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The Nevanlinna matrix of the truncated Hausdorff matrix moment problem via orthogonal matrix polynomials on $[a, b]$ for the case of an even number of moments

The scalar moment problem was first introduced by T. J. Stieltjes in his work "Recherches sur les fractions continues", Annals of the Faculty of Sciences of Toulouse 8, 1–122, (1895). He formulated it as follows: Given the moments of order k ($k = 0, 1, 2, \dots$), find a positive mass distribution on the half-line $[0, +\infty)$.

The study of matrix and operator moment problems was initiated by M. G. Krein in his seminal paper "Fundamental aspects of the representation theory of Hermitian operators with deficiency index (m, m)", Translations of the American Mathematical Society, Series II, 97, 75–143, (1949).

This paper is related to the truncated Hausdorff matrix moment (THMM) problem: the truncated moment problem on a compact interval $[a, b]$ in contrast to the Stieltjes moment problem on $[0, +\infty)$ and the Hamburger moment problem on $(-\infty, +\infty)$. Our approach relies on V. P. Potapov's method, which reformulates interpolation and moment problems as equivalent matrix inequalities and introduces auxiliary matrices that satisfy the $\widetilde{J}_q$–inner function property of the Potapov class, together with a system of column pairs.

The method begins by constructing Hankel matrices from the prescribed moments. If these matrices are positive semidefinite, the THMM problem is solvable. In the strictly positive definite case, known as the non-degenerate case, we transform the associated matrix inequalities to derive the Nevanlinna (or resolvent) matrix of the THMM problem, which characterizes its solutions.

This framework has been extensively applied, for instance in A. E. Choque Rivero, Yu. M. Dyukarev, B. Fritzsche, and B. Kirstein, "A truncated matricial moment problem on a finite interval", in Interpolation, Schur Functions and Moment Problems, Operator Theory: Advances and Applications, Birkhäuser, Basel, 165, 121–173, (2006).

The main contribution of the present work is to represent the Nevanlinna matrix of the THMM problem in terms of orthogonal matrix polynomials (OMP) and their associated polynomials of the second kind at point b. Note that the representation at point a was obtained earlier in A. E. Choque Rivero, "From the Potapov to the Krein–Nudel'man representation of the resolvent matrix of the truncated Hausdorff matrix moment problem" Bulletin of the Mexican Mathematical Society, 21(2), 233–259 (2015).

In addition, we establish new identities involving OMP and reformulate an explicit relationship between the Nevanlinna matrices of the THMM problem at points a and b, through OMP.

Keywords: **Truncated Hausdorff matrix moment problem; Nevanlinna matrix; orthogonal matrix polynomials.**

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1. Introduction

The truncated Hausdorff matrix moment (THMM) problem is stated as follows: Given an interval $[a, b]$ on the real axis and a finite sequence of $q \times q$ matrices $(s_j)_{j=0}^{m}$, where q and m are natural numbers, find the set $\mathcal{M}_{\geq}^{q}[[a, b], \mathfrak{B} \cap [a, b]; (s_j)_{j=0}^{m}]$ of all nonnegative Hermitian $q \times q$ measures σ defined on the σ-algebra of all Borel subsets of the interval $[a, b]$ such that

$$s_j = \int_{[a,b]} t^j \, d\sigma(t) \tag{1}$$

is valid for each integer j with $0 \leq j \leq m$.

The criteria for solving the THMM problem with an even number of moments (resp. an odd number of moments) are provided in [12, Theorem 1.3] (resp. [13, Theorem 1.3]). Following these results, for $m = 2n + 1$ (resp. $m = 2n$), the perturbed moments are defined as follows:

$$\begin{aligned}
s_j^{(1)} &:= s_j, && 0 \leq j \leq m, \\
s_j^{(2)} &:= -ab s_j + (a + b) s_{j+1} - s_{j+2}, && 0 \leq j \leq m - 2, \\
s_j^{(3)} &:= b s_j - s_{j+1}, && 0 \leq j \leq m - 1, \\
s_j^{(4)} &:= -a s_j + s_{j+1}, && 0 \leq j \leq m - 1.
\end{aligned}$$

