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Formal power series solutions of the differential equation $ay' = by^m$ over Dedekind domains of characteristic zero

Let D be a Dedekind domain of characteristic zero, let $K = \mathrm{Frac}(D)$ be its fraction field, and consider the Cauchy problem $ay' = by^m$, $y(0) = c_0$, in the ring $D[[x]]$, where $a, b, c_0 \in D$, $a, b \neq 0$, and $m \in \mathbb{N}$. The paper studies when the unique formal solution $y \in K[[x]]$ with initial value c_0 actually has all coefficients in D, and therefore belongs to $D[[x]]$. The argument starts from the coefficient recursion in $K[[x]]$ and derives an explicit formula for the coefficients of the unique solution. This reduces the existence problem in $D[[x]]$ to an arithmetic integrality question governed by the valuations attached to nonzero prime ideals of D. The main point is that over a Dedekind domain the relevant obstruction is not ordinary divisibility by prime elements, but comparison of valuations of fractional ideals after localization at prime ideals.

For the linear case $m = 1$, the zero initial value always gives the zero solution. For $c_0 \neq 0$, the coefficients contain factorial denominators, and this produces a global obstruction unless only finitely many rational primes remain nonunits in D. In that finite-prime situation the exact criterion for a nonzero initial value is the ideal containment $(b) \subseteq (a)\mathfrak{r}_1$, where $\mathfrak{r}_1$ is an explicitly described correction ideal built from the prime ideals lying over the rational primes that are nonunits in D. For the nonlinear case $m \geq 2$, writing $d = m-1$, the coefficients are expressed through the product $C_k(d) = \prod_{i=0}^{k-1}(di + 1)$. The paper shows that prime ideals over rational primes not dividing d create no additional obstruction beyond the basic condition $(bc_0^d) \subseteq (a)$, while the prime ideals over rational primes dividing d contribute a correction ideal $\mathfrak{r}(d)$. As a result, the exact existence criterion becomes $(bc_0^d) \subseteq (a)\mathfrak{r}(d)$. In particular, for $m = 2$ one has $\mathfrak{r}(1) = D$, so the condition reduces to $(bc_0) \subseteq (a)$. Several examples are included to illustrate the theorem over $\mathbb{Z}$, localizations of $\mathbb{Z}$, Gaussian integers, and the Dedekind domain $\mathbb{Z}[\sqrt{-5}]$, which is not a unique factorization domain.

Keywords: **formal power series; Dedekind domain; Cauchy problem; prime-ideal valuation; integrality criterion.**

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1. Introduction

Let D be a Dedekind domain of characteristic zero. We recall the definition of a Dedekind domain in Section 3, together with the prime-ideal valuation conventions used later. Since $\mathrm{char}(D) = 0$, we view $\mathbb{Z}$ as a subring of D. Fix

$$a, b, c_0 \in D, \qquad a, b \neq 0, \qquad m \in \mathbb{N}.$$

We consider the Cauchy problem in the ring of formal power series $D[[x]]$:

$$ay' = by^m, \qquad y(0) = c_0.$$

We seek a solution in the form

$$y = \sum_{k=0}^{\infty} c_k x^k.$$

Our goal is to determine exactly when this problem has a solution in $D[[x]]$. The analytic part is easy over the fraction field $K = \mathrm{Frac}(D)$: once c_0 is fixed, the coefficient recursion determines a unique solution $y \in K[[x]]$. Section 2 records this computation, while Section 3 introduces the prime-ideal valuations and arithmetic notation used in the descent criterion. The real issue is descent from $K[[x]]$ to $D[[x]]$, and for a Dedekind domain that descent is governed by valuations at nonzero prime ideals rather than divisibility by prime elements. Related algebraic approaches to differential equations over commutative rings appear in [3, 4, 5, 6].

Main result preview. For $m = 1$, the zero initial value always gives the zero solution. For $c_0 \neq 0$, the factorial denominators in

$$c_k = \frac{c_0 b^k}{k! a^k}$$

force an infinite-prime obstruction unless only finitely many rational primes remain nonunits in D. In that finite-prime case, the exact criterion for a nonzero initial value is $(b) \subseteq (a)\mathfrak{r}_1$, where $\mathfrak{r}_1$ is the correction ideal defined in Section 4 and encoding the obstruction coming from the rational primes that remain nonunits in D. For $m \geq 2$, writing $d = m - 1$, the only extra obstruction beyond the basic containment $(bc_0^d) \subseteq (a)$ comes from the prime ideals lying over the rational primes dividing d, and the final criterion is

$$(bc_0^d) \subseteq (a)\mathfrak{r}(d).$$

Here $\mathfrak{r}(d)$, also defined in Section 4, is the corresponding correction ideal for the primes dividing d. When D is also a unique factorization domain (UFD), these ideal containments reduce to the familiar divisibility statements $ar_1 \mid b$ and $ar(d) \mid bc_0^d$. In particular, in this UFD notation, for $m = 2$ one has $r(1) = 1$, so the condition becomes simply $a \mid bc_0$.

2. The Unique Solution in $K[[x]]$ and Its Coefficients

In this section we prove that the Cauchy problem has a unique solution $y \in K[[x]]$, compute its coefficients in K, and reduce existence in $D[[x]]$ to coefficient integrality.

Let $K = \mathrm{Frac}(D)$. Any solution in $D[[x]]$ is also a solution in $K[[x]]$, and in K the element a is invertible. Therefore all identities involving division by a below are understood inside $K[[x]]$.

Write

$$y = \sum_{k=0}^{\infty} c_k x^k.$$

Then

$$y^m = \sum_{n=0}^{\infty} \left(\sum_{\substack{i_1,\ldots,i_m \in \mathbb{Z}_{\geq 0} \\ i_1+\cdots+i_m=n}} c_{i_1} \cdots c_{i_m} \right) x^n.$$

Comparing coefficients in $ay' = by^m$, we obtain for every $n \in \mathbb{Z}_{\geq 0}$,

$$a(n+1)c_{n+1} = b \sum_{\substack{i_1,\ldots,i_m \in \mathbb{Z}_{\geq 0} \\ i_1+\cdots+i_m=n}} c_{i_1} \cdots c_{i_m}.$$

The right-hand side depends only on $c_0, \ldots, c_n$, and $a(n+1)$ is invertible in K. Therefore the recurrence determines $c_1, c_2, \ldots$ uniquely from the initial value c_0, and hence defines a unique solution in $K[[x]]$. The general formal existence-uniqueness statement for $y' = F(x, y)$, after translating the initial value to 0, is classical; see [2, Ch. VII, §1, no. 2, Prop. 2.1]. Although stated there in the analytic setting, the proof is purely coefficientwise, so the same argument works over any field. We keep the short derivation for the present equation.

