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The Jacobi operator and the stability of vertical minimal surfaces in the sub-Riemannian Lie group $\widetilde{\mathrm{SL}(2,\mathbb{R})}$

We consider oriented immersed minimal surfaces in three-dimensional sub-Riemannian manifolds which are vertical, i.e., perpendicular to the two-dimensional horizontal distribution of the sub-Riemannian structure. We showed earlier that a vertical surface is minimal in the sub-Riemannian sense if and only if it is minimal in the Riemannian sense and that its sub-Riemannian stability implies its Riemannian stability. We introduce the sub-Riemannian version of the Jacobi operator for such surfaces and prove a sufficient condition for the stability of vertical minimal surfaces similar to a theorem of Fischer-Colbrie and Schoen: if a surface allows a positive function with the vanishing Jacobi operator then it is stable.

Next, we use the Jacobi operator technique to investigate vertical minimal surfaces in the Lie group $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ that can be described as the universal covering of the unit tangent bundle of the hyperbolic plane with the standard left-invariant Sasaki metric (that corresponds to one of the Thurston geometries) and with two different types of sub-Riemannian structures. First, we consider a family of non-left-invariant structures defined by some parameters, find the values of parameters for which vertical minimal surfaces exist, and describe such complete connected surfaces. These are Euclidean half-planes and cylinders, and they all are stable in the sub-Riemannian sense and thus in the Riemannian sense. In particular, this gives us examples of structures that do not allow vertical minimal surfaces. Then, we describe complete connected vertical minimal surfaces for another sub-Riemannian structure that is left-invariant. These are half-planes and helicoidal surfaces that also appear to be stable in the sub-Riemannian sense and thus in the Riemannian sense.

Keywords: **sub-Riemannian manifold; left-invariant metric; minimal surface; Jacobi operator; stability.**

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1. Introduction

The Jacobi operator of a minimal submanifold in a Riemannian manifold (see, e.g., [1]) is a very useful tool that allows one to determine whether this submanifold is stable. In particular, a well-known theorem by Fischer-Colbrie and Schoen ([2]) states that a complete non-compact minimal hypersurface is stable if and only if there exists a positive function for which the Jacobi operator vanishes. It also can be of use in sub-Riemannian geometry. For example, in [6] the Jacobi operator of a minimal surface in the three-dimensional sub-Riemannian Heisenberg group plays a crucial role in the proof of a Bernstein-like theorem. Note that, contrary to the Riemannian case, for submanifolds in sub-Riemannian manifolds the first and second sub-Riemannian volume variation formulas are not universal: they depend on a sub-Riemannian structure and can be quite complicated. See [6] and [4] for the examples of the Heisenberg group and the universal covering $\widetilde{\mathrm{E}(2)}$ of the group of orientation-preserving Euclidean plane isometries respectively. Hence, the Jacobi operators also depend on such structure.

Taking this into account, in [4] we started to look into so-called vertical minimal surfaces in three-dimensional sub-Riemannian manifolds, a relatively simple, but interesting class of surfaces. In [5] we found the first and second sub-Riemannian area variation formulas for such surfaces (Proposition 1 in the next section) showing that they can be written down in a way independent of a sub-Riemannian structure. That allowed us to consider various classes of sub-Riemannian manifolds and establish the stability of vertical minimal surfaces in them. Here we continue this work. First of all, we derive the Jacobi operator for a vertical minimal surface in any three-dimensional sub-Riemannian manifold (Proposition 2) and prove a sufficient condition for stability similar to the one of Fischer-Colbrie and Schoen: if a surface allows a positive function with the vanishing Jacobi operator then it is stable (Theorem 1). Then we apply it to the study of vertical minimal surfaces in the Lie group $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ with two different classes of sub-Riemannian structures (Theorems 2 and 3) obtaining some new classes of such stable surfaces (note that they are also minimal and stable in the Riemannian sense) and finding examples of structures that do not allow vertical minimal surfaces.

2. Preliminaries and the Jacobi operator

A *sub-Riemannian manifold* is a smooth manifold M together with a completely non-integrable smooth distribution $\mathcal{H}$ on M (a *horizontal distribution*) and a smooth field of Euclidean scalar products $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ on $\mathcal{H}$ (a *sub-Riemannian metric*). In particular, $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ can be the restriction of some Riemannian metric $\langle\cdot,\cdot\rangle$ on M to $\mathcal{H}$. Here we will assume that all sub-Riemannian structures are of this form. We will call a sub-Riemannian structure on a Lie group M *left-invariant* if both $\mathcal{H}$ and $\langle\cdot,\cdot\rangle$ are left-invariant.

Let Σ be an oriented immersed surface (without boundary) in a three-dimensional sub-Riemannian manifold M with a two-dimensional horizontal distribution. If N_h is the orthogonal projection of the Riemannian unit normal field N of Σ onto $\mathcal{H}$ and $d\Sigma$ is the Riemannian area form of Σ, then the *sub-Riemannian area* of a domain $D \subset \Sigma$ (see, e.g., [3]) is defined as

$$A(D) = \int_D |N_h| \, d\Sigma.$$

The *normal variation* of the surface Σ defined by a smooth function u with compact support is the map

$$\varphi : \Sigma \times I \to M : \varphi_s(p) = exp_p(su(p)N(p)),$$

where I is an open neighborhood of 0 in $\mathbb{R}$ and exp_p is the Riemannian exponential map at p. Therefore, we construct the variation by drawing the Riemannian geodesic through each point $p \in \Sigma$ in the direction of $u(p)N(p)$. Denote

$$A(s) = \int_{\Sigma_s} |N_h| \, d\Sigma_s,$$

where $\Sigma_s = \varphi_s(\Sigma)$. Then $A'(0)$ is called the first *(normal) sub-Riemannian area variation* defined by φ, and $A''(0)$ is called the *second* one. A surface Σ is called *minimal* if $A'(0) = 0$ for any normal variations with compact support in $\Sigma \setminus \Sigma_0$, where $\Sigma_0 = \{p \in \Sigma \mid N_h(p) = 0\}$ is the *singular set* of Σ. A minimal surface Σ is called *stable* if $A''(0) \geqslant 0$ for any normal variations with compact support in $\Sigma \setminus \Sigma_0$.

We will call a surface Σ in a three-dimensional sub-Riemannian manifold *vertical* if $T_p\Sigma$ is perpendicular to $\mathcal{H}_p$ for each $p \in \Sigma$, i.e., the normal vectors of these planes are orthogonal. In particular, for such surfaces $N_h = N$ and $\Sigma_0 = \varnothing$. In [5] we proved the following.

