ISSN 2221-5646(Print) 2523-4641(Online)
Visnyk of V.N.Karazin Kharkiv National University
Ser. "Mathematics, Applied Mathematics and Mechanics"
2025, Vol. 102, p. 57–67
DOI: 10.26565/2221-5646-2025-102-04
УДК 517.986.32

Вісник Харківського національного університету імені В.Н. Каразіна
Серія "Математика, прикладна математика і механіка"
2025, Том 102, с. 57–67

Roman Skurikhin
PhD student
Department of Pure Mathematics
V. N. Karazin Kharkiv National University
4 Svobody Sq., Kharkiv, Ukraine, 61022
roman.skurikhin@karazin.ua *http://orcid.org/0009-0005-0504-8518*

Sergiy Gefter
PhD in mathematics, Senior Research Fellow
B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine
47 Nauky Ave., Kharkiv, Ukraine, 61103
gefter@karazin.ua *http://orcid.org/0000-0001-5301-5087*

Eugene Karolinsky
PhD in mathematics, Associate Professor
Department of Pure Mathematics
V. N. Karazin Kharkiv National University
4 Svobody Sq., Kharkiv, Ukraine, 61022
karolinsky@karazin.ua *http://orcid.org/0000-0002-1109-4247*

Closed equivalence relations on compact spaces and pairs of commutative C^*-algebras: a categorical approach

In this paper, we study a categorical extension of the classical Gelfand–Naimark duality between compact Hausdorff spaces and commutative unital C^*-algebras. We establish an equivalence between the category of compact Hausdorff spaces with closed equivalence relations and the category of pairs consisting of a commutative unital C^*-algebra together with one of its unital C^*-subalgebras. The motivation is that Gelfand duality can be enriched by additional structure: closed equivalence relations encode quotient spaces and invariance on the topological side, while subalgebras reflect restrictions and symmetries on the algebraic side. Shilov's theorem, which identifies closed unital self-adjoint subalgebras of $C(X)$ with algebras of functions invariant under closed equivalence relations, provides an essential link between these settings. We introduce the category **EqRel**, whose objects are compact Hausdorff spaces with closed equivalence relations and whose morphisms are continuous trajectory-preserving maps, and the category $\mathbf{C^*Pairs}$, whose objects are pairs (A, B) with A a commutative unital C^*-algebra and $B \subset A$ a unital C^*-subalgebra, with morphisms given by unital

$*$-homomorphisms preserving B. Contravariant functors are defined in both directions: $(X, R) \mapsto (C(X), B_R)$, where B_R consists of functions constant on R-classes, and $(A, B) \mapsto (\Sigma(A), R_B)$, where $\Sigma(A)$ is the spectrum and R_B relates characters agreeing on B. We verify that these constructions are functorial and compatible with composition of morphisms. Using the Kolmogorov–Gelfand theorem, the Gelfand transform, and Shilov's theorem, we show that these functors are mutually inverse up to morphism of functors and thus prove the categorical equivalence **EqRel** $\simeq$ $\mathbf{C^*Pairs}^{\mathrm{op}}$. This result demonstrates that the geometric notion of closed equivalence relations on compact spaces is in perfect correspondence with the algebraic notion of unital subalgebras of commutative C^*-algebras.

Keywords: **categorical equivalence; Gelfand duality; closed equivalence relation; commutative C^*-algebra; invariant subalgebra; Shilov theorem.**

2020 Mathematics Subject Classification: 46L05; 46M15.

1. Introduction

This paper presents a categorical correspondence between two seemingly distinct mathematical objects: topological spaces with closed equivalence relations and pairs consisting of a commutative C^*-algebra and its subalgebra.

Our motivation stems from the classical duality between compact Hausdorff spaces and commutative unital C^*-algebras established by the Gelfand–Naimark theorem. Extending this idea, we explore how additional structure — in particular, closed equivalence relations or subalgebras — can be encoded categorically and translated between the topological and algebraic frameworks.

The paper is structured as follows. In Section 2, we review essential notions from category theory, topology, and C^*-algebras. Section 3 introduces two categories: one based on compact spaces with closed equivalence relations, another based on commutative C^*-algebra pairs, and defines natural functors between them. Finally, in Section 4 we prove that the above mentioned functors establish an equivalence of categories.