Based on these perturbed moments, we construct the block Hankel matrices

$$H_{r,j} := (s_{j+k}^{(r)})_{j,k=0}^{j} = \begin{pmatrix} s_0^{(r)} & s_1^{(r)} & \cdots & s_j^{(r)} \\ s_1^{(r)} & s_2^{(r)} & \cdots & s_{j+1}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ s_j^{(r)} & s_{j+1}^{(r)} & \cdots & s_{2j}^{(r)} \end{pmatrix} \tag{2}$$

for $r = 1, 2, 3$ and 4.

It was proven in [13, Theorem 1.3] (resp. [12, Theorem 1.3]) that the THMM problem has a solution if and only if the block matrices $H_{1,n}$ and $H_{2,n-1}$ (resp. $H_{3,n}$ and $H_{4,n}$) are both nonnegative Hermitian.

To characterize the solution set of the THMM problem $\mathcal{M}^q_{\geq}[[a,b],\mathfrak{B}\cap[a,b];(s_j)^m_{j=0}]$ for $m = 2n$ and $m = 2n+1$, the problem is usually reformulated by identifying an associated class of holomorphic matrix-valued functions given by

$$\mathfrak{S}^q_{\geq}[[a,b],\mathfrak{B}\cap[a,b];(s_j)^m_{j=0}] := \left\{ s(z) = \int_{[a,b]} \frac{d\sigma(t)}{t-z},\ \sigma \in \mathcal{M}^q_{\geq}[[a,b],\mathfrak{B}\cap[a,b];(s_j)^m_{j=0}] \right\}.$$

A matrix function $s(z) \in \mathfrak{S}^q_{\geq}[[a,b],\mathfrak{B}\cap[a,b];(s_j)^m_{j=0}]$ is called the associated solution to the THMM problem. This technique, commonly referred to as V. P. Potapov's method [26], has been successfully applied in numerous works, including [4, 5, 6, 14, 19, 20, 21] and others.

The THMM problem is said to be non-degenerate when both block matrices $H_{1,n}$ and $H_{2,n-1}$ (resp. $H_{3,n}$ and $H_{4,n}$) are positive definite Hermitian.

A description of the solution set of the THMM problem, which encompasses both degenerate and non-degenerate cases, is provided in [24] through a function-theoretic Schur-Nevanlinna-type algorithm. An algebraic version of this procedure, which is applicable to (finite or infinite) sequences of complex $q \times q$ matrices, was developed based on the Schur analysis of matrix Hausdorff moment sequences [22, 23]. See also [25].

Henceforth, we focus exclusively on the non-degenerate case.

Definition 1. *[15, Definition 1.1]. Let $[a,b]$ be a finite interval on real axis $\mathbb{R}$. The sequence of $q \times q$ matrices $(s_j)^{2n}_{j=0}$ (resp. $(s_j)^{2n+1}_{j=0}$) is called a Hausdorff positive definite sequence on $[a,b]$ if the block Hankel matrices $H_{1,n}$ and $H_{2,n-1}$ (resp. $H_{3,n}$ and $H_{4,n}$) are both positive definite matrices.*

Throughout this paper, we restrict our attention to sequences that are Hausdorff positive definite on $[a,b]$.

According to Definition 1, the THMM problem is also considered non-degenerate when the sequence $(s_j)^m_{j=0}$, for $m = 2n$ and $m = 2n+1$ is positive definite on $[a,b]$. In such cases, the corresponding solution $s(z)$ to the THMM problem is given by

$$s(z) = \left(\alpha^{(m)}(z)\,\mathbf{p}(z) + \beta^{(m)}(z)\,\mathbf{q}(z)\right)\left(\gamma^{(m)}(z)\,\mathbf{p}(z) + \delta^{(m)}(z)\,\mathbf{q}(z)\right)^{-1}, \qquad (3)$$