Proposition 1. *A vertical surface Σ in a three-dimensional sub-Riemannian manifold is minimal in the sub-Riemannian sense if and only if it is minimal in the Riemannian sense. In this case its second sub-Riemannian area variation is*

$$A''(0) = \int_\Sigma -\left(X(u) - \langle \nabla_N X, N\rangle u\right)^2 + |\nabla_\Sigma u|^2 - \left(\mathrm{Ric}\,(N,N) + |B|^2\right) u^2 \, d\Sigma,$$

where u is a smooth function with compact support that defines the normal variation, ∇ and Ric *are the Riemannian connection and the Ricci tensor of M respectively, X is the unit normal vector field of $\mathcal{H}$ (which is tangent to Σ because it is vertical), ∇_Σ and B are the Riemannian gradient and the second fundamental form of Σ respectively. It follows that if Σ is stable in the sub-Riemannian sense then it is also stable in the Riemannian sense.*

Define the *characteristic vector field* Z on Σ as the right angle rotation of N in $\mathcal{H}$ (in the orientation defined by X). Then $\{X, Z\}$ is an orthonormal frame on Σ, so $|\nabla_\Sigma u|^2 = X(u)^2 + Z(u)^2$ and the second variation formula takes the form

$$\begin{aligned} A''(0) = \int\limits_\Sigma & Z(u)^2 + 2\langle \nabla_N X, N\rangle u X(u) - \\ & - \left(\langle \nabla_N X, N\rangle^2 + \mathrm{Ric}\,(N,N) + |B|^2\right) u^2 \, d\Sigma, \end{aligned} \tag{1}$$

Note that the divergence of the field $\langle \nabla_N X, N\rangle u^2 X$ on Σ has the vanishing integral by the Stokes' theorem because u is with compact support. On the other hand, this divergence equals

$$\begin{aligned} \mathrm{div}_\Sigma \left(\langle \nabla_N X, N\rangle u^2 X\right) &= 2\langle \nabla_N X, N\rangle u X(u) + \\ &+ \left(X\left(\langle \nabla_N X, N\rangle\right) + \langle \nabla_N X, N\rangle \,\mathrm{div}_\Sigma X\right) u^2, \end{aligned}$$

where $\mathrm{div}_\Sigma X = \langle \nabla_X X, X\rangle + \langle \nabla_Z X, Z\rangle = \langle \nabla_Z X, Z\rangle$ due to the orthonormality of $\{X, Z\}$. It means that (1) can be rewritten as

$$A''(0) = \int\limits_\Sigma Z(u)^2 - f\, u^2 \, d\Sigma \tag{2}$$

for some function f.

Proposition 2. *Let Σ be a minimal surface in a three-dimensional sub-Riemannian manifold whose second variation is of the form (2). Then it also has a form*

$$A''(0) = -\int\limits_\Sigma u L(u)\, d\Sigma \tag{3}$$

where L is the Jacobi operator on the space of smooth functions on Σ:

$$L(u) = Z(Z(u)) + \langle \nabla_X Z, X\rangle\, Z(u) + f\, u. \tag{4}$$

Proof. Note that, similarly to $\mathrm{div}_\Sigma X$ above, $\mathrm{div}_\Sigma Z = \langle \nabla_X Z, X\rangle + \langle \nabla_Z Z, Z\rangle = \langle \nabla_X Z, X\rangle$, so in (4)

$$L(u) = Z(Z(u)) + Z(u)\,\mathrm{div}_\Sigma Z + f\, u = \mathrm{div}_\Sigma (Z(u) Z) + f\, u.$$

From this, as u has compact support,

$$\begin{aligned} 0 = \int\limits_\Sigma \mathrm{div}_\Sigma (u Z(u) Z)\, d\Sigma &= \int\limits_\Sigma Z(u)^2 + u\,\mathrm{div}_\Sigma (Z(u) Z)\, d\Sigma = \\ &= \int\limits_\Sigma Z(u)^2 + u(L(u) - f\, u)\, d\Sigma, \end{aligned}$$

and that implies (3).

In particular, this Jacobi operator indeed is a linear operator on $C^\infty(M)$ as in the Riemannian case. Now we will show that an analogue of the sufficiency part in the Fischer-Colbrie–Schoen theorem ([2]) is true for this operator.

Theorem 1. *Let Σ be a minimal surface in a three-dimensional sub-Riemannian manifold with the second variation of the form (2) and the Jacobi operator L from (4). If there exists a smooth function $u > 0$ on Σ such that $L(u) = 0$ then Σ is stable.*

Proof. As $u > 0$, we can define $v = \ln u$ on Σ with derivatives

$$Z(v) = \frac{Z(u)}{u}, \; Z(Z(v)) = \frac{Z(Z(u))}{u} - \frac{Z(u)^2}{u^2}.$$

This, (4), and $L(u) = 0$ imply that

$$\operatorname{div}_\Sigma(Z(v)Z) = Z(Z(v)) + \langle \nabla_X Z, X\rangle\, Z(v) = -Z(v)^2 - f. \tag{5}$$

For any smooth function w on Σ with compact support

$$\operatorname{div}_\Sigma\left(w^2 Z(v)Z\right) = \operatorname{div}_\Sigma(Z(v)Z)\, w^2 + 2Z(v)Z(w)w$$

The integral of this divergence on Σ vanishes, thus by (5) and the Cauchy–Schwarz inequality we have

$$\int\limits_\Sigma \left(f + Z(v)^2\right) w^2\, d\Sigma = -\int\limits_\Sigma \operatorname{div}_\Sigma(Z(v)Z)\, w^2\, d\Sigma =$$
$$= \int\limits_\Sigma 2Z(v)Z(w)w\, d\Sigma \leqslant \int\limits_\Sigma Z(v)^2 w^2 + Z(w)^2\, d\Sigma,$$

hence for the variation defined by w the second variation (2) is non-negative:

$$A''(0) = \int\limits_\Sigma Z(w)^2 - f\, w^2\, d\Sigma \geqslant 0,$$

and this means the stability of Σ.

Note that the statement also stays true for $u > 0$ with $L(u) \leqslant 0$ with almost the same proof. It is interesting whether the necessity (hard) part of the Fischer-Colbrie–Schoen theorem is also true for complete non-compact Σ, that is, whether the stability implies the existence of $u > 0$ with $L(u) = 0$. Here and in the next session by the completeness of a surface we mean the Riemannian completeness.

3. Vertical minimal surfaces in $\widetilde{\mathrm{SL}(2,\mathbb{R})}$

The three-dimensional Thurston geometry $\widetilde{SL(2,\mathbb{R})}$ can be described (see [8]) as the universal covering of the unit tangent bundle of the hyperbolic plane $\mathbb{H}^2$ with the Sasaki metric, that is, the half-space $\{(x, y, z) \in \mathbb{R}^3 \mid y > 0\}$ with the following orthonormal frame of vector fields:

$$\begin{gathered} X_1 = y\cos z\frac{\partial}{\partial x} + y\sin z\frac{\partial}{\partial y} - \cos z\frac{\partial}{\partial z}, \\ X_2 = -y\sin z\frac{\partial}{\partial x} + y\cos z\frac{\partial}{\partial y} + \sin z\frac{\partial}{\partial z}, X_3 = \frac{\partial}{\partial z}. \end{gathered} \tag{6}$$

This manifold also is a Lie group (the universal covering of the special linear group $\mathrm{SL}(2,\mathbb{R})$) and the fields $\{X_1, X_2, X_3\}$ are left-invariant. The corresponding left-invariant metric is $\frac{1}{y^2}\left(dx^2+dy^2+(dx+y\,dz)^2\right)$, so we can consider a simpler orthonormal frame of

$$\begin{gathered} Y_1 = y\frac{\partial}{\partial x} - \frac{\partial}{\partial z} = \cos z\, X_1 - \sin z\, X_2, \\ Y_2 = y\frac{\partial}{\partial y} = \sin z\, X_1 + \cos z\, X_2,\ Y_3 = \frac{\partial}{\partial z} = X_3, \end{gathered} \tag{7}$$

where the fields Y_1 and Y_2 are not left-invariant. The non-zero Lie brackets of the fields (6) are

$$\begin{gathered} [X_1, X_2] = -[X_2, X_1] = -X_3,\ [X_2, X_3] = -[X_3, X_2] = X_1, \\ [X_3, X_1] = -[X_1, X_3] = X_2, \end{gathered}$$