To compute its coefficients, write

$$d = m - 1.$$

For $k \in \mathbb{Z}_{\geq 0}$, define

$$C_k(d) = \prod_{i=0}^{k-1} (di + 1) \qquad (C_0(d) = 1).$$

Lemma 1. *For every $k \in \mathbb{Z}_{\geq 0}$,*

$$y^{(k)} = \frac{C_k(d) b^k}{a^k} y^{dk+1}.$$

Proof. We use induction on k. For $k = 0$, the identity is trivial.

Assume it holds for some $k \in \mathbb{Z}_{\geq 0}$. Then

$$\begin{aligned} y^{(k+1)} &= \left(\frac{C_k(d) b^k}{a^k} y^{dk+1} \right)' \\ &= \frac{C_k(d) b^k}{a^k} (dk+1) y^{dk} y' \\ &= \frac{C_k(d)(dk+1) b^{k+1}}{a^{k+1}} y^{d(k+1)+1} \\ &= \frac{C_{k+1}(d) b^{k+1}}{a^{k+1}} y^{d(k+1)+1}. \end{aligned}$$

This proves the claim. □

Evaluating at $x = 0$ and dividing by $k!$, we obtain for every $k \in \mathbb{Z}_{\geq 0}$,

$$c_k = \frac{y^{(k)}(0)}{k!} = \frac{C_k(d) c_0 (b c_0^d)^k}{k! a^k}.$$

For $m = 1$, we have $d = 0$ and $C_k(0) = 1$, so this reduces to

$$c_k = \frac{c_0 b^k}{k! a^k}.$$

The existence problem in $D[[x]]$ is therefore equivalent to deciding when these explicit coefficient formulas actually lie in D. From this point on, the problem is purely arithmetic.

3. Prime-Ideal Valuations and Arithmetic Lemmas

For a rational prime $p \in \mathbb{Z}$, we write

$$v_p(n) = \max\{k \in \mathbb{Z}_{\geq 0} : p^k \mid n\} \qquad (n \in \mathbb{Z},\ n \neq 0).$$

This is the usual p-adic valuation on the integers. It is this valuation that appears in Legendre's formula and in the estimates for $C_n(d)$ below.

The next lemmas isolate the arithmetic input needed for the descent argument. The first step is to replace divisibility by prime elements with valuations at nonzero prime ideals. After that, the only genuinely quantitative ingredients are the p-adic behavior of the product

$$C_n(d) = \prod_{i=0}^{n-1} (di + 1)$$

and the estimates for $v_p(n!)$.

Background conventions. Standard background on Dedekind domains, fractional ideals, and prime-ideal valuations may be found in [1, Ch. 9], [8, Ch. I, §3], and [8, Ch. I, §11].

Definition 1 (Dedekind domain). *A domain D is called a Dedekind domain if it is noetherian, integrally closed in its fraction field, and every nonzero prime ideal of D is maximal; see [1, Ch. 9] and [8, Ch. I, §3]. Equivalently, every nonzero proper ideal of D factors uniquely into prime ideals.*

Definition 2 (Fractional ideals). *A nonzero fractional ideal of D is a nonzero D-submodule $I \subseteq K$ such that $\alpha I \subseteq D$ for some nonzero $\alpha \in D$.*

Definition 3 (Prime-ideal valuations). *Since D is Dedekind, every nonzero fractional ideal factors uniquely in the form*

$$I = \prod_{\mathfrak{p} \neq 0} \mathfrak{p}^{n_{\mathfrak{p}}},$$

where the product runs over the nonzero prime ideals of D, all but finitely many exponents $n_{\mathfrak{p}} \in \mathbb{Z}$ are zero, and the factorization is unique. For a nonzero fractional ideal I, we define

$$v_{\mathfrak{p}}(I) = n_{\mathfrak{p}}.$$

For $x \in K \setminus \{0\}$, we define

$$v_{\mathfrak{p}}(x) = v_{\mathfrak{p}}(xD),$$

and we set $v_{\mathfrak{p}}(0) = +\infty$.

Standard facts. We will use the following standard consequences of the factorization of fractional ideals; see again [1, Ch. 9], [8, Ch. I, §3], and [8, Ch. I, §11].

- For nonzero fractional ideals I, J, one has

  $$v_{\mathfrak{p}}(IJ) = v_{\mathfrak{p}}(I) + v_{\mathfrak{p}}(J)$$

  for every nonzero prime ideal $\mathfrak{p}$.

- For a nonzero fractional ideal I, only finitely many valuations $v_{\mathfrak{p}}(I)$ are nonzero.

- For nonzero fractional ideals I, J, one has

  $$I \subseteq J \qquad \Longleftrightarrow \qquad v_{\mathfrak{p}}(I) \geq v_{\mathfrak{p}}(J)$$

  for every nonzero prime ideal $\mathfrak{p}$.

- For nonzero $x, y \in K$ and $n \in \mathbb{Z}_{\geq 0}$,

  $$v_{\mathfrak{p}}(xy) = v_{\mathfrak{p}}(x) + v_{\mathfrak{p}}(y), \qquad v_{\mathfrak{p}}(x^n) = n v_{\mathfrak{p}}(x).$$

Lemma 2. *Let $x \in K$. Then $x \in D$ if and only if*

$$v_{\mathfrak{p}}(x) \geq 0$$

for every nonzero prime ideal $\mathfrak{p}$ of D.

Proof. This is the standard valuation criterion for membership in a Dedekind domain; see [8, Ch. I, §11]. □

For a rational prime p, we write $\mathfrak{p} \mid p$ if $p \in \mathfrak{p}$, equivalently if $\mathfrak{p} \cap \mathbb{Z} = (p)$. Because (p) is a nonzero ideal of the Dedekind domain D, it has only finitely many prime divisors.