2. Preliminaries

Notions from Category Theory. We recall some standard definitions in category theory (see, e.g., [1, Chapter II]).

Definition 1. *A covariant functor $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ assigns*

- *to each object $A \in \mathcal{C}$, an object $F(A) \in \mathcal{D}$,*
- *to each morphism $\varphi : A \to B$ in $\mathcal{C}$, a morphism $F(\varphi) : F(A) \to F(B)$ in $\mathcal{D}$,*

such that

1. $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$,
2. $F(\varphi \circ \psi) = F(\varphi) \circ F(\psi)$ *for all composable morphisms ψ, φ.*

Definition 2. *A* contravariant functor $F : \mathcal{C} \to \mathcal{D}$ *assigns*

- *to each object* $A \in \mathcal{C}$*, an object* $F(A) \in \mathcal{D}$*,*
- *to each morphism* $\varphi : A \to B$ *in* $\mathcal{C}$*, a morphism* $F(\varphi) : F(B) \to F(A)$ *in* $\mathcal{D}$*,*

such that

1. $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$,
2. $F(\varphi \circ \psi) = F(\psi) \circ F(\varphi)$ *for all composable morphisms.*

Definition 3. *Given a category* $\mathcal{C}$*, its* dual category *(or* opposite category*), denoted by* $\mathcal{C}^{\mathrm{op}}$*, is defined as follows:*

- *the objects of* $\mathcal{C}^{\mathrm{op}}$ *are the same as those of* $\mathcal{C}$*,*
- *for each morphism* $\varphi : A \to B$ *in* $\mathcal{C}$*, there is a corresponding morphism* $\varphi^{\mathrm{op}} : B \to A$ *in* $\mathcal{C}^{\mathrm{op}}$*,*
- *composition in* $\mathcal{C}^{\mathrm{op}}$ *is given by reversing the order of composition in* $\mathcal{C}$*, i.e.* $(\varphi \circ \psi)^{\mathrm{op}} = \psi^{\mathrm{op}} \circ \varphi^{\mathrm{op}}$.

Definition 4. *Let* $F, G : \mathcal{C} \to \mathcal{D}$ *be two functors. A* morphism of functors $\eta : F \to G$ *is a family of morphisms* $\{\eta_X : F(X) \to G(X)\}_{X \in \mathrm{Ob}(\mathcal{C})}$ *such that for every morphism* $f : X \to Y$ *in* $\mathcal{C}$*, the following diagram commutes:*

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

Definition 5. *A morphism* $\varphi : X \to Y$ *in a category* $\mathcal{C}$ *is an* isomorphism *if there exists a morphism* $\psi : Y \to X$ *such that* $\psi \circ \varphi = \mathrm{id}_X$ *and* $\varphi \circ \psi = \mathrm{id}_Y$*.*

Definition 6. *Two categories* $\mathcal{C}$ *and* $\mathcal{D}$ *are said to be* equivalent *if there exist functors* $F : \mathcal{C} \to \mathcal{D}$ *and* $G : \mathcal{D} \to \mathcal{C}$ *together with isomorphisms of functors*

$$\eta : G \circ F \to \mathrm{id}_{\mathcal{C}}, \qquad \epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}.$$

Remark 1. *A contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ *can be viewed as an ordinary covariant functor* $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$*. In particular, when studying equivalences, it is common to consider equivalences between a category and the dual of another. Thus, equivalence can also be formulated in terms of contravariant functors once one passes to opposite categories.*

C^*-Algebras and Their Spectra. Let us recall some basic definitions from the theory of Banach and C^*-algebras (see, e.g., [3, Chapter 1]).

Definition 7. *A* Banach algebra *is a complex associative algebra A equipped with a norm $\|\cdot\|$ such that*

- *A is a Banach space,*
- *$\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$,*
- *A has a multiplicative identity 1_A such that $1_A a = a = a 1_A$ for all $a \in A$.*

Definition 8. *A* character *on a Banach algebra A is a linear functional $\varphi : A \to \mathbb{C}$ such that $\varphi(1_A) = 1$ and $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in A$.*

Definition 9. *The* spectrum *of a commutative Banach algebra A is the set $\Sigma(A)$ of all characters of A, equipped with the weak* topology (pointwise convergence).*

It is well known that the spectrum $\Sigma(A)$ is a compact Hausdorff space.