and the only non-zero Lie bracket of (7) is $[Y_1, Y_2] = -Y_1 - Y_3$. Using the Koszul formula as in [7], we derive from this that the Riemannian connection ∇ of the left-invariant metric is defined by

$$\begin{gathered} \nabla_{X_1}X_2 = -\nabla_{X_2}X_1 = -\frac{X_3}{2},\ \nabla_{X_2}X_3 = -\frac{X_1}{2},\ \nabla_{X_3}X_2 = -\frac{3X_1}{2}, \\ \nabla_{X_3}X_1 = \frac{3X_2}{2},\ \nabla_{X_1}X_3 = \frac{X_2}{2},\ \nabla_{X_1}X_1 = \nabla_{X_2}X_2 = \nabla_{X_3}X_3 = 0, \end{gathered} \tag{8}$$

hence

$$\begin{gathered} \nabla_{Y_1}Y_2 = -Y_1 - \frac{Y_3}{2},\ \nabla_{Y_2}Y_1 = \frac{Y_3}{2},\ \nabla_{Y_2}Y_3 = \nabla_{Y_3}Y_2 = -\frac{Y_1}{2}, \\ \nabla_{Y_3}Y_1 = \nabla_{Y_1}Y_3 = \frac{Y_2}{2},\ \nabla_{Y_1}Y_1 = Y_2,\ \nabla_{Y_2}Y_2 = \nabla_{Y_3}Y_3 = 0. \end{gathered} \tag{9}$$

It also follows from [7] that the Ricci tensor of this metric is defined by

$$\begin{gathered} \mathrm{Ric}(X_1,X_1) = \mathrm{Ric}(X_2,X_2) = \mathrm{Ric}(Y_1,Y_1) = \mathrm{Ric}(Y_2,Y_2) = -\frac{3}{2}, \\ \mathrm{Ric}(X_3,X_3) = \mathrm{Ric}(Y_3,Y_3) = \frac{1}{2},\ \mathrm{Ric}(X_i,X_j) = \mathrm{Ric}(Y_i,Y_j) = 0,\ i \neq j. \end{gathered} \tag{10}$$

It follows from the Lie brackets above that the left-invariant distribution orthogonal to $X_3 = Y_3$ is completely non-integrable and so defines a sub-Riemannian structure such that this distribution is horizontal. For this structure complete connected vertical surfaces are well-known: they are cylinders over geodesics in $\mathbb{H}^2$ (see, e.g., [9]). We showed in [5] that these surfaces are stable in the sub-Riemannian sense and thus in the Riemannian sense. Let us consider here a more general situation.

Theorem 2. *A two-dimensional horizontal distribution $\mathcal{H} = X^{\perp}$, whose unit normal field X is a linear combination of the fields Y_1, Y_2, Y_3 with constant coefficients, defines a sub-Riemannian structure on* $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ *(i.e., is its horizontal* *distribution) if and only if X is of the form $\frac{1}{\sqrt{\lambda^2+\mu^2+1}}(\lambda Y_1+\mu Y_2+Y_3)$, where $\lambda\neq -1$. This sub-Riemannian structure allows vertical minimal surfaces only for $\lambda=0$ and $\lambda=1$.*

If $\mu\neq 0$ then a complete connected vertical surface is minimal if and only if it is a Euclidean half-plane $x=x_0$ for $\lambda=0$ or a Euclidean half-plane $z=z_0$ for $\lambda=1$.

If $\mu=0$ and $\lambda=1$ then a complete connected vertical surface is minimal if and only if it is either a Euclidean half-plane $z=z_0$ or a cylinder that can be parameterized as

$$r(s,t)=\left(s, y_0\cos t, z_0+\sqrt{2}\,t\right),\ s\in\mathbb{R},\ t\in\left(-\frac{\pi}{2}+2\pi k, \frac{\pi}{2}+2\pi k\right), \tag{11}$$

where $k\in\mathbb{Z}$.

If $\mu=\lambda=0$ then a complete connected vertical surface is minimal if and only if is a cylinder over a geodesic in $\mathbb{H}^2$.

All these surfaces are stable in the sub-Riemannian sense and thus in the Riemannian sense.

Proof. If X is of the form $\lambda Y_1+\mu Y_2$ then Y_3 belongs to its orthogonal distribution. As $[Y_1,Y_3]=[Y_2,Y_3]=0$, this distribution is integrable. So indeed it should be $X=\frac{1}{\sqrt{\lambda^2+\mu^2+1}}(\lambda Y_1+\mu Y_2+Y_3)$ for $\mathcal{H}=X^{\perp}$ to define a sub-Riemannian structure. In this case $\{Y_1-\lambda Y_3, Y_2-\mu Y_3\}$ is a frame of $\mathcal{H}$. The Lie bracket $-Y_1-Y_3$ of these fields forms with them a linearly independent triple if and only if $\lambda\neq -1$. This is the condition for $\mathcal{H}$ to be completely non-integrable, so we get the desired form of X. Substituting (7) into it yields

$$X=\frac{1}{\sqrt{\lambda^2+\mu^2+1}}\left(\lambda y\frac{\partial}{\partial x}+\mu y\frac{\partial}{\partial y}+(-\lambda+1)\frac{\partial}{\partial z}\right). \tag{12}$$

In the case $\mu\neq 0$ integral curves of this field are transversal to Euclidean planes $y=y_0$ (recall that $y>0$), so we can build any complete connected vertical surface Σ of the sub-Riemannian structure by drawing these integral curves through points of a curve $t\mapsto(x(t)+\lambda/\mu, 1, z(t))$ and obtain the following parameterization for Σ:

$$r(s,t)=\left(x(t)+\frac{\lambda}{\mu}e^{\mu s},\ e^{\mu s},\ z(t)+(-\lambda+1)s\right). \tag{13}$$

Taking derivatives, we get

$$r_s=\sqrt{\lambda^2+\mu^2+1}\,X,\ r_t=(x',\,0,\,z')=x'e^{-\mu s}Y_1+(x'e^{-\mu s}+z')Y_3.$$

From this and (9), the covariant derivatives are

$$\begin{gathered}\nabla_{r_s}r_s=(\lambda+1)(-\mu Y_1+\lambda Y_2),\\ \nabla_{r_t}r_s=-\frac{\mu}{2}(3x'e^{-\mu s}+z')Y_1+\frac{1}{2}(3\lambda x'e^{-\mu s}+\lambda z'+x'e^{-\mu s})Y_2-\frac{\mu}{2}x'e^{-\mu s}Y_3,\\ \nabla_{r_t}r_t=x''e^{-\mu s}Y_1+(2(x')^2e^{-2\mu s}+x'z'e^{-\mu s})Y_2+(x''e^{-\mu s}+z'')Y_3.\end{gathered}$$

The unit normal field of Σ is

$$N = \frac{1}{\Delta}\left((\mu x' e^{-\mu s} + \mu z')Y_1 - (\lambda x' e^{-\mu s} + \lambda z' - x' e^{-\mu s})Y_2 - \mu x' e^{-\mu s} Y_3\right),$$

where $\Delta = \sqrt{(\mu x' e^{-\mu s} + \mu z')^2 + (\lambda x' e^{-\mu s} + \lambda z' - x' e^{-\mu s})^2 + (\mu x' e^{-\mu s})^2}$. So, the coefficients of the second fundamental form of Σ are

$$\begin{aligned} b_{11} = \langle \nabla_{r_s} r_s, N\rangle &= \frac{\lambda+1}{\Delta}\left(-(\lambda^2+\mu^2)(x' e^{-\mu s} + z') + \lambda x' e^{-\mu s}\right), \\ b_{12} = \langle \nabla_{r_t} r_s, N\rangle &= \frac{1}{2\Delta}\left(-(\lambda^2+\mu^2)(3x' e^{-\mu s} + z')(x' e^{-\mu s} + z') + \right. \\ &\left. +(1+2\lambda+\mu^2)(x')^2 e^{-2\mu s}\right), \\ b_{22} = \langle \nabla_{r_t} r_t, N\rangle &= \frac{1}{\Delta}\left(-x'(2x' e^{-\mu s} + z')((\lambda-1)x' e^{-\mu s} + \lambda z')e^{-\mu s} + \right. \\ &\left. +\mu(x'' z' - x' z'')e^{-\mu s}\right). \end{aligned}$$