Lemma 3. *Let $\mathfrak{p}$ be a nonzero prime ideal of D. Then exactly one of the following holds.*

1. *$\mathfrak{p} \cap \mathbb{Z} = (p)$ for a rational prime p, and then for every nonzero $n \in \mathbb{Z}$,*

$$v_{\mathfrak{p}}(n) = v_{\mathfrak{p}}(p)\, v_p(n).$$

2. *$\mathfrak{p} \cap \mathbb{Z} = (0)$, and then for every nonzero $n \in \mathbb{Z}$,*

$$v_{\mathfrak{p}}(n) = 0.$$

Proof. This follows from the general facts on contraction of prime ideals and on the valuations attached to prime ideals; see [1, Ch. 1] and [8, Ch. I, §11]. □

For the coefficients in nonlinear cases of the main theorem, the key quantity is

$$C_n(d) = \prod_{i=0}^{n-1} (di + 1),$$

because it is exactly the factor that appears in

$$c_n = \frac{C_n(d) c_0 (b c_0^d)^n}{n! a^n}.$$

The next lemma measures the p-adic cancellation between $C_n(d)$ and $n!$ at the level of ordinary integers.

Lemma 4. *Let $d, n \in \mathbb{N}$, and define*

$$C_n(d) = \prod_{i=0}^{n-1} (di + 1).$$

Let p be a rational prime.

1. *If $p \nmid d$, then*

$$v_p(n!) \le v_p(C_n(d)) \le v_p(n!) + \log_p n + \log_p d.$$

2. *If $p \mid d$, then*

$$v_p(C_n(d)) = 0.$$

Proof. For $k \geq 1$, let

$$N_k = \# \left\{ 0 \leq i \leq n-1 : p^k \mid di + 1 \right\}.$$

Thus N_k counts how many factors in $C_n(d)$ are divisible by p^k. A factor with p-adic valuation r contributes 1 to each of $N_1, \ldots, N_r$ and nothing to the later terms. Therefore

$$v_p(C_n(d)) = \sum_{i=0}^{n-1} v_p(di+1) = \sum_{k \geq 1} N_k.$$

This is the same counting principle as in Legendre's formula

$$v_p(n!) = \sum_{k \geq 1} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

Suppose first that $p \nmid d$. Then d is invertible modulo p^k for every $k \geq 1$, so the congruence

$$di + 1 \equiv 0 \pmod{p^k}$$

has exactly one solution modulo p^k. Hence among any p^k consecutive values of i, exactly one satisfies $p^k \mid di + 1$. The interval $0, 1, \ldots, n-1$ consists of $\lfloor n/p^k \rfloor$ complete blocks of length p^k, and possibly one final incomplete block. Each full block contributes exactly one such index, and the incomplete block contributes at most one. Therefore

$$\left\lfloor \frac{n}{p^k} \right\rfloor \leq N_k \leq \left\lceil \frac{n}{p^k} \right\rceil.$$

Summing the lower bounds over $k \geq 1$, we get

$$v_p(C_n(d)) = \sum_{k \geq 1} N_k \geq \sum_{k \geq 1} \left\lfloor \frac{n}{p^k} \right\rfloor = v_p(n!).$$

For the upper bound, note that every factor $di + 1$ is at most $d(n-1) + 1$. Hence $N_k = 0$ as soon as $p^k > d(n-1) + 1$. If

$$K = \left\lfloor \log_p(d(n-1)+1) \right\rfloor,$$

then

$$
\begin{aligned}
v_p(C_n(d)) &= \sum_{k=1}^{K} N_k \\
&\le \sum_{k=1}^{K} \left\lceil \frac{n}{p^k} \right\rceil \\
&\le \sum_{k=1}^{K} \left(\left\lfloor \frac{n}{p^k} \right\rfloor + 1 \right) \\
&\le v_p(n!) + K \\
&\le v_p(n!) + \log_p(d(n-1)+1) \\
&\le v_p(n!) + \log_p n + \log_p d.
\end{aligned}
$$

Finally, if $p \mid d$, then $di + 1 \equiv 1 \pmod{p}$ for every i. So none of the factors in $C_n(d)$ is divisible by p, and therefore $v_p(C_n(d)) = 0$. □

Lemma 5. *Let $d, n \in \mathbb{N}$, let*

$$C_n(d) = \prod_{i=0}^{n-1} (di + 1),$$

and let $\mathfrak{p}$ be a nonzero prime ideal of D.

1. *If $\mathfrak{p} \mid p$ for a rational prime $p \nmid d$, then*

$$v_{\mathfrak{p}}(n!) \le v_{\mathfrak{p}}(C_n(d)) \le v_{\mathfrak{p}}(n!) + v_{\mathfrak{p}}(p)(\log_p n + \log_p d).$$

2. *If $\mathfrak{p} \mid p$ for a rational prime $p \mid d$, then*

$$v_{\mathfrak{p}}(C_n(d)) = 0.$$

3. *If $\mathfrak{p} \cap \mathbb{Z} = (0)$, then*

$$v_{\mathfrak{p}}(n!) = v_{\mathfrak{p}}(C_n(d)) = 0.$$

Proof. If $\mathfrak{p} \cap \mathbb{Z} = (0)$, then Lemma 3 gives $v_{\mathfrak{p}}(n!) = 0$, and each factor $di + 1$ is a nonzero integer, so again Lemma 3 gives $v_{\mathfrak{p}}(C_n(d)) = 0$.

Now assume $\mathfrak{p} \mid p$. By Lemma 3,

$$v_{\mathfrak{p}}(n!) = v_{\mathfrak{p}}(p)\, v_p(n!), \qquad v_{\mathfrak{p}}(C_n(d)) = v_{\mathfrak{p}}(p)\, v_p(C_n(d)).$$

If $p \nmid d$, the desired inequalities follow from Lemma 4 after multiplication by $v_{\mathfrak{p}}(p)$. If $p \mid d$, the same lemma gives $v_p(C_n(d)) = 0$, hence $v_{\mathfrak{p}}(C_n(d)) = 0$. □

The final ingredient is the bound on $v_p(n!)$ in a crude asymptotic form.