Definition 10. *A Banach algebra A is called a* C^*-algebra *if it is equipped with an involution $a \mapsto a^*$ satisfying*

1. $(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*$,
2. $(ab)^* = b^* a^*$,
3. $(a^*)^* = a$,
4. $\|a^*\| = \|a\|$,
5. $\|aa^*\| = \|a\|^2$,

for all $a, b \in A$ and $\alpha, \beta \in \mathbb{C}$.

Example 1. *Let X be a compact Hausdorff space. The set $C(X)$ of complex-valued continuous functions on X is a commutative unital C^*-algebra with*

- *pointwise operations,*
- *involution $f^*(x) := \overline{f(x)}$,*
- *norm $\|f\| := \max_{x \in X} |f(x)|$.*

Structure Theorems for Commutative C^*-Algebras. The following classical results establish a deep duality between commutative C^*-algebras and compact Hausdorff spaces.

Theorem 1 (Kolmogorov-Gelfand [4])**.** *Let X be a compact Hausdorff space. For each $x \in X$, define a character $\varphi_x : C(X) \to \mathbb{C}$ by $\varphi_x(f) := f(x)$. Then the map*

$$I : X \to \Sigma(C(X)), \quad I(x) = \varphi_x$$

is a homeomorphism. Thus, the spectrum of $C(X)$ is naturally identified with the space X.

Theorem 2 (Gelfand-Naimark [2], Theorem 11.18)**.** *Let A be a commutative unital C^*-algebra. The Gelfand transform*

$$\Gamma : A \to C(\Sigma(A)), \quad a \mapsto \hat{a}, \quad \hat{a}(\varphi) := \varphi(a),$$

is an isometric $*$-isomorphism of C^*-algebras.

Categorical Reformulation of the Gelfand Duality. Let us make this duality precise in categorical terms. Following [1, Section II.10], we write **Haus** for the category whose objects are compact Hausdorff spaces and whose morphisms are continuous maps, and **Ban** for the category whose objects are commutative unital C^*-algebras and whose morphisms are unital $*$-homomorphisms.

Theorem 3. *[1, Section II.10], The categories* **Haus** *and* $\mathbf{Ban}^{\mathrm{op}}$ *are equivalent.*

Proof. Define the contravariant functor

$$C : \mathbf{Haus} \to \mathbf{Ban}, \quad X \mapsto C(X), \quad f \mapsto C(f) : g \mapsto g \circ f.$$

Also define the contravariant functor

$$\Sigma : \mathbf{Ban} \to \mathbf{Haus}, \quad A \mapsto \Sigma(A), \quad L \mapsto \Sigma(L) : \varphi \mapsto \varphi \circ L.$$

These functors satisfy the required properties:

- $\Sigma(C(X)) \cong X$ via the map $I(x) := \varphi_x$ (Theorem 1),
- $C(\Sigma(A)) \cong A$ via the Gelfand transform Γ (Theorem 2),
- both isomorphisms are functorial.

Hence, $\mathbf{Haus} \simeq \mathbf{Ban}^{\mathrm{op}}$. □

Shilov's Theorem on Invariant Subalgebras. Let us recall the definition of closed equivalence relation:

Definition 11. *Let X be a compact Hausdorff space. An* equivalence relation *$R \subset X \times X$ is said to be* closed *if R is a closed subset of $X \times X$ [6, p. 52].*

Example 2. *Let $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$ (the closed unit disk), and define $(x, y) \in R$ if $x_1^2 + x_2^2 = y_1^2 + y_2^2$. Then*

$$R = \{((x_1, x_2), (y_1, y_2)) \in X \times X : x_1^2 + x_2^2 = y_1^2 + y_2^2\}$$

is a closed equivalence relation.