Taking into account the coefficients of the first fundamental form of Σ

$$g_{11} = \lambda^2 + \mu^2 + 1,\ g_{12} = (\lambda+1)x' e^{-\mu s} + z',\ g_{22} = (x')^2 e^{-2\mu s} + (x' e^{-\mu s} + z')^2,$$

we can rewrite the minimality condition $H = 0$, that is, $b_{11}g_{22} - 2b_{12}g_{12} + b_{22}g_{11} = 0$, in the form

$$f_3(t)e^{-3\mu s} + f_2(t)e^{-2\mu s} + f_1(t)e^{-\mu s} + f_0(t) = 0, \tag{14}$$

where $f_3 = (x')^3(\lambda-1)((\lambda-1)^2+\mu^2)$, so it should be $x = x_0$ or $\lambda = 1$.

If $x = x_0$ then the regularity of Σ implies $z' \neq 0$, so we can put without loss of generality $z(t) = t$. Then in (14) we have $f_0 = \lambda(\lambda^2+\mu^2)$, so $\lambda = 0$. Thus, $\Delta = |\mu|$ and for $N = Y_1$

$$b_{11} = -\mu,\ b_{12} = -\frac{\mu}{2},\ b_{22} = 0,\ g_{11} = \mu^2 + 1,\ g_{12} = g_{22} = 1,$$

that clearly implies $H = 0$. In this case (13) takes the form

$$r(s,t) = (x_0,\ e^{\mu s},\ t + s).$$

This is a parameterization of a half-plane $x = x_0$, $y > 0$. The characteristic field Z should be such that the frame $\{X, Z, N\}$ is orthonormal. Then

$$Z = \frac{1}{\sqrt{\mu^2+1}}(-Y_2 + \mu Y_3) = -\frac{1}{\mu\sqrt{\mu^2+1}}\, r_s + \frac{\sqrt{\mu^2+1}}{\mu}\, r_t,$$

so

$$\langle B(Z), X\rangle = -\frac{1}{\mu(\mu^2+1)}\, b_{11} + \frac{1}{\mu}\, b_{12} = \frac{-\mu^2+1}{2(\mu^2+1)}$$

and, taking into account $\langle B(X), X\rangle + \langle B(Z), Z\rangle = 2H = 0$,

$$\langle B(Z), Z\rangle = -\langle B(X), X\rangle = -\frac{1}{\mu^2+1}\, b_{11} = \frac{\mu}{\mu^2+1}.$$

Therefore,

$$|B|^2 = \langle B(X), X\rangle^2 + 2\langle B(X), Z\rangle^2 + \langle B(Z), Z\rangle^2 = \frac{1}{2}.$$

From (10) we have $\operatorname{Ric}(N, N) = -\frac{3}{2}$, and

$$\nabla_N X = \frac{1}{\sqrt{\mu^2+1}}\left(-\mu Y_1 + \frac{1}{2}Y_2 - \frac{\mu}{2}Y_3\right)$$

from (9), so $\langle \nabla_N X, N\rangle = -\frac{\mu}{\sqrt{\mu^2+1}}$. Hence, the second variation formula (1) takes the form

$$A''(0) = \int_\Sigma Z(u)^2 - \frac{2\mu}{\sqrt{\mu^2+1}}\, uX(u) + \frac{1}{\mu^2+1}\, u^2\, d\Sigma.$$

Let us rewrite this expression using the divergence, as was explained in a remark after (1). From (9), $\nabla_Z X = \frac{-\mu^2+1}{2(\mu^2+1)}Y_1$, thus $\operatorname{div}_\Sigma X = \langle \nabla_Z X, Z\rangle = 0$ and so $\operatorname{div}_\Sigma\left(u^2 X\right) = 2uX(u)$. For a function u with compact support the integral of this divergence over Σ vanishes, which finally implies

$$A''(0) = \int_\Sigma Z(u)^2 + \frac{1}{\mu^2+1}\, u^2\, d\Sigma \geqslant 0,$$

which means that Σ is stable. Note that its Riemannian stability follows also from the results of [5], but here we have shown its stability with respect to sub-Riemannian structures different from the one considered in that paper.

In the case when $\lambda = 1$ it appears that $f_0 = (z')^3(\mu^2+1)$ in (14), thus, similarly to the previous case, $z = z_0$, $x(t) = t$, and (13) becomes

$$r(s,t) = \left(t + \frac{1}{\mu}e^{\mu s},\, e^{\mu s},\, z_0\right).$$

It means that Σ is a half-plane $z = z_0$, $y > 0$. Here we have $\Delta = \sqrt{2}|\mu|e^{-\mu s}$ and

$$b_{11} = -\sqrt{2}\mu,\; b_{12} = -\frac{\mu e^{-\mu s}}{\sqrt{2}},\; b_{22} = 0,\; g_{11} = \mu^2+2,\; g_{12} = 2e^{-\mu s},\; g_{22} = 2e^{-2\mu s}.$$

for $N = \frac{1}{\sqrt{2}}(Y_1 - Y_3)$. Again we clearly have the minimality of Σ. The characteristic field is

$$Z = \frac{1}{\sqrt{2}\sqrt{\mu^2+2}}\left(\mu Y_1 - 2Y_2 + \mu Y_3\right) = -\frac{\sqrt{2}}{\mu\sqrt{\mu^2+2}}\, r_s + \frac{e^{\mu s}\sqrt{\mu^2+2}}{\sqrt{2}\mu}\, r_t.$$

From this we get

$$\langle B(Z), X\rangle = -\frac{\sqrt{2}}{\mu(\mu^2+2)}\, b_{11} + \frac{e^{\mu s}}{\sqrt{2}\mu}\, b_{12} = \frac{-\mu^2+2}{2(\mu^2+2)},$$
$$\langle B(Z), Z\rangle = -\langle B(X), X\rangle = -\frac{1}{\mu^2+2}\, b_{11} = \frac{\sqrt{2}\mu}{\mu^2+2},$$

thus

$$|B|^2 = 2\langle B(X), X\rangle^2 + 2\langle B(X), Z\rangle^2 = \frac{1}{2}.$$

According to (9),

$$\nabla_N X = \frac{1}{\sqrt{2}\sqrt{\mu^2+2}}\left(-\frac{\mu}{2}Y_1 + Y_2 - \frac{\mu}{2}Y_3\right),$$

which implies $\langle \nabla_N X, N\rangle = 0$. As $\mathrm{Ric}(N,N) = -\frac{1}{2}$ from (10), (1) now takes the form

$$A''(0) = \int\limits_{\Sigma} Z(u)^2\, d\Sigma \geqslant 0,$$

and the stability of Σ follows.