where $\mathbf{p}$ and $\mathbf{q}$ denote $q \times q$ matrix-valued functions of the complex variable z, which are defined in an appropriate domain in the complex plane. See [12, Definition 5.2] and [13, Definition 5.2]. The functions $\alpha^{(m)}(z)$, $\beta^{(m)}(z)$, $\gamma^{(m)}(z)$, and $\delta^{(m)}(z)$ are matrix-valued polynomials constructed from the given moment sequence $\{s_j\}^m_{j=0}$. These matrices $\alpha^{(m)}(z)$, $\beta^{(m)}(z)$, $\gamma^{(m)}(z)$, and $\delta^{(m)}(z)$ collectively constitute the entries of the Nevanlinna matrix

$$U^{(m)}(z) := \begin{bmatrix} \alpha^{(m)}(z) & \beta^{(m)}(z) \\ \gamma^{(m)}(z) & \delta^{(m)}(z) \end{bmatrix}, \tag{4}$$

which is linked to the THMM problem. The Nevanlinna matrix was first generally defined in [1, Definition 2.4.3, p. 55]. Within the THMM problem, this matrix is also called the resolvent matrix associated with the THMM problem. The Nevanlinna matrix $U^{(m)}(z)$ is a $2q \times 2q$ matrix polynomial defined on the entire complex plane. This matrix is vital in analyzing the solution to the THMM problem; see Equations (3) and (4).

As presented in [18], the Nevanlinna matrix of the THMM problem was constructed regarding to point $z = 0$. In the same work [18], both even and odd number of moments were considered. In [12] and [13], the Nevanlinna matrix for the THMM problem was examined at point $z = a$, specifically for the even and odd cases of moments, respectively. Furthermore, [15] introduced a novel Nevanlinna matrix that includes both even and odd moment cases and is constructed with respect to point $z = b$.

Similar procedures to those described in [12] and [13] can help construct the Nevanlinna matrix regarding to point $z = b$

$$\widehat{V}^{(m)}(z) := \begin{bmatrix} \widehat{\alpha}^{(m)}(z) & \widehat{\beta}^{(m)}(z) \\ \widehat{\gamma}^{(m)}(z) & \widehat{\delta}^{(m)}(z) \end{bmatrix}. \tag{5}$$

The representation of the Nevanlinna matrix at point $z = b$ is crucial, as its components define the solution set of the THMM problem (Equation (3)). Furthermore, constructing the Nevanlinna matrix at $z = b$ enables the derivation of new relationships between OMP, Dyukarev-Stieltjes parameters, matrix continued fractions (see [2, 3, 4, 7, 8]), and Blaschke-Potapov factors [5, 6]. Moreover, the admissible control problem and the time optimal control problem may be solved by using the Nevanlinna matrix with respect to point $z = b$. See [9], [10], and [11].

For $m = 2n + 1$ (resp. $m = 2n$), an explicit relationship was established between the Nevanlinna matrix $U^{(m)}$ regarding to point $z = a$, introduced in [12, Proposition 6.10] (resp. [13, Proposition 6.10]), and the Nevanlinna matrix $\widehat{V}^{(m)}$ constructed in [15, Definition 4.1] (resp. [15, Definition 3.1]) with respect to point $z = b$. This relation takes the form

$$U^{(m)}(z)\,\mathfrak{D}^{(m)} = \widehat{V}^{(m)}(z), \tag{6}$$

where $\mathfrak{D}^{(m)}$ is a constant invertible matrix. The relation (6) was proven in [15, Theorem 4.3] (resp. [15, Theorem 3.8]).

Under specific conditions, an additional explicit connection was established between the Nevanlinna matrices: one evaluated at point $z = a$ [12], and the other at $z = 0$ [18].

Next, regarding [14], we provide a brief review of general notation related to matrix polynomials.

We will use $\mathbb{R}$, and $\mathbb{N}_0$ to denote the set of real numbers, and nonnegative integers, respectively. Through 0_q, and I_q, we denote the $q \times q$ zero matrix, and the $q \times q$ identity matrix, respectively.