Lemma 6. *Let p be a rational prime and $n \in \mathbb{N}$. Then*

$$v_p(n!) \le \frac{n}{p-1}, \qquad v_p(n!) \ge \frac{n}{p-1} - \frac{p}{p-1} - \log_p n.$$

Proof. Using Legendre's formula,

$$v_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

The upper bound follows from

$$\sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor \le \sum_{k=1}^{\infty} \frac{n}{p^k} = \frac{n}{p-1}.$$

For the lower bound, let $K = \lfloor \log_p n \rfloor$. Then

$$\begin{aligned} v_p(n!) &= \sum_{k=1}^{K} \left\lfloor \frac{n}{p^k} \right\rfloor \\ &\ge \sum_{k=1}^{K} \left(\frac{n}{p^k} - 1 \right) \\ &= \sum_{k=1}^{K} \frac{n}{p^k} - K. \end{aligned}$$

Moreover,

$$\sum_{k=1}^{K} \frac{n}{p^k} = \frac{n}{p-1} - \frac{n}{p^{K+1}} \cdot \frac{p}{p-1} > \frac{n}{p-1} - \frac{p}{p-1},$$

because $p^{K+1} > n$. Therefore

$$v_p(n!) \ge \frac{n}{p-1} - \frac{p}{p-1} - \log_p n.$$

$\square$

4. Main Descent Theorem

The explicit formulas from Section 2 reduce existence in $D[[x]]$ to a valuation problem. The correction ideals introduced below record the residual obstruction left by the factorial denominator.

For $d \in \mathbb{N}$, define the nonzero ideal

$$\mathfrak{r}(d) = \prod_{p \mid d} \prod_{\mathfrak{p} \mid p} \mathfrak{p}^{\left\lceil \frac{v_{\mathfrak{p}}(p)}{p-1} \right\rceil},$$

with the convention $\mathfrak{r}(1) = D$. This product is finite because d has only finitely many rational prime divisors and each (p) has only finitely many prime divisors in D.

If only finitely many rational primes are nonunits in D, define also

$$\mathfrak{r}_1 = \prod_{\substack{p \text{ prime} \\ p \text{ nonunit in } D}} \prod_{\mathfrak{p}|p} \mathfrak{p}^{\left\lceil \frac{v_{\mathfrak{p}}(p)}{p-1} \right\rceil}.$$

Theorem 1. *Let $a, b, c_0 \in D$ with $a, b \neq 0$, and let $m \in \mathbb{N}$. Consider the Cauchy problem*

$$ay' = by^m, \qquad y(0) = c_0.$$

The existence of a solution $y \in D[[x]]$ for this problem is characterized as follows.

1. ***Linear case** $(m = 1)$.*

   (а) *If $c_0 = 0$, then the unique solution is the zero series.*

   (б) *Assume $c_0 \neq 0$. If infinitely many rational primes are nonunits in D, then no solution exists in $D[[x]]$.*

   (в) *Assume $c_0 \neq 0$ and only finitely many rational primes are nonunits in D. Then a solution exists in $D[[x]]$ if and only if*

   $$(b) \subseteq (a)\mathfrak{r}_1.$$

2. ***Nonlinear case** $(m \geq 2)$. Write $d = m - 1$.*

   (а) *If $c_0 = 0$, then the unique solution is the zero series.*

   (б) *If $c_0 \neq 0$, then a solution exists in $D[[x]]$ if and only if*

   $$(bc_0^d) \subseteq (a)\mathfrak{r}(d).$$

For $m = 2$, the nonlinear criterion becomes

$$(bc_0) \subseteq (a),$$

because $\mathfrak{r}(1) = D$.

Remark 1. *If D is a unique factorization domain, then, since D is Dedekind, it is also a principal ideal domain. Hence $\mathfrak{r}(d)$ and $\mathfrak{r}_1$ are principal ideals. Writing*

$$\mathfrak{r}(d) = (r(d)), \qquad \mathfrak{r}_1 = (r_1),$$

the conditions in the theorem become the familiar element-divisibility statements

$$ar(d) \mid bc_0^d, \qquad ar_1 \mid b.$$

Before the detailed proof in Section 5, we briefly explain the main idea behind the three cases.

Case $m = 1$. The coefficient formula is

$$c_k = \frac{c_0 b^k}{k! a^k}.$$

The obstruction comes from the factorial denominators in $k!$. If infinitely many rational primes are nonunits in D, only the zero initial value descends. In the finite-prime case, the final criterion is $c_0 = 0$, or else $(b) \subseteq (a)\mathfrak{r}_1$.

Case $m = 2$. The coefficient formula is

$$c_k = c_0 \left(\frac{bc_0}{a} \right)^k.$$

Here one has complete cancellation $C_k(1) = k!$, so only the basic ideal containment remains, namely $(bc_0) \subseteq (a)$.

Case $m \geq 3$. The coefficient formula is

$$c_k = \frac{C_k(d) c_0 (bc_0^d)^k}{k! a^k}, \qquad d = m - 1.$$

In this range only the prime ideals lying over rational primes dividing d contribute the correction ideal $\mathfrak{r}(d)$, and the final criterion is $(bc_0^d) \subseteq (a)\mathfrak{r}(d)$.

5. Proof of the Main Theorem

5.1. Case $m = 1$.

We first separate the linear proof into two parts. If infinitely many rational primes remain nonunits in D, then some factorial denominator survives and blocks every nonzero solution. If only finitely many do, the problem becomes a valuation criterion encoded by the ideal $\mathfrak{r}_1$.

Zero initial value.

If $c_0 = 0$, then $y = 0$ is a solution. By Section 2, the solution in $K[[x]]$ with initial value 0 is unique. Hence $y = 0$ is also the unique solution in $D[[x]]$.

Infinite-prime obstruction.