If $\mu = \lambda = 0$ then, as was mentioned earlier, Σ is minimal if and only if is a cylinder over a geodesic in $\mathbb{H}^2$, and all such cylinders are stable in the sub-Riemannian sense by known results. So we will assume $\lambda \neq 0$ from now on. In this case integral trajectories of X are transversal to $x = x_0$ (see (12)), so we can draw them through points of a curve $t \mapsto (0, y(t), z(t))$ to get the parameterization

$$r(s,t) = (\lambda y(t)s,\ y(t),\ z(t) + (-\lambda+1)s) \tag{15}$$

of a vertical complete connected surface Σ. Now we have

$$r_s = \sqrt{\lambda^2+1}X = \lambda Y_1 + Y_3, r_t = (\lambda y's,\ y',\ z') = \frac{\lambda y' s Y_1 + y' Y_2 + (\lambda y' s + y z')Y_3}{y},$$
$$g_{11} = \lambda^2+1, g_{12} = \frac{\lambda(\lambda+1)y's + yz'}{y}, g_{22} = \frac{(y')^2(2\lambda^2 s^2+1) + 2\lambda y y' z' s + y^2(z')^2}{y^2},$$
$$N = \frac{1}{\Delta}\left(-y'Y_1 - \lambda((\lambda-1)y's + yz')Y_2 + \lambda y' Y_3\right),$$

where $\Delta = \sqrt{(y')^2 + \lambda^2((\lambda-1)y's + yz')^2 + \lambda^2(y')^2}$. From (9),

$$\nabla_{r_s} r_s = \lambda(\lambda+1)Y_2,$$
$$\nabla_{r_t} r_s = \frac{1}{2y}\left(-y'Y_1 + \lambda((3\lambda+1)y's + yz')Y_2 + \lambda y' Y_3\right),$$
$$\nabla_{r_t} r_t = \frac{1}{y^2}\big((\lambda(yy'' - 3(y')^2)s - yy'z')Y_1 + (yy'' - (y')^2 + \lambda y'(2\lambda y's + yz')s)Y_2 +$$
$$+ (\lambda(yy'' - (y')^2)s + y^2 z'')Y_3\big),$$

and so

$$b_{11} = -\frac{\lambda^2(\lambda+1)}{\Delta}((\lambda-1)y's + yz'),$$
$$b_{12} = \frac{1}{2\Delta y}\left((\lambda^2+1)(y')^2 - \lambda^2((3\lambda+1)y's + yz')((\lambda-1)y's + yz')\right),$$
$$b_{22} = \frac{1}{\Delta y^2}\big(-y'(\lambda(yy'' - 3(y')^2)s - yy'z') - \lambda(yy'' - (y')^2 + \lambda y'(2\lambda y's + yz')s)\cdot$$
$$\cdot((\lambda-1)y's + yz') + \lambda y'(\lambda(yy'' - (y')^2)s + y^2 z'')\big).$$

In this case the minimality condition $b_{11}g_{22} - 2b_{12}g_{12} + b_{22}g_{11} = 0$ can be written down as

$$h_3(t)s^3 + h_2(t)s^2 + h_1(t)s + h_0(t) = 0, \tag{16}$$

where $h_3 = (y'\lambda(\lambda-1))^3$, so for a minimal Σ it should be $y = y_0$ or $\lambda = 1$. But for the first of these cases (putting $z(t) = t$ and $N = -Y_2$)

$$b_{11} = -\lambda(\lambda+1),\ b_{12} = -\frac{\lambda}{2},\ b_{22} = 0,\ g_{11} = \lambda^2 + 1,\ g_{12} = g_{22} = 1,$$

and from $H = 0$ we get $\lambda = 0$, a contradiction. Therefore, $\lambda = 1$. Then, checking that in this case $h_1 = h_2 = 0$ and calculating h_0 in (16), we obtain the minimality condition for Σ:

$$2(y'z'' - y''z') = y(z')^3.$$

We already know that half-planes $z = z_0$ are minimal, and the proof of their stability above stays correct for $\mu = 0$: $|B^2| + \mathrm{Ric}(N,N) = 0$ independently of a sub-Riemannian structure, and $\langle \nabla_N X, N\rangle = 0$ (where $N = \frac{1}{\sqrt{2}}(Y_1 - Y_3)$) is true for $\lambda = 1$ and any μ, so the second variation stays the same. Therefore, in the rest of this proof we can assume $z' \neq 0$ and rewrite the previous equation as $y'' = -\frac{y}{2}$ for $y = y(z)$. Hence, $y = y_0 \cos\frac{z-z_0}{\sqrt{2}}$. It means that we can put $y = y_0 \cos t$ and $z = z_0 + \sqrt{2}\,t$ into (15), where $y_0 > 0$ and z_0 denote the values of the corresponding functions at 0 and t is such that $y > 0$. Note that Σ is a cylinder, whose parameterization can be rewritten as (11) by changing s, but here we will continue using (15):

$$r(s,t) = \left(sy_0\cos t, y_0\cos t, z = z_0 + \sqrt{2}\,t\right).$$

We now have $\Delta = \sqrt{2}\,y_0$, and, from the previous formulas,

$$\begin{aligned}
N &= \frac{1}{\sqrt{2}}\left(\sin t\, Y_1 - \sqrt{2}\cos t\, Y_2 - \sin t\, Y_3\right),\\
b_{11} &= -2\cos t,\ b_{12} = \frac{\sqrt{2}\,s\sin 2t - \cos 2t}{\sqrt{2}\cos t},\\
Z = \frac{1}{\sqrt{2}}\left(-\cos t\, Y_1 - \sqrt{2}\sin tY_2 + \cos t\, Y_3\right) &= \frac{\sqrt{2}\,s\sin t - \cos t}{\sqrt{2}}\, r_s + \cos t\, r_t,\\
\langle B(Z), X\rangle &= \frac{\sqrt{2}\,s\sin t - \cos t}{2}\, b_{11} + \frac{\cos t}{\sqrt{2}}\, b_{12} = \frac{1}{2},\\
\langle B(Z), Z\rangle &= -\langle B(X), X\rangle = -\frac{1}{2}\, b_{11} = \cos t,
\end{aligned}$$

so we get $|B|^2 = \frac{1+4\cos^2 t}{2}$ and $\mathrm{Ric}(N,N) = -\frac{1+2\cos^2 t}{2}$ by (10). Finally, from (9),

$$\nabla_N X = \frac{1}{2\sqrt{2}}\left(\cos t\, Y_1 + \sqrt{2}\sin t\, Y_2 - \cos t\, Y_3\right),$$

thus $\langle \nabla_N X, N\rangle = 0$. Therefore, here we have the second variation

$$A''(0) = \int\limits_\Sigma Z(u)^2 - \cos^2 t\, u^2 \, d\Sigma.$$

Let us use here the sub-Riemannian Jacobi operator of Σ. By (9),

$$\nabla_X Z = \frac{1}{2\sqrt{2}}\left(3\sin t\, Y_1 - \sqrt{2}\cos t\, Y_2 + \sin t\, Y_3\right),$$

thus $\langle \nabla_X Z, X\rangle = \sin t$. From Proposition 2, the Jacobi operator can then be written down as

$$L(u) = Z(Z(u)) + \sin t\, Z(u) + \cos^2 t\, u = \frac{(\sqrt{2}\, s\sin t - \cos t)^2}{2} u_{ss} + \cos^2 t\, u_{tt} + \\ + \frac{2\cos t(\sqrt{2}\, s \sin t - \cos t)}{\sqrt{2}} u_{st} + \frac{\sqrt{2}\, s(1+\sin^2 t) - \sin t\cos t}{\sqrt{2}} u_s + \cos^2 t\, u.$$

In particular, we have for functions $u = u(t)$ that are independent of s

$$L(u) = \cos^2 t(u_{tt} + u),$$

so among solutions $u(t) = C_1 \cos t + C_2 \sin t$ of the equation $L(u) = 0$ there is $u(t) = \cos t > 0$. Therefore, Σ is stable by Theorem 1, and this concludes the proof.