Example 3. *Let $X = \{z \in \mathbb{C} : |z| = 1\}$ (the unit circle), and fix $\theta \in \mathbb{R}$ such that $\theta/\pi \notin \mathbb{Q}$. Define $z_1 \sim z_2$ if $z_2 = e^{in\theta} z_1$ for some $n \in \mathbb{Z}$. Then the orbit of any point under this relation is dense in X, and the graph of the relation is not closed in $X \times X$.*

The following theorem is central in identifying the subalgebra B with the algebra of functions invariant under R_B, and it plays a key role in the proof of our main result.

Theorem 4 (Shilov [5])**.** *Let X be a compact Hausdorff space, and let $B \subset C(X)$ be a closed unital self-adjoint subalgebra. Define an equivalence relation $R_B \subset X \times X$ by*

$$(x, y) \in R_B \qquad \Longleftrightarrow \qquad \forall f \in B: \ f(x) = f(y).$$

Then:

1. *R_B is a closed equivalence relation,*
2. *B coincides with the algebra of functions invariant under R_B, i.e.,*

$$B = \{f \in C(X) : f(x) = f(y) \ \forall (x, y) \in R_B\} =: B_{R_B}.$$

3. Closed Equivalence Relations and Pairs of C^*-Algebras

Definition 12. *Let (X_1, R_1) and (X_2, R_2) be compact spaces equipped with equivalence relations. A continuous map $f : X_1 \to X_2$ is said to be* trajectory-preserving *if whenever (x, y) belongs to R_1, the pair $(f(x), f(y))$ belongs to R_2.*

We define the category **EqRel** as follows:

- objects are pairs (X, R), where X is a compact Hausdorff space and R is a closed equivalence relation on X,
- morphisms are continuous trajectory-preserving maps.

Also, consider the category **C*Pairs** where

- objects are pairs (A, B), where A is a commutative unital C^*-algebra, and $B \subset A$ a unital C^*-subalgebra,
- morphisms $L : (A_1, B_1) \to (A_2, B_2)$ are unital $*$-homomorphisms $L : A_1 \to A_2$ satisfying $L(B_1) \subset B_2$.

Let (X, R) be an object in **EqRel**. Define

$$B_R := \{f \in C(X) : f(x) = f(y) \text{ whenever } (x, y) \in R\}.$$

I.e., B_R is the set of functions on X which are constant on each equivalence class.

Lemma 1. *B_R is a unital C^*-subalgebra of $C(X)$.*

Proof. It is closed under addition, multiplication, scalar multiplication, and involution. The unit function 1 is clearly in B_R. Uniform limits of invariant functions are invariant, so B_R is closed. □

Now let $f : (X_1, R_1) \to (X_2, R_2)$ be a morphism in **EqRel**. Then f induces a unital $*$-homomorphism

$$C(f) : C(X_2) \to C(X_1), \quad C(f)(\varphi) := \varphi \circ f.$$

Lemma 2. *The map $C(f)$ maps B_{R_2} into B_{R_1}. Hence, $C(f)$ is a morphism of pairs:*

$$C(f) : (C(X_2), B_{R_2}) \to (C(X_1), B_{R_1}).$$

Proof. Let $\varphi \in B_{R_2}$ and $(x, y) \in R_1$. Since f is trajectory-preserving, we have $(f(x), f(y)) \in R_2$, so

$$C(f)(\varphi)(x) = \varphi(f(x)) = \varphi(f(y)) = C(f)(\varphi)(y),$$

hence $C(f)(\varphi) \in B_{R_1}$. □

Theorem 5. *The assignment $(X, R) \mapsto (C(X), B_R)$ and $f \mapsto C(f)$ defines a contravariant functor*

$$C : \mathbf{EqRel} \to \mathbf{C^*Pairs}.$$

Proof. To prove that C defines a contravariant functor, we must verify two properties:

(i) **Identity morphisms:** For each object (X, R) in **EqRel**, the identity map $\mathrm{id}_X : X \to X$ is trajectory-preserving. Then $C(\mathrm{id}_X) : C(X) \to C(X)$ is given by

$$C(\mathrm{id}_X)(\varphi) = \varphi \circ \mathrm{id}_X = \varphi.$$

Clearly, $C(\mathrm{id}_X) = \mathrm{id}_{C(X)}$, and it maps B_R to itself.