A matrix polynomial is an expression of the form $P(t) = A_0 t^n + A_1 t^{n-1} + \ldots + A_{n-1}t + A_n$, where $t \in \mathbb{R}$ and each coefficient A_k is a $q \times q$ matrix, with $A_0 \neq 0$. Here, the degree of P is n, denoted by $\deg P := n$. If A_0 equals the identity matrix, the polynomial is called monic. Note that if $P(t) \equiv 0_q$ for all $t \in \mathbb{R}$, then $\deg P := -\infty$. When $\deg P = n \geq 0$, the matrix A_n is referred to as the leading coefficient of P. For all $\ell \in \mathbb{N}_0$ and $\kappa \in \mathbb{N}_0$ with $\ell \leq \kappa$, we define the index set $\mathbb{Z}_{\ell,\kappa} := \{n \in \mathbb{N}_0 \mid \ell \leq n \leq \kappa\}$.The following remark was partially reproduced from [14, Definition 3.2] and [14, Remark 3.6].

Remark 1. *Let $n \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{2n}$ be a Hausdorff positive definite sequence: The corresponding block Hankel matrix $H_{1,n}$ is positive definite. Let σ be a nonnegative Hermitian $q \times q$ measure on $\mathbb{R}$ satisfying (1) for $0 \leq j \leq 2n$. A sequence $(P_j)_{j=0}^{n}$ of complex $q \times q$ matrix polynomials is called a monic left orthogonal system of matrix polynomials with respect to σ if the three conditions below are fulfilled.*

(I) $\deg P_j = j$ *for all* $j \in \mathbb{Z}_{0,n}$.

(II) P_j *has the leading coefficient* I_q *for all* $j \in \mathbb{Z}_{0,n}$.

(III) The following equality is satisfied:

$$\int_{[a,b]} P_j d\sigma P_k^* = \begin{cases} \widehat{H}_{1,j}, & \text{if } j = k, \\ 0_q, & \text{if } j \neq k \end{cases}$$

for all $0 \leq j,\ k \leq n$ *where* $\widehat{H}_{1,j}$ *denotes the Schur complement of* $H_{1,j-1}$ *in* $H_{1,j}$. *See Definition 4.*

Let σ be a nonnegative Hermitian measure on the Borel sets of $[a, b]$, and let B be a Borel set of $[a, b]$. Denote $\sigma_1 := \sigma$. Let us introduce the following perturbed measures:

$$\sigma_2(B) := \int_B (b-t)(t-a)\sigma(dt),$$
$$\sigma_3(B) := \int_B (t-a)\sigma(dt),$$
$$\sigma_4(B) := \int_B (b-t)\sigma(dt).$$

In Definitions 2 and 3, we introduced four monic orthogonal systems of matrix polynomials, $(P_{r,j})_{j=0}^{n}$ for $r = 1, 2, 3, 4$. As shown in Proposition 1, $(P_{r,j})_{j=0}^{n}$ for

$r = 1, 3, 4$ (resp. $(P_{2,j})_{j=0}^{n-1}$) are orthogonal with respect to σ_r for $r = 1, 3, 4$, (resp. σ_2).

We now revisit key aspects of the Nevanlinna matrix (4) in relation to OMP. The Nevanlinna matrix associated with the THMM problem was initially formulated by using OMP for an even number of moments in [28]; an odd number of moments was first explored in [3]. Furthermore, [7] introduced alternative representations of the Nevanlinna matrix via OMP, specifically at point $z = a$.

Moreover, explicit relationships between Nevanlinna matrices expressed with OMP have been established.

In [17], an explicit relationship between Nevanlinna matrices through OMP was presented. In this relationship, the Nevanlinna matrix, which was obtained in [18] regarding to point $z = 0$, was considered. Additionally, this relation involved the Nevanlinna matrices introduced in [12] and [13], both with respect to point $z = a$.

Main results of the work

In this work, we consider the case of an even number of given moments.

a) Every block of the Nevanlinna matrix of the THMM problem at point $z = b$ admits an explicit representation via OMP on $[a, b]$ and their polynomials of the second kind; see Theorem 2.

b) In [15, Theorem 4.3], an explicit relationship was obtained between the Nevanlinna matrices of the THMM problem regarding to point $z = a$ and point $z = b$. In the present work, we establish an explicit relation between the Nevanlinna matrices of the THMM problem with respect to point $z = a$ and point $z = b$ via OMP.