In particular, this theorem gives examples of sub-Riemannian structures that do not admit vertical minimal surfaces.

Note that sub-Riemannian structures from the previous theorem are not left-invariant except for the case $\mu = \lambda = 0$. On the other hand, as $[X_2, X_3] = X_1$, the horizontal distribution $\mathcal{H} = X_1^\perp$ defines a left-invariant sub-Riemannian structure on $\widetilde{SL(2,\mathbb{R})}$. For its vertical minimal surfaces we have the following (in fact, a similar description up to an isometry will take place for any sub-Riemannian structure of the form $\mathcal{H} = (\lambda X_1 + \mu X_2)^\perp$).

Theorem 3. *A complete connected vertical surface in* $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ *with the left-invariant sub-Riemannian structure defined by the horizontal distribution* $\mathcal{H} = X_1^\perp$ *is minimal if and only if it is either a half-plane* $z = \frac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$ *or a helicoidal surface with one of the following parameterizations:*

$$\begin{array}{ll} r(s,t) = (x_0 - t\sin s, t\cos s, s), & t \in (0, +\infty), \\ r(s,t) = (x_0 \pm t - t\sin s, t\cos s, s), & t \in (0, +\infty), \\ r(s,t) = (x_0 + y_0 \sinh t - y_0 \cosh t \sin s, y_0 \cosh t \cos s, s), & t \in \mathbb{R}, \\ r(s,t) = (x_0 \pm y_0 \cosh t - y_0 \sinh t \sin s, y_0 \sinh t \cos s, s), & t \in (0, +\infty), \\ s \in \left(-\frac{\pi}{2} + 2\pi k, \frac{\pi}{2} + 2\pi k\right), \quad k \in \mathbb{Z}. & \end{array} \tag{17}$$

All these surfaces are stable in the sub-Riemannian sense and thus in the Riemannian sense.

Proof. A vertical surface Σ for this structure is formed by integral curves of

$$X = X_1 = y\cos z\frac{\partial}{\partial x} + y\sin z\frac{\partial}{\partial y} - \cos z\frac{\partial}{\partial z}. \tag{18}$$

Integrating this field, we get for the third coordinate $z' = -\cos z$, that is, either $z = \frac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$ or $z(\sigma) = \frac{\pi}{2} - 2\arctan Ce^{\sigma} + 2\pi k$ for $C > 0$ and $k \in \mathbb{Z}$ that monotonically decreases from $\frac{\pi}{2} + 2\pi k$ to $-\frac{\pi}{2} + 2\pi k$, where σ is a natural parameter. That means that in the latter case we can use z as the parameter $z = s \in \left(-\frac{\pi}{2} + 2\pi k, \frac{\pi}{2} + 2\pi k\right)$ of this curve. In the former case Σ is a half-plane $z = z_0$ that was already considered in the previous theorem. In particular, we have shown that these half-planes are indeed minimal and that for them $|B^2| + \mathrm{Ric}(N, N) = 0$, where the unit normal field is

$$N = \frac{1}{\sqrt{2}}(Y_1 - Y_3) = \frac{1}{\sqrt{2}}(\cos z\, X_1 - \sin z\, X_2 - X_3) = \frac{1}{\sqrt{2}}(\pm X_2 - X_3)$$

due to (7) and $z = \frac{\pi}{2} + \pi k$. It then follows from (8) that $\nabla_N X = \frac{1}{2\sqrt{2}}(-3X_2 \pm X_3)$, hence $\langle \nabla_N X, N\rangle = \mp 1$, and (1) becomes

$$A''(0) = \int_{\Sigma} Z(u)^2 \mp 2uX(u) - u^2\, d\Sigma.$$

Note that the field $X = X_1$ has zero divergence in $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ by (8). On the other hand, we can calculate this divergence at points of Σ using the orthonormal frame $\{X, Z, N\}$ and taking into account that $\langle \nabla_X X, X\rangle = 0$ because $|X| = 1$: $0 = \langle \nabla_Z X, Z\rangle + \langle \nabla_N X, N\rangle$. Therefore, $\mathrm{div}_\Sigma X = \langle \nabla_Z X, Z\rangle = -\langle \nabla_N X, N\rangle = \pm 1$. From this we get that $\mathrm{div}_\Sigma\left(u^2 X\right) = 2uX(u) \pm u^2$, and the integral of it vanishes for functions u with compact supports, which implies

$$A''(0) = \int_{\Sigma} Z(u)^2\, d\Sigma \geqslant 0,$$

so Σ is stable.

Thus, from now on we can consider surfaces Σ built from integral curves of (18) with $z = s$ as a parameter. In particular, these curves are transversal to $z = z_0$, so we can draw them through points of a curve $t \mapsto (x(t), y(t), 0)$, where $y(t) > 0$. Integrating (18), we can get a parameterization

$$r(s,t) = (x(t) - y(t)\sin s,\ y(t)\cos s,\ s) \tag{19}$$

of Σ (note that y in (18) corresponds to $y(t)\cos s$ here). Then, according to (6),

$$\begin{aligned} r_s &= (-y\cos s,\ -y\sin s,\ 1) = -\frac{1}{\cos s}X_1 = -\frac{1}{\cos s}X, \\ r_t &= (x' - y'\sin s,\ y'\cos s,\ 0) = \frac{x'\cos s\, X_1 + (y' - x'\sin s)X_2 + (x' - y'\sin s)X_3}{y\cos s}, \\ N &= \frac{1}{\Delta}\left(-(x' - y'\sin s)X_2 + (y' - x'\sin s)X_3\right), \end{aligned}$$

where $\Delta = \sqrt{(x' - y'\sin s)^2 + (y' - x'\sin s)^2}$. From (8) we get the covariant derivatives

$$\nabla_{r_s} r_s = -\frac{\sin s}{\cos^2 s} X_1,$$
$$\nabla_{r_t} r_s = -\frac{1}{2y\cos^2 s}\left(3(x' - y'\sin s)X_2 + (y' - x'\sin s)X_3\right),$$
$$\nabla_{r_t} r_t = \frac{1}{y^2\cos^2 s}\left(((x''y - x'y')\cos^2 s - 2(y' - x'\sin s)(x' - y'\sin s))X_1\right.$$
$$((y'' - x''\sin s)y - (y' - x'\sin s)y' + 2x'(x' - y'\sin s))\cos s\, X_2$$
$$\left.((x'' - y''\sin s)y - (x' - y'\sin s)y')\cos s\, X_3\right),$$

and thus the coefficients of the second fundamental form

$$b_{11} = 0,\ b_{12} = \frac{1}{2\Delta y\cos^2 s}\left(3(x' - y'\sin s)^2 - (y' - x'\sin s)^2\right),$$
$$b_{22} = \frac{1}{\Delta y^2\cos s}\left((x''y' - x'y'')y\cos^2 s - 2x'(x' - y'\sin s)^2\right).$$