(ii) **Composition:** Let

$$f : (X_1, R_1) \to (X_2, R_2), \quad g : (X_2, R_2) \to (X_3, R_3)$$

be morphisms in **EqRel**, i.e., both f and g are continuous and trajectory-preserving. Then so is $g \circ f$, and for any $\varphi \in C(X_3)$ we have

$$C(g \circ f)(\varphi) = \varphi \circ g \circ f = (C(f) \circ C(g))(\varphi).$$

Furthermore, if $\varphi \in B_{R_3}$, then $C(g)(\varphi) \in B_{R_2}$ and $C(f)(C(g)(\varphi)) \in B_{R_1}$. Hence, $C(g \circ f)$ maps B_{R_3} into B_{R_1}, and

$$C(g \circ f) = C(f) \circ C(g)$$

as morphisms in $\mathbf{C^*Pairs}$.

Therefore, C is a contravariant functor from **EqRel** to **C*Pairs**. ☐

On the other hand, let (A, B) be a pair in **C*Pairs**, i.e., A is a commutative unital C^*-algebra and $B \subset A$ is a unital C^*-subalgebra.

Define an equivalence relation R_B on the spectrum $\Sigma(A)$ by

$$(\varphi, \psi) \in R_B \quad \Longleftrightarrow \quad \forall b \in B: \ \varphi(b) = \psi(b).$$

Lemma 3. *R_B is a closed equivalence relation on the compact Hausdorff space $\Sigma(A)$.*

Proof. Reflexivity and symmetry are immediate. Transitivity follows from the equality condition on B.

To prove closedness, suppose $(\varphi_\alpha, \psi_\alpha) \in R_B$ is a net converging to (φ, ψ) in $\Sigma(A) \times \Sigma(A)$ with the product of weak* topologies. Then for every $b \in B$,

$$\varphi(b) = \lim_\alpha \varphi_\alpha(b) = \lim_\alpha \psi_\alpha(b) = \psi(b),$$

so $(\varphi, \psi) \in R_B$. ☐

Now let $L : (A_1, B_1) \to (A_2, B_2)$ be a morphism in **C*Pairs**. Then $L : A_1 \to A_2$ is a unital $*$-homomorphism with $L(B_1) \subset B_2$. Define

$$\Sigma(L) : \Sigma(A_2) \to \Sigma(A_1), \quad \Sigma(L)(\varphi) := \varphi \circ L.$$

Lemma 4. *$\Sigma(L)$ is continuous and trajectory-preserving with respect to the relations R_{B_2} and R_{B_1}.*

Proof. Continuity of $\Sigma(L)$ follows from standard functional analysis: composition with a continuous map is continuous in the weak* topology.

Let $(\varphi, \psi) \in R_{B_2}$, i.e., $\varphi(b_2) = \psi(b_2)$ for all $b_2 \in B_2$. Then for any $b_1 \in B_1$, since $L(b_1) \in B_2$, we have

$$(\Sigma(L)(\varphi))(b_1) = \varphi(L(b_1)) = \psi(L(b_1)) = (\Sigma(L)(\psi))(b_1),$$

so $(\Sigma(L)(\varphi), \Sigma(L)(\psi)) \in R_{B_1}$. ☐

Theorem 6. *The assignment $(A, B) \mapsto (\Sigma(A), R_B)$ and $L \mapsto \Sigma(L)$ defines a contravariant functor*

$$\Sigma : \mathbf{C^*Pairs} \to \mathbf{EqRel}.$$

Proof. As before, we verify the two functorial properties.

- **Identity:** For $\mathrm{id}_A : A \to A$, we have $\Sigma(\mathrm{id}_A)(\varphi) = \varphi$, so $\Sigma(\mathrm{id}_A) = \mathrm{id}_{\Sigma(A)}$.
- **Composition:** Let $L : A_1 \to A_2$ and $M : A_2 \to A_3$ be morphisms in **C*Pairs**. Then for any $\varphi \in \Sigma(A_3)$,

  $$\Sigma(M \circ L)(\varphi) = \varphi \circ M \circ L = \Sigma(L)(\Sigma(M)(\varphi)),$$

  so $\Sigma(M \circ L) = \Sigma(L) \circ \Sigma(M)$.