This paper is organized as follows. In Sections 2 and 3, notations and algebraic identities are introduced, respectively. Furthermore, the orthogonality of the polynomials introduced in Definitions 2 and 3 appear in Section 4. In Section 5, we represent the Nevanlinna matrix of the THMM problem at point $z = b$ through OMP for an even number of moments. In Section 6, we obtain identities related to the OMP defined in Definitions 2 and 3. Finally, Section 7, presents an explicit relation for an even number of moments between the Nevanlinna matrices of the THMM problem regarding to point $z = a$ and point $z = b$ via OMP.

2. Notations and preliminaries

In this section, we reproduce some matrix notations from [17] that appear throughout this work.

We will use $\mathbb{C}$ to denote the set of complex numbers. Through $\mathbb{C}^{p\times q}$, and $0_{p\times q}$, we denote the $p \times q$ complex-valued matrices, and the $p \times q$ zero matrix, respectively. Let us recall that 0_q, and I_q, denote the $q \times q$ zero matrix, and the $q \times q$ identity matrix, respectively. In cases where the sizes of the null and the identity matrix are clear, we will omit the indices.

Let $R_j : \mathbb{C} \to \mathbb{C}^{(j+1)q\times(j+1)q}$ be defined by

$$R_j(z) := (I_{(j+1)q} - zT_j)^{-1}, \qquad j \in \mathbb{N}_0, \tag{7}$$

with

$$T_0 := 0_q, \quad T_j := \begin{pmatrix} 0_{q\times jq} & 0_q \\ I_{jq} & 0_{jq\times q} \end{pmatrix}, \qquad j \in \mathbb{N}_0. \tag{8}$$

Additionally, for $j \in \mathbb{N}_0$ let

$$v_0 := I_q, \quad v_j := \operatorname{column}(I_q,\ 0_{jq\times q}). \tag{9}$$

For each positive integer j such that $1 \le j \le n$, let

$$L_{1,j} := (\delta_{i,k+1} I_q)_{\substack{i=0,\dots,j\\ k=0,\dots,j-1}} \quad \text{and} \quad L_{2,j} := (\delta_{i,k} I_q)_{\substack{i=0,\dots,j\\ k=0,\dots,j-1}}, \tag{10}$$

where $\delta_{i,k}$ denotes the Kronecker symbol defined by $\delta_{i,k} := 1$ if $i = k$, and $\delta_{i,k} := 0$ if $i \neq k$.
For $0 \le j \le k$, we set

$$y_{[j,k]} := \operatorname{column}(s_j, s_{j+1}, \dots, s_k), \quad \widehat{y}_{[j,k]} := \operatorname{column}\left(s_j^{(2)}, s_{j+1}^{(2)}, \dots, s_k^{(2)}\right). \tag{11}$$

For $j \in \mathbb{N}_0$, we define the following auxiliary matrices:

$$\widetilde{H}_{1,j} := (s_{k+\ell+1})_{\ell,k=0}^{j}, \quad \widetilde{H}_{2,j} := (s_{k+\ell+2})_{\ell,k=0}^{j}, \tag{12}$$

$$u_j := \operatorname{column}(-s_0, -s_1, \dots, -s_j). \tag{13}$$

Let $n \in \mathbb{N}$, and let $(s_j)_{j=0}^{2n}$ be a sequence of complex $q \times q$ matrices. Define

$$u_{1,0} := 0_q, \quad u_{1,j} := \operatorname{column}\left(0_q,\ -y_{[0,j-1]}\right), \quad 1 \le j \le n, \tag{14}$$

$$u_{2,0} := -(a+b)s_0 + s_1,$$

$$u_{2,j-1} := \operatorname{column}\left(u_{2,0},\ -\widehat{y}_{[0,j-2]}\right), \quad 1 \le j \le n-1. \tag{15}$$