In particular, for minimal surfaces $\langle B(Z), Z\rangle = -\langle B(X), X\rangle = 0$, hence $|B|^2 = 2\langle B(Z), X\rangle^2$. As the coefficients of the first fundamental form are

$$g_{11} = \frac{1}{\cos^2 s},\ g_{12} = -\frac{x'}{y\cos s},$$
$$g_{22} = \frac{1}{y^2\cos^2 s}\left((x')^2\cos^2 s + (y' - x'\sin s)^2 + (x' - y'\sin s)^2\right),$$

the minimality condition $b_{11}g_{22} - 2b_{12}g_{12} + b_{22}g_{11} = 0$ is equivalent to

$$\left(x''y' - x'y''\right)y + x'\left((x')^2 - (y')^2\right) = 0. \tag{20}$$

First, let us consider the solution $x = x_0$. Then we can put $y(t) = t$ for $t > 0$ and get from (19) the first parameterization in (17). In this case $N = \frac{1}{\Delta}(\sin s\, X_2 + X_3)$, where $\Delta = \sqrt{1 + \sin^2 s}$, and

$$b_{11} = b_{22} = 0,\ b_{12} = \frac{3\sin^2 s - 1}{2\Delta t\cos^2 s},\ Z = \frac{1}{\Delta}(X_2 - \sin s\, X_3) = \frac{t\cos s}{\Delta} r_t,$$
$$|B|^2 = 2\langle B(Z), X\rangle^2 = \frac{2t^2\cos^4 s}{\Delta^2} b_{12}^2 = \frac{(3\sin^2 s - 1)^2}{2\Delta^4}.$$

We also have $\mathrm{Ric}(N, N) = \frac{-3\sin^2 s + 1}{2\Delta^2}$ by (10) and $\nabla_N X = \frac{1}{2\Delta}(3X_2 + \sin s\, X_3)$ from (8), hence $\langle\nabla_N X, N\rangle = \frac{2\sin s}{\Delta^2}$. Therefore, the second variation is

$$A''(0) = \int\limits_\Sigma Z(u)^2 + \frac{4\sin s}{1 + \sin^2 s} uX(u) - \frac{3\sin^4 s + 1}{(1 + \sin^2 s)^2} u^2\, d\Sigma.$$

by (1). Again, $\mathrm{div}_\Sigma X = \langle\nabla_Z X, Z\rangle = -\langle\nabla_N X, N\rangle = -\frac{2\sin s}{\Delta^2}$. Thus,

$$\mathrm{div}_\Sigma\left(\frac{\sin s}{1 + \sin^2 s} u^2 X\right) = \left(X\left(\frac{\sin s}{1 + \sin^2 s}\right) + \frac{\sin s}{1 + \sin^2 s}\mathrm{div}_\Sigma X\right)u^2 +$$
$$+ \frac{2\sin s}{1 + \sin^2 s} uX(u) = -\frac{\sin^4 s + 1}{(1 + \sin^2 s)^2} u^2 + \frac{2\sin s}{1 + \sin^2 s} uX(u).$$

As the integral of this expression vanishes for u with compact support,

$$A''(0) = \int_\Sigma Z(u)^2 + \frac{\cos^2 s}{1+\sin^2 s}\, u^2\, d\Sigma \geqslant 0,$$

and Σ is stable.

For $x' \neq 0$ let us rewrite (20) for the function $y = y(x)$ and get $yy'' + (y')^2 = 1$, that is, $(y^2)'' = 2$, so $y^2 = (x - x_0)^2 + C$. For $C = 0$ we can put $x(t) = x_0 \pm t$ and $y(t) = t > 0$, thus getting the second parameterization in (17) from (19). For $C > 0$ we obtain the third parameterization in (17) with $x(t) = x_0 + y_0 \sinh t$ and $y(t) = y_0 \cosh t$, where $y_0 > 0$, and so $C = y_0^2 > 0$. Finally, the fourth parameterization corresponds to $x(t) = x_0 \pm y_0 \cosh t$ and $y(t) = y_0 \sinh t$ for $t > 0$, thus $C = -y_0^2 < 0$. Therefore, in these last two cases the curves (x, y) are hyperbolas in the half-plane $y > 0$.

For $x(t) = x_0 \pm t$ and $y(t) = t$ from the general formulas we obtain $\Delta = \sqrt{2}(1 \mp \sin s)$, $N = \frac{1}{\sqrt{2}}(\mp X_2 + X_3)$, and

$$\begin{gathered}
b_{11} = 0,\ b_{12} = \frac{1 \mp \sin s}{\sqrt{2}\, t \cos^2 s},\ b_{22} = \frac{\mp 2(1 \mp \sin s)}{\sqrt{2}\, t^2 \cos s}, \\
Z = \frac{1}{\sqrt{2}}(X_2 \pm X_3) = \frac{\cos s\,(\cos s\, r_s + t\, r_t)}{\sqrt{2}(1 \mp \sin s)}, \\
|B|^2 = 2\,\langle B(Z), X\rangle^2 = \frac{\cos^4 s\,(\cos s\, b_{11} + t\, b_{12})^2}{(1 \mp \sin s)^2} = \frac{1}{2}.
\end{gathered}$$

As N is (up to a sign) the same as in the case $z = \frac{\pi}{2} + \pi k$ above, here also $\operatorname{Ric}(N, N) = -\frac{1}{2}$ and $\langle \nabla_N X, N\rangle = \mp 1$. The rest of the stability proof for Σ is also literally the same as in that case.

For the third parameterization in (17) we have $x(t) = x_0 + y_0 \sinh t$, $y(t) = y_0 \cosh t$, and for the fourth one we have $x(t) = x_0 \pm y_0 \cosh t$, $y(t) = y_0 \sinh t$. Let us denote $\alpha = x' - y' \sin s$, $\beta = y' - x' \sin s$ for all these cases. Then

$$r_t = \frac{x' \cos s\, X_1 + \beta X_2 + \alpha X_3}{y \cos s},\ N = \frac{1}{\Delta}(-\alpha X_2 + \beta X_3),$$

where $\Delta = \sqrt{\alpha^2 + \beta^2}$. Note that $\alpha^2 - \beta^2 = y_0^2 \cos^2 s$ for the third parameterization in (17) and $\alpha^2 - \beta^2 = -y_0^2 \cos^2 s$ for the fourth one. From the general formulas above,

$$b_{11} = 0,\ b_{12} = \frac{3\alpha^2 - \beta^2}{2y\Delta \cos^2 s},\ b_{22} = -\frac{3\alpha^2 - \beta^2}{y\Delta \cos s}.$$

Recall that $r_s = -\frac{1}{\cos s} X_1 = -\frac{1}{\cos s} X$, so

$$\begin{gathered}
Z = \frac{1}{\Delta}(\beta X_2 + \alpha X_3) = \frac{\cos s\,(x' \cos s\, r_s + y\, r_t)}{\Delta}, \\
|B|^2 = 2\,\langle B(Z), X\rangle^2 = \frac{\cos^4 s\,(x' \cos s\, b_{11} + y b_{12})^2}{\Delta^2} = \frac{\left(3\alpha^2 - \beta^2\right)^2}{2\Delta^4}.
\end{gathered}$$

As $\mathrm{Ric}(N,N) = \frac{-3\alpha^2+\beta^2}{2\Delta^2}$ from (10) and $\nabla_N X = \frac{1}{\Delta}\left(3\beta X_2 - \alpha X_3\right)$ from (8), which implies $\langle \nabla_N X, N\rangle = -\frac{2\alpha\beta}{\Delta^2}$, the second variation (1) takes the form

$$A''(0) = \int\limits_{\Sigma} Z(u)^2 - \frac{4\alpha\beta}{\Delta^2}\,uX(u) - \frac{\left(3\alpha^2-\beta^2\right)\left(\alpha^2-\beta^2\right)+4\alpha^2\beta^2}{\Delta^4}\,u^2\, d\Sigma.$$