Therefore, Σ is a contravariant functor from **C*Pairs** to **EqRel**. ☐

4. Main Result

Theorem 7. *The functors*

$$C : \mathbf{EqRel} \to \mathbf{C^*Pairs}, \qquad \Sigma : \mathbf{C^*Pairs} \to \mathbf{EqRel}$$

establish an equivalence of categories

$$\mathbf{EqRel} \simeq \mathbf{C^*Pairs}^{\mathrm{op}}.$$

Proof. We construct isomorphisms of functors in both directions.

(1) Let (X, R) be an object in **EqRel**. Consider the canonical map

$$I_X : X \to \Sigma(C(X)), \quad x \mapsto \varphi_x, \text{ where } \varphi_x(f) := f(x).$$

This is a homeomorphism by the Kolmogorov–Gelfand theorem. We now show it is also an isomorphism in the category **EqRel**.

Let $(x, y) \in R$, and let $f \in B_R$ (i.e., invariant under R). Then

$$\varphi_x(f) = f(x) = f(y) = \varphi_y(f),$$

so $(\varphi_x, \varphi_y) \in R_{B_R}$. Hence, I_X maps R into R_{B_R}, and similarly its inverse does the reverse.

Thus, $I_X : (X, R) \to (\Sigma(C(X)), R_{B_R})$ is an isomorphism in **EqRel**.

(2) Let (A, B) be an object in **C*Pairs**. Consider the Gelfand transform

$$\Gamma_A : A \to C(\Sigma(A)), \quad a \mapsto \hat{a}, \quad \hat{a}(\varphi) := \varphi(a).$$

This is an isometric $*$-isomorphism. Moreover, by Shilov's Theorem 4, we have $\Gamma_A(B) = B_{R_B}$, hence

$$\Gamma_A : (A, B) \to (C(\Sigma(A)), B_{R_B})$$

is an isomorphism in **C*Pairs**.

(3) Naturality. Both families $\{I_X\}$ and $\{\Gamma_A\}$ are compatible with morphisms in their respective categories. In particular, for any morphism $f : (X, R_1) \to (Y, R_2)$ in **EqRel**, the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle I_X}\downarrow & & \downarrow{\scriptstyle I_Y} \\ \Sigma(C(X)) & \xrightarrow{\Sigma(C(f))} & \Sigma(C(Y)) \end{array}$$

and for any morphism $L : (A_1, B_1) \to (A_2, B_2)$ in **C*Pairs**, the following diagram also commutes:

$$\begin{array}{ccc} A_1 & \xrightarrow{L} & A_2 \\ {\scriptstyle \Gamma_{A_1}}\downarrow & & \downarrow{\scriptstyle \Gamma_{A_2}} \\ C(\Sigma(A_1)) & \xrightarrow{C(\Sigma(L))} & C(\Sigma(A_2)) \end{array}$$

Hence, the functors C and Σ establish an equivalence of categories. □

Conflicts of Interest: The authors declare no conflict of interest.

REFERENCES

1. S. I. Gelfand, Yu. I. Manin. Methods of homological algebra. 2nd ed. 2003. Springer Monographs in Mathematics. Springer-Verlag, Berlin-Heidelberg. 372p. DOI: 10.1007/978-3-662-12492-5

2. W. Rudin. Functional Analysis. 2nd ed. 1991. International Series in Pure and Applied Mathematics. McGraw-Hill, New York. 424 p. — ISBN-10: 0-07-054236-8; ISBN-13: 978-0-07-054236-5

3. K. R. Davidson. C^*-algebras by example. Fields Institute Monographs. – 1996. – Vol. 6. American Mathematical Society, Providence, RI. DOI: 10.1090/fim/006

4. A. N. Kolmogorov. On Rings of Continuous Functions on Topological Spaces. In: Selected Works of A. N. Kolmogorov. Springer, Dordrecht. – 1991. – P. 291–297.