Assume now that $c_0 \neq 0$. By Section 2, the unique solution in $K[[x]]$ has coefficients

$$c_k = \frac{c_0 b^k}{k! a^k}.$$

Suppose first that infinitely many rational primes are nonunits in D. The principal ideal (bc_0) has only finitely many prime divisors, so only finitely many rational primes p have the property that some prime ideal $\mathfrak{p} \mid p$ divides (bc_0). Choose a rational prime p that is a nonunit in D and has no such prime ideal above it. Let $\mathfrak{p} \mid p$. Then

$$v_{\mathfrak{p}}(c_0 b^p) = 0.$$

On the other hand, Lemmas 3 and 6 give

$$v_{\mathfrak{p}}(p!a^p) \geq v_{\mathfrak{p}}(p!) = v_{\mathfrak{p}}(p)v_p(p!) = v_{\mathfrak{p}}(p) > 0.$$

Hence the coefficient $c_p = \frac{c_0 b^p}{p!a^p}$ does not lie in D, so no nonzero solution exists.

Now assume that only finitely many rational primes are nonunits in D, and define $\mathfrak{r}_1$ as above.

Necessity.

We show that every solution in $D[[x]]$ must satisfy

$$(b) \subseteq (a)\mathfrak{r}_1.$$

Let $y \in D[[x]]$. By Lemma 2, the coefficient formula lies in $D[[x]]$ if and only if

$$v_{\mathfrak{p}}(k!a^k) \leq v_{\mathfrak{p}}(c_0 b^k)$$

for every nonzero prime ideal $\mathfrak{p}$ of D and every $k \geq 1$.

Fix $\mathfrak{p}$. If $\mathfrak{p} \mid p$ for a rational prime p, then

$$v_{\mathfrak{p}}(p)v_p(k!) + kv_{\mathfrak{p}}(a) \leq v_{\mathfrak{p}}(c_0) + kv_{\mathfrak{p}}(b).$$

Using the lower bound from Lemma 6,

$$v_{\mathfrak{p}}(p)\left(\frac{k}{p-1} - \frac{p}{p-1} - \log_p k\right) + kv_{\mathfrak{p}}(a) \leq v_{\mathfrak{p}}(c_0) + kv_{\mathfrak{p}}(b).$$

After dividing by k and letting $k \to \infty$, we obtain

$$v_{\mathfrak{p}}(a) + \frac{v_{\mathfrak{p}}(p)}{p-1} \leq v_{\mathfrak{p}}(b),$$

hence

$$v_{\mathfrak{p}}(a) + \left\lceil \frac{v_{\mathfrak{p}}(p)}{p-1} \right\rceil \leq v_{\mathfrak{p}}(b).$$

If $\mathfrak{p} \cap \mathbb{Z} = (0)$, then Lemma 3 gives $v_{\mathfrak{p}}(k!) = 0$, so

$$kv_{\mathfrak{p}}(a) \leq v_{\mathfrak{p}}(c_0) + kv_{\mathfrak{p}}(b),$$

and therefore $v_{\mathfrak{p}}(a) \leq v_{\mathfrak{p}}(b)$. Since $v_{\mathfrak{p}}(\mathfrak{r}_1) = 0$ in this case, this is exactly

$$v_{\mathfrak{p}}((a)\mathfrak{r}_1) \leq v_{\mathfrak{p}}(b).$$

Combining both cases, we get

$$v_{\mathfrak{p}}((a)\mathfrak{r}_1) \leq v_{\mathfrak{p}}(b)$$

for every nonzero prime ideal $\mathfrak{p}$. Hence

$$(b) \subseteq (a)\mathfrak{r}_1.$$

Sufficiency.

Conversely, suppose

$$(b) \subseteq (a)\mathfrak{r}_1.$$

Equivalently,

$$v_{\mathfrak{p}}(b) \geq v_{\mathfrak{p}}((a)\mathfrak{r}_1)$$

for every nonzero prime ideal $\mathfrak{p}$.

If $\mathfrak{p} \mid p$, then

$$v_{\mathfrak{p}}(b) \geq v_{\mathfrak{p}}(a) + \left\lceil \frac{v_{\mathfrak{p}}(p)}{p-1} \right\rceil.$$

By Lemma 6,

$$v_{\mathfrak{p}}(k!) = v_{\mathfrak{p}}(p)v_p(k!) \leq \frac{v_{\mathfrak{p}}(p)k}{p-1} \leq k \left\lceil \frac{v_{\mathfrak{p}}(p)}{p-1} \right\rceil.$$

Thus

$$v_{\mathfrak{p}}(c_0 b^k) = v_{\mathfrak{p}}(c_0) + k v_{\mathfrak{p}}(b) \geq v_{\mathfrak{p}}(k! a^k).$$

If $\mathfrak{p} \cap \mathbb{Z} = (0)$, then $v_{\mathfrak{p}}(k!) = 0$, so

$$v_{\mathfrak{p}}(c_0 b^k) = v_{\mathfrak{p}}(c_0) + k v_{\mathfrak{p}}(b) \geq k v_{\mathfrak{p}}(a) = v_{\mathfrak{p}}(k! a^k).$$

Hence every coefficient $\frac{c_0 b^k}{k! a^k}$ lies in D, so $y \in D[[x]]$. This proves the linear case.

5.2. Case $m \geq 2$.

We again start from the explicit coefficients and reduce the problem to integrality in D. Once that reduction is in place, the proof splits according to the only three possible prime-ideal types: prime ideals over rational primes not dividing d, prime ideals over rational primes dividing d, and prime ideals whose contraction to $\mathbb{Z}$ is (0).

Zero initial value.

Write $d = m - 1$, and let $\mathfrak{r}(d)$ be the correction ideal from the theorem. If $c_0 = 0$, then $y = 0$ is a solution. By Section 2, the solution in $K[[x]]$ with initial value 0 is unique. Hence $y = 0$ is also the unique solution in $D[[x]]$.

So assume from now on that $c_0 \neq 0$.

Reduction to coefficient integrality.

By Section 2, the unique solution in $K[[x]]$ has coefficients

$$c_k = \frac{C_k(d) c_0 (b c_0^d)^k}{k! a^k}.$$

Thus, by Lemma 2, $y \in D[[x]]$ if and only if for every nonzero prime ideal $\mathfrak{p}$ of D and every $k \geq 1$,

$$v_{\mathfrak{p}}(k! a^k) \leq v_{\mathfrak{p}}\left(C_k(d) c_0 (b c_0^d)^k\right). \tag{1}$$

We now analyze (1) by prime type. Type I and Type III lead only to the basic inequality $v_{\mathfrak{p}}(a) \le v_{\mathfrak{p}}(bc_0^d)$, while Type II is exactly where the extra factor $\mathfrak{r}(d)$ appears.