Once again, $\mathrm{div}_\Sigma X = \langle \nabla_Z X, Z\rangle = -\langle \nabla_N X, N\rangle = \frac{2\alpha\beta}{\Delta^2}$. By direct computation we get $X\left(\frac{\alpha\beta}{\Delta^2}\right) = \frac{y_0^4\cos^4 s}{\Delta^4} = \frac{(\alpha^2-\beta^2)^2}{\Delta^4}$, hence

$$\mathrm{div}_\Sigma\left(\frac{\alpha\beta}{\Delta^2}\,u^2 X\right) = \frac{(\alpha^2-\beta^2)^2+2\alpha^2\beta^2}{\Delta^4}\,u^2 + \frac{2\alpha\beta}{\Delta^2}\,uX(u),$$

and the integral of this expression vanishes for u with compact support, so finally

$$A''(0) = \int\limits_{\Sigma} Z(u)^2 - \frac{\alpha^2-\beta^2}{\Delta^2}\,u^2\, d\Sigma.$$

For the fourth case in (17) the expression under this integral is always non-negative, thus we already have the stability of Σ. For the third one we again will use the sub-Riemannian Jacobi operator. As $\nabla_X Z = \frac{1}{2\Delta}\left(\alpha X_2 - \beta X_3\right)$ from (8), $\langle \nabla_X Z, X\rangle = 0$. By Proposition 2, the Jacobi operator of Σ then is

$$\begin{gathered}
L(u) = Z(Z(u)) + \frac{y_0^2\cos^2 s}{\Delta^2}\,u = \\
= \frac{y_0\cosh t\,\cos s}{\Delta}\left(\cos s\,\frac{\partial}{\partial s} + \frac{\partial}{\partial t}\right)\left(\frac{y_0\cosh t\,\cos s}{\Delta}\left(\cos s\,u_s + u_t\right)\right) + \frac{y_0^2\cos^2 s}{\Delta^2}\,u = \\
= \frac{y_0^2\cos^2 s}{\Delta^2}\left(\cosh^2 t\,\cos^2 s\,u_{ss} + 2\cosh^2 t\,\cos s\,u_{st} + \cosh^2 t\,u_{tt} + \right. \\
\left. + \cos s\,\cosh t(\sinh t - \sin s\,\cosh t)u_s + \cosh t\,\sinh t\,u_t + u\right)
\end{gathered}$$

Again, let us restrict L to functions of the form $u = u(t)$. For them $L(u) = 0$ if and only if $\cosh^2 t\,u_{tt} + \sinh t\cosh t\,u_t + u = 0$. Among solutions $u(t) = \frac{C_1}{\cosh t} + C_2\tanh t$ of this Sturm–Liouville equation there is $u(t) = \frac{1}{\cosh t} > 0$. By Theorem 1, this implies that Σ is stable, thus concluding the proof.

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REFERENCES

1. T. H. Colding, W. P. Minicozzi II. A Course in Minimal Surfaces. – American Mathematical Society. – 2011. DOI: 10.1090/gsm/121

2. D. Fischer-Colbrie, R. Schoen. The structure of complete stable minimal surface in 3-manifolds of non-negative scalar curvature, Comm. Pure Appl. Math. – 1980. – Vol. **33**, No **2**. – P. 199–211. DOI: 10.1002/cpa.3160330206

3. N. Garofalo, D.-M. Nhieu. Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces, Comm. Pure Appl. Math. – 1996. – Vol. **42**, No **3**. – P. 1081–1144. DOI: 10.1002/(SICI)1097-0312(199610)49:10<1081::AID-CPA3>3.0.CO;2-A

4. I. Havrylenko, E. Petrov. Stability of minimal surfaces in the sub-Riemannian manifold $\widetilde{E(2)}$, Visnyk of V.N. Karazin Kharkiv National University, Ser. Math., Appl. Math., Mech. – 2023. – Vol. **98**. – P. 50–67. DOI: 10.26565/2221-5646-2023-98-04

5. I. Havrylenko, E. Petrov. Stability of vertical minimal surfaces in three-dimensional sub-Riemannian manifolds, Proceedings of the International Geometry Center. – 2023. – Vol. **18**, No **2**. – P. 159–182. DOI: 10.15673/pigc.v18i2.3009

6. A. Hurtado, M. Ritoré, C. Rosales. The classification of complete stable area-stationary surfaces in the Heisenberg group $\mathbb{H}^1$, Adv. in Math. – 2010. – Vol. **224**, No **2**. – P. 561–600. DOI: 10.1016/j.aim.2009.12.002

7. J. Milnor. Curvatures of left invariant metrics on Lie groups, Adv. in Math. – 1976. – Vol. **21**, No **3**. – P. 293–329. DOI: 10.1016/S0001-8708(76)80002-3

8. P. Scott. The geometries of 3-manifolds, Bull. Lond. Math. Soc. – 1983. – Vol. **15**, No **5**. – P. 401–487. DOI: 10.1112/blms/15.5.401

9. R. Younes. Minimal surfaces in $\widetilde{PSL_2}(\mathbb{R})$, Ill. J. Math. – 2010. – Vol. **54**, No **2**. – P. 671–712. DOI: 10.1215/ijm/1318598677

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Оператор Якобі та стійкість вертикальних мінімальних поверхонь у субрімановій групі Лі $\widetilde{\mathrm{SL}(2,\mathbb{R})}$

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Ми розглядаємо орієнтовані занурені мінімальні поверхні у тривимірних субріманових многовидах, які є вертикальними, тобто перпендикулярними до двовимірного горизонтального розподілу субріманової структури. Раніше ми показали, що вертикальна поверхня є мінімальною в субрімановому сенсі тоді й тільки тоді, коли вона мінімальна в рімановому сенсі, і що з її субріманової стійкості випливає її ріманова стійкість. Ми вводимо субріманову версію оператора Якобі для таких поверхонь і доводимо достатню умову стійкості вертикальних мінімальних поверхонь, що аналогічна до теореми Фішер-Колбрі та Шоена: якщо поверхня допускає додатну функцію з нульовим оператором Якобі, то вона є стійкою.

Далі ми використовуємо техніку операторів Якобі для дослідження вертикальних мінімальних поверхонь у групі Лі $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, яку можна описати як універсальне накриття розшарування одиничних дотичних векторів гіперболічної площини зі стандартною лівоінваріантною метрикою Сасакі (що відповідає одній з геометрій Терстона) та з двома різними типами субріманових структур. Спочатку ми розглядаємо сім'ю нелівоінваріантних структур, визначених деякими параметрами, знаходимо значення параметрів, для яких існують вертикальні мінімальні поверхні, та описуємо такі повні зв'язні поверхні. Це евклідові напівплощини та циліндри, й усі вони є стійкими в субрімановому сенсі, а отже і в рімановому сенсі. Зокрема, це дає нам приклади структур, що не допускають вертикальних мінімальних поверхонь. Потім ми описуємо повні зв'язні вертикальні мінімальні поверхні для іншої субріманової структури, що є лівоінваріантною. Це напівплощини та гелікоїдальні поверхні, які також виявляються стійкими в субрімановому сенсі, а отже й у рімановому сенсі.
Ключові слова: **субрімановий многовид; лівоінваріантна метрика; мінімальна поверхня; оператор Якобі; стійкість.**

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