5. G. E. Shilov. Ideals and Subrings of the Ring of Continuous Functions. Dokl. Akad. Nauk SSSR. – 1939. – Vol. *22*. P. 7–10.

6. N. Bourbaki. General Topology. Elements of Mathematics, Part I. 1989. Springer-Verlag, Berlin. ISBN: 3-540-19374-X

Article history: Received: 10 October 2025; Final form: 19 November 2025
Accepted: 20 November 2025. Published 11 December 2025.

How to cite this article:

R. Skurikhin, S. Gefter, E. Karolinsky, Closed equivalence relations on compact spaces and pairs of commutative C^*-algebras: a categorical approach, Visnyk of V. N. Karazin Kharkiv National University. Ser. Mathematics, Applied Mathematics and Mechanics, Vol. 102, 2025, p. 57–67. DOI: 10.26565/2221-5646-2025-102-04

Замкнені відношення еквівалентності на компактних просторах і пари комутативних C^*-алгебр: категорний підхід

Р. Скуріхін[1], С. Гефтер[2], Є. Каролінський[1]
[1]*Кафедра фундаментальної математики*
Харківський національний університет імені В. Н. Каразіна
площа Свободи, 4, Харків, Україна, 61022
[2]*Фізико-технічний інститут низьких температур*
ім. Б.І. Веркіна Національної академії наук України
проспект Науки, 47, Харків, Україна, 61103

У цій статті ми досліджуємо категорне узагальнення класичної дуальності Гельфанда-Наймарка між компактними хаусдорфовими просторами та комутативними унітальними C^*-алгебрами. Ми встановлюємо еквівалентність між категорією компактних хаусдорфових просторів із замкненими відношеннями еквівалентності та категорією пар, що складаються з комутативної унітальної C^*-алгебри та однієї з її унітальних C^*-підалгебр. Мотивація полягає в тому, що дуальність Гельфанда може бути збагачена додатковою структурою: замкнені відношення еквівалентності кодують фактор-простори та інваріантність з топологічного боку, тоді як підалгебри відображають обмеження та симетрії з алгебраїчного боку. Теорема Шилова, яка ототожнює замкнені самоспряжені підалгебри з одиницею $C(X)$ з алгебрами функцій, інваріантних відносно замкнених відношень еквівалентності, забезпечує ключовий зв'язок між цими підходами. Ми вводимо категорію **EqRel**, об'єктами якої є компактні хаусдорфові простори із замкненими відношеннями еквівалентності, а морфізмами — неперервні відображення, що зберігають траєкторії, та категорію $\mathbf{C^*Pairs}$, об'єктами якої є пари (A, B), де A — комутативна C^*-алгебра з одиницею, а $B \subset A$ — C^*-підалгебра з одиницею, причому морфізмами є $*$-гомоморфізми, що зберігають B. У обох напрямах визначаються контраваріантні функтори: $(X, R) \mapsto (C(X), B_R)$, де B_R складається з функцій, сталих на R-класах, і $(A, B) \mapsto (\Sigma(A), R_B)$, де $\Sigma(A)$ — спектр, а R_B пов'язує характери, що збігаються на B. Ми перевіряємо, що ці побудови є функторіальними та сумісними зі складанням морфізмів. Використовуючи теорему Колмогорова-Гельфанда, перетворення Гельфанда та теорему Шилова, ми показуємо, що ці функтори є взаємно оберненими з точністю до ізоморфізму функторів, і тим самим доводимо що $\mathbf{EqRel} \simeq \mathbf{C^*Pairs}^{\mathrm{op}}$. Цей результат показує, що геометричне поняття замкнених відношень еквівалентності на компактних просторах перебуває в повній відповідності з алгебраїчним поняттям унітальних підалгебр комутативних C^*-алгебр.

Ключові слова: **еквівалентність категорій; дуальність Гельфанда; замкнене відношення еквівалентності; комутативна C^*-алгебра; інваріантна підалгебра; теорема Шилова.**

Історія статті: отримана: 10 жовтня 2025; останній варіант: 19 листопада 2025
прийнята: 20 листопада 2025. Оприлюднено 11 грудня 2025.