Type I. Assume that $\mathfrak{p} \mid p$ for a rational prime $p \nmid d$. By Lemma 5,

$$v_{\mathfrak{p}}(k!) \le v_{\mathfrak{p}}(C_k(d)) \le v_{\mathfrak{p}}(k!) + v_{\mathfrak{p}}(p)(\log_p k + \log_p d).$$

Necessity. If (1) holds for all k, then

$$\begin{aligned} v_{\mathfrak{p}}(k!) + kv_{\mathfrak{p}}(a) &\le v_{\mathfrak{p}}(C_k(d)) + v_{\mathfrak{p}}(c_0) + kv_{\mathfrak{p}}(bc_0^d) \\ &\le v_{\mathfrak{p}}(k!) + v_{\mathfrak{p}}(p)(\log_p k + \log_p d) \\ &\quad + v_{\mathfrak{p}}(c_0) + kv_{\mathfrak{p}}(bc_0^d). \end{aligned}$$

Dividing by k and letting $k \to \infty$, we obtain

$$v_{\mathfrak{p}}(a) \le v_{\mathfrak{p}}(bc_0^d).$$

Sufficiency. Conversely, if $v_{\mathfrak{p}}(a) \le v_{\mathfrak{p}}(bc_0^d)$, then Lemma 5 gives

$$\begin{aligned} v_{\mathfrak{p}}(k!a^k) &= v_{\mathfrak{p}}(k!) + kv_{\mathfrak{p}}(a) \\ &\le v_{\mathfrak{p}}(C_k(d)) + kv_{\mathfrak{p}}(bc_0^d) \\ &\le v_{\mathfrak{p}}\left(C_k(d)c_0(bc_0^d)^k\right), \end{aligned}$$

because $v_{\mathfrak{p}}(c_0) \ge 0$. So Type I prime ideals impose no extra correction factor.

Type II. Assume that $\mathfrak{p} \mid p$ for a rational prime $p \mid d$. By Lemma 5,

$$v_{\mathfrak{p}}(C_k(d)) = 0,$$

so (1) becomes

$$v_{\mathfrak{p}}(k!) + kv_{\mathfrak{p}}(a) \le v_{\mathfrak{p}}(c_0) + kv_{\mathfrak{p}}(bc_0^d).$$

Necessity. Since

$$v_{\mathfrak{p}}(k!) = v_{\mathfrak{p}}(p)v_p(k!),$$

Lemma 6 yields

$$v_{\mathfrak{p}}(p)\left(\frac{k}{p-1} - \frac{p}{p-1} - \log_p k\right) + kv_{\mathfrak{p}}(a) \le v_{\mathfrak{p}}(c_0) + kv_{\mathfrak{p}}(bc_0^d).$$

After dividing by k and letting $k \to \infty$, we get

$$v_{\mathfrak{p}}(a) + \frac{v_{\mathfrak{p}}(p)}{p-1} \le v_{\mathfrak{p}}(bc_0^d),$$

hence

$$v_{\mathfrak{p}}(a) + \left\lceil \frac{v_{\mathfrak{p}}(p)}{p-1} \right\rceil \le v_{\mathfrak{p}}(bc_0^d).$$

Sufficiency. Conversely, if this ceiling inequality holds, then Lemma 6 gives

$$\begin{aligned} v_{\mathfrak{p}}(k!) + kv_{\mathfrak{p}}(a) &\le \frac{v_{\mathfrak{p}}(p)k}{p-1} + kv_{\mathfrak{p}}(a) \\ &\le k\left\lceil \frac{v_{\mathfrak{p}}(p)}{p-1} \right\rceil + kv_{\mathfrak{p}}(a) \\ &\le kv_{\mathfrak{p}}(bc_0^d) \\ &\le v_{\mathfrak{p}}(c_0) + kv_{\mathfrak{p}}(bc_0^d), \end{aligned}$$

so (1) follows. Thus Type II prime ideals are exactly the source of $\mathfrak{r}(d)$.
Type III. Assume that $\mathfrak{p} \cap \mathbb{Z} = (0)$. Lemma 5 gives

$$v_{\mathfrak{p}}(k!) = v_{\mathfrak{p}}(C_k(d)) = 0,$$

so (1) is simply

$$kv_{\mathfrak{p}}(a) \le v_{\mathfrak{p}}(c_0) + kv_{\mathfrak{p}}(bc_0^d).$$

Letting $k \to \infty$, we obtain

$$v_{\mathfrak{p}}(a) \le v_{\mathfrak{p}}(bc_0^d),$$

and this inequality is also sufficient. So Type III prime ideals again impose only the basic containment.

Combining the three prime types, we conclude that $y \in D[[x]]$ if and only if

$$v_{\mathfrak{p}}((a)\mathfrak{r}(d)) \le v_{\mathfrak{p}}(bc_0^d)$$

for every nonzero prime ideal $\mathfrak{p}$, which is equivalent to

$$(bc_0^d) \subseteq (a)\mathfrak{r}(d).$$

This proves the nonlinear case and therefore the theorem.

6. Corollaries

We now record three immediate consequences of the main theorem.

Corollary 1. *Let $m \ge 2$, and write $d = m - 1$. The equation*

$$y' = y^m$$

admits a solution in $D[[x]]$ for every initial value $y(0) = c_0 \in D$ if and only if $\mathfrak{r}(d) = D$. In particular, the equation $y' = y^2$ has a solution in $D[[x]]$ for every initial value $y(0) = c_0 \in D$.

Proof. Here $a = b = 1$, so the nonlinear part of the theorem says that a solution exists exactly when

$$(c_0^d) \subseteq \mathfrak{r}(d).$$

If this holds for every $c_0 \in D$, then taking $c_0 = 1$ gives $D \subseteq \mathfrak{r}(d)$. Since $\mathfrak{r}(d) \subseteq D$, we obtain $\mathfrak{r}(d) = D$. The converse is immediate. When $m = 2$, we have $d = 1$ and by convention $\mathfrak{r}(1) = D$. □

Corollary 2. *Over $D = \mathbb{Z}$, the equation*

$$y' = y^m$$

has a solution in $\mathbb{Z}[[x]]$ for every initial value $y(0) = c_0 \in \mathbb{Z}$ if and only if $m = 2$.

