатні числа. Крім того, було встановлено, що стабілізацію можна досягти у деякому спеціальному випадку нестрого спадаючих степенів при додаткових умовах на коефіцієнти керування. У цій роботі ми розв'язуємо задачу лінійної стабілізації у раніше недослідженому випадку нестрого спадаючих степенів для тривимірних систем. Ми пропонуємо конструктивний підхід до знаходження умов на коефіцієнти лінійного керування, що гарантують асимптотичну стійкість. Наш підхід ґрунтується на методі функції Ляпунова. Ми також використовуємо стабілізацію з застосуванням нелінійної апроксимації, щоб узагальнити отриманий результат. Ефективність запропонованого підходу продемонстровано та підтверджено чисельними прикладами.
Ключові слова: **стабілізація; нелінійні системи; метод функції Ляпунова; критичний випадок; керування зі зворотним зв'язком; лінійна стабілізація.**

Конфлікт інтересів: Автори повідомляють про відсутність конфлікту інтересів.

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