Proof. If $m = 1$, then the linear part of the theorem applies with $a = b = 1$, and since infinitely many rational primes are nonunits in $\mathbb{Z}$, only the initial value $c_0 = 0$ is possible.

Now assume $m \geq 2$. The preceding corollary shows that existence for every $c_0 \in \mathbb{Z}$ is equivalent to $\mathfrak{r}(m-1) = \mathbb{Z}$. Over $\mathbb{Z}$, one has

$$\mathfrak{r}(m-1) = \prod_{p \mid (m-1)} (p),$$

so $\mathfrak{r}(m-1) = \mathbb{Z}$ if and only if $m - 1 = 1$, that is, $m = 2$. □

Corollary 3. *Let $m \geq 2$. Then the equation*

$$y' = y^m$$

has a nonzero solution in $D[[x]]$.

Proof. If $m = 2$, choose any nonzero $c_0 \in D$; then $(c_0) \subseteq D = \mathfrak{r}(1)$.

Now assume $m \geq 3$, and write $d = m - 1$. The ideal $\mathfrak{r}(d)$ is nonzero, so it contains a nonzero element c_0. Then $(c_0) \subseteq \mathfrak{r}(d)$, hence

$$(c_0^d) \subseteq \mathfrak{r}(d)^d \subseteq \mathfrak{r}(d).$$

The nonlinear part of the theorem therefore gives a solution in $D[[x]]$ with nonzero constant term c_0. □

Corollary 4. *Let $m = 1$. Then the equation*

$$ay' = by$$

either has a solution in $D[[x]]$ for every initial value $c_0 \in D$, or has a solution only for $c_0 = 0$.

Proof. By the linear part of the theorem, $c_0 = 0$ always gives the zero solution. For $c_0 \neq 0$, existence is characterized by a condition depending only on a and b: either infinitely many rational primes are nonunits in D, in which case no nonzero initial value is possible, or only finitely many are, in which case existence is equivalent to $(b) \subseteq (a)\mathfrak{r}_1$, which again is independent of c_0. □

7. Examples

The examples show how the same coefficient formulas behave as the arithmetic of the coefficient ring changes. In each example the theorem applies directly because the rings under consideration are Dedekind domains; when they are principal, we translate the ideal criterion into ordinary divisibility.

7.1. Linear case $m = 1$.

If $D = \mathbb{Z}$, then infinitely many rational primes are nonunits, so the equation $ay' = by$ has a solution in $\mathbb{Z}[[x]]$ only for $c_0 = 0$.

If $D = \mathbb{Z}_{(2)} = \left\{\frac{u}{v} \in \mathbb{Q} : u, v \in \mathbb{Z},\ 2 \nmid v\right\}$, the localization of $\mathbb{Z}$ at (2), then the only rational prime that is not a unit is 2, and $\mathfrak{r}_1 = (2)$. Since $\mathbb{Z}_{(2)}$ is a principal ideal domain, the criterion becomes

$$2a \mid b.$$

For example, if $a = 1$, $b = 2$, and $c_0 = 1$, then

$$y(x) = \sum_{k=0}^{\infty} \frac{2^k}{k!} x^k \in \mathbb{Z}_{(2)}[[x]].$$

This series solves the Cauchy problem

$$y' = 2y, \qquad y(0) = 1$$

in $\mathbb{Z}_{(2)}[[x]]$.

7.2. Quadratic case $m = 2$.

In the quadratic case, complete cancellation $C_k(1) = k!$ reduces the theorem to the simple condition

$$(bc_0) \subseteq (a),$$

or equivalently $a \mid bc_0$ in a principal ideal domain.

For instance, if $D = \mathbb{Z}$, $a = 2$, $b = 3$, and $c_0 = 4$, then $2 \mid 12$, so the Cauchy problem

$$2y' = 3y^2, \qquad y(0) = 4,$$

has a solution in $\mathbb{Z}[[x]]$, namely

$$y(x) = 4 \sum_{k=0}^{\infty} 6^k x^k \in \mathbb{Z}[[x]].$$

By contrast, if $a = 4$, $b = 1$, and $c_0 = 2$, then $4 \nmid 2$, so the Cauchy problem

$$4y' = y^2, \qquad y(0) = 2,$$

has no solution in $\mathbb{Z}[[x]]$, even though the unique solution in $\mathbb{Q}[[x]]$ has coefficients

$$c_k = 2^{1-k}.$$

7.3. Higher-Order case $m \geq 3$.

To keep the higher-order discussion concrete, we now fix $m = 3$, so $d = m-1 = 2$. This is the first nonlinear case in which the correction ideal is nontrivial, and it isolates the role of the rational prime 2.

Over $\mathbb{Z}$.

Here $\mathfrak{r}(2) = (2)$, so the criterion becomes $2a \mid bc_0^2$. For the Cauchy problem

$$y' = y^3, \qquad y(0) = 1,$$

this condition fails because $2 \nmid 1$. Hence no solution exists in $\mathbb{Z}[[x]]$. The unique solution in $\mathbb{Q}[[x]]$ is

$$y(x) = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{x^k}{2^k},$$

so the factor 2^k in the denominator is exactly the obstruction.

Over $\mathbb{Z}[i]$.

Here

$$\mathbb{Z}[i] = \{u + vi : u, v \in \mathbb{Z}\}$$

is the ring of Gaussian integers. It is a principal ideal domain, hence a Dedekind domain. Since

$$(2) = (1+i)^2.$$

the case $d = 2$ again gives the principal correction ideal $\mathfrak{r}(2) = (2)$, so the criterion becomes $2a \mid bc_0^2$. Take

$$a = 1 + i, \qquad b = 1 - i, \qquad c_0 = 1 + i.$$

Then

$$bc_0^2 = (1-i)(1+i)^2 = 2(1+i) = 2a,$$

so the Cauchy problem

$$(1+i)y' = (1-i)y^3, \qquad y(0) = 1 + i,$$

has a solution in $\mathbb{Z}[i][[x]]$, namely

$$y(x) = (1+i)\sum_{k=0}^{\infty} \binom{2k}{k} x^k.$$

Over $\mathbb{Z}\left[\frac{1}{2}\right]$. Here

$$\mathbb{Z}\left[\frac{1}{2}\right] = \left\{\frac{a}{2^n} : a \in \mathbb{Z},\ n \in \mathbb{Z}_{\geq 0}\right\}.$$

This ring is a principal ideal domain, and 2 is a unit in it, so $\mathfrak{r}(2) = D$. Therefore the same Cauchy problem that failed over $\mathbb{Z}$,

$$y' = y^3, \qquad y(0) = 1,$$

does have a solution in $\mathbb{Z}\left[\frac{1}{2}\right][[x]]$, namely

$$y(x) = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{x^k}{2^k} \in \mathbb{Z}\left[\frac{1}{2}\right][[x]].$$

Together these three exact examples show that in the higher-order case the same prime 2 may block descent, permit it, or disappear entirely, depending on the arithmetic of the coefficient ring.

7.4. A non-UFD example: $\mathbb{Z}[\sqrt{-5}]$.

Let

$$D = \mathbb{Z}[\sqrt{-5}] = \{u + v\sqrt{-5} : u, v \in \mathbb{Z}\},$$

the ring of integers of $\mathbb{Q}(\sqrt{-5})$; see [7, Cor. 2 to Thm. 1]. Hence D is a Dedekind domain by [7, Thm. 14, Ch. 3], but it is not a unique factorization domain. Indeed, the classical nonunique factorization

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

already shows that factorization of elements behaves differently here; see [7, Ch. 1, Ex. 15].

Take $m = 2$, $a = 1 + \sqrt{-5}$, $b = 1 - \sqrt{-5}$, and $c_0 = 1 + \sqrt{-5}$. Since $\mathfrak{r}(1) = D$, the theorem says that the equation

$$(1 + \sqrt{-5})y' = (1 - \sqrt{-5})y^2, \qquad y(0) = 1 + \sqrt{-5},$$

has a solution in $D[[x]]$ if and only if

$$(bc_0) \subseteq (a).$$

Here

$$bc_0 = (1 - \sqrt{-5})(1 + \sqrt{-5}) = 6,$$

so

$$(bc_0) = (6) \subseteq (1 + \sqrt{-5}) = (a).$$

Hence a solution exists in $D[[x]]$. In fact,

$$y(x) = (1 + \sqrt{-5}) \sum_{k=0}^{\infty} (1 - \sqrt{-5})^k x^k \in D[[x]]$$

solves the displayed Cauchy problem.

Thus the theorem applies in the non-UFD Dedekind domain $D = \mathbb{Z}[\sqrt{-5}]$. The chosen data are built from the classical factorization

$$6 = (1 + \sqrt{-5})(1 - \sqrt{-5}),$$

while the existence criterion is still expressed in terms of ideals.

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Формальні степеневі ряди для диференціального рівняння $ay' = by^m$ над областями Дедекінда нульової характеристики

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У роботі досліджено задачу Коші $ay' = by^m$, $y(0) = c_0$, у кільці формальних степеневих рядів $D[[x]]$, де D — область Дедекінда нульової характеристики, $K = \mathrm{Frac}(D)$ — її поле часток, $a, b, c_0 \in D$, $a, b \neq 0$, $m \in \mathbb{N}$. Встановлюються точні умови, за яких єдиний формальний розв'язок $y \in K[[x]]$ із початковим значенням c_0 насправді має всі коефіцієнти в D, тобто належить $D[[x]]$. Для цього спершу в $K[[x]]$ виводиться рекурентне співвідношення для коефіцієнтів розв'язку і одержується явна формула для них. Це зводить задачу існування розв'язку в $D[[x]]$ до арифметичної задачі про цілість коефіцієнтів, яку природно формулювати мовою нормувань, пов'язаних з ненульовими простими ідеалами області D. Основна ідея полягає в тому, що для областей Дедекінда перешкода спуску з $K[[x]]$ до $D[[x]]$ визначається не звичайною подільністю на прості елементи, а порівнянням нормувань дробових ідеалів після локалізації за простими ідеалами.

У лінійному випадку $m = 1$ нульове початкове значення завжди дає нульовий розв'язок. Для $c_0 \neq 0$ у формулах для коефіцієнтів з'являються факторіальні знаменники, що створює глобальну перешкоду, якщо в D не є оборотними нескінченно багато раціональних простих чисел. Якщо ж таких простих лише скінченна кількість, то точний критерій існування для ненульового початкового значення має вигляд $(b) \subseteq (a)\mathfrak{r}_1$, де $\mathfrak{r}_1$ — явно побудований коригувальний ідеал, що залежить від простих ідеалів над відповідними раціональними простими. У нелінійному випадку $m \geq 2$, поклавши $d = m - 1$, дістаємо вираз для коефіцієнтів через добуток $C_k(d) = \prod_{i=0}^{k-1}(di + 1)$. Доведено, що прості ідеали над раціональними простими, які не ділять d, не створюють додаткових перешкод, крім базової умови $(bc_0^d) \subseteq (a)$, тоді як прості ідеали над простими, що ділять d, задають коригувальний ідеал $\mathfrak{r}(d)$. У результаті точний критерій існування набуває вигляду $(bc_0^d) \subseteq (a)\mathfrak{r}(d)$. Зокрема, для $m = 2$ маємо $\mathfrak{r}(1) = D$, і умова спрощується до $(bc_0) \subseteq (a)$. Наведено приклади для кілець $\mathbb{Z}$, локалізацій кільця $\mathbb{Z}$, кільця цілих гаусових чисел та дедекіндової області $\mathbb{Z}[\sqrt{-5}]$, яка не є факторіальним кільцем, що демонструють роботу одержаного критерію в різних арифметичних ситуаціях.

Ключові слова: **формальні степеневі ряди; область Дедекінда; задача Коші; нормування за простими ідеалами; критерій цілості коефіцієнтів.**

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