Now let $(s_j)_{j=0}^{2n+1}$ be a sequence of complex $q \times q$ matrices. We set

$$u_{3,0} := s_0, \quad u_{3,j} := y_{[0,j]} - b\operatorname{column}\left(0_q,\ y_{[0,j-1]}\right), \quad 1 \le j \le n, \tag{16}$$

$$u_{4,0} := -s_0, \quad u_{4,j} := -y_{[0,j]} + a\operatorname{column}\left(0_q,\ y_{[0,j-1]}\right), \quad 1 \le j \le n. \tag{17}$$

Let $y_{[j,k]}$ and $\widehat{y}_{[j,k]}$ be as in (11). Define

$$Y_{1,j} := y_{[j,2j-1]}, \quad 1 \le j \le n, \quad Y_{2,j} := \widehat{y}_{[j,2j-1]}, \quad 1 \le j \le n-1, \tag{18}$$

$$Y_{3,j} := b\, y_{[j,2j-1]} - y_{[j+1,2j]}, \quad Y_{4,j} := -a\, y_{[j,2j-1]} + y_{[j+1,2j]}, \quad 1 \le j \le n. \tag{19}$$

Finally, let $H_{r,j}$ and $Y_{r,j}$, for $r = 1, 2, 3$, and 4 be as in (2), (18), and (19). We denote

$$\Sigma_{r,j} := \operatorname{column}\left(-H_{r,j-1}^{-1} Y_{r,j},\ I_q\right) \tag{20}$$

for $r = 1, 3, 4$ (resp. $r = 2$), with $1 \le j \le n$ (resp. $1 \le j \le n-1$).

In Theorem 2 we obtain a representation of the Nevanlinna matrix of the THHM problem in terms of the matrix polynomials introduced below. These polynomials were first defined in [3]. Their orthogonality will be discussed later in Proposition 1.

Definition 2. *Let* $(s_k)_{k=0}^{2j}$ *be a sequence that is Hausdorff positive definite on* $[a,b]$*. Let* $u_{r,j}$*,* $\Sigma_{r,j}$ *for* $r=1,2$*,* R_j*, and* v_j *be defined by (14), (15), (20), (7), and (9), respectively. We define for all* $z\in\mathbb{C}$

$$P_{1,0}^*(\bar z) := I_q, \quad Q_{1,0}^*(\bar z) := 0_q, \quad P_{2,0}^*(\bar z) := I_q, \quad Q_{2,0}^*(a,b,\bar z) := -(u_{2,0}^* + zs_0), \tag{21}$$

$$P_{1,j}^*(\bar z) := v_j^* R_j^*(\bar z)\Sigma_{1,j}, \qquad 1\le j\le n, \tag{22}$$

$$Q_{1,j}^*(\bar z) := -u_{1,j}^* R_j^*(\bar z)\Sigma_{1,j}, \qquad 1\le j\le n, \tag{23}$$

$$P_{2,j}^*(a,b,\bar z) := v_j^* R_j^*(\bar z)\Sigma_{2,j}, \qquad 1\le j\le n-1, \tag{24}$$

$$Q_{2,j}^*(a,b,\bar z) := -(u_{2,j}^* + zs_0 v_j^*) R_j^*(\bar z)\Sigma_{2,j}, \qquad 1\le j\le n-1. \tag{25}$$

The matrix polynomials $Q_{1,j}$ *and* $Q_{2,j}$ *are polynomials of the second kind with respect to* $P_{1,j}$ *and* $P_{2,j}$*.*

Remark 2. *For brevity, we will frequently omit the parameters* a *and* b *in* $P_{2,j}^*$ *and* $Q_{2,j}^*$*. Specifically, rather than writing* $P_{2,j}^*(a,b,\bar z)$ *and* $Q_{2,j}^*(a,b,\bar z)$*, we use the simplified expressions* $P_{2,j}^*(\bar z)$ *and* $Q_{2,j}^*(\bar z)$*. In particular, when* $z=a$ *or* $z=b$*, we will write* $P_{2,j}^*(a)$*,* $Q_{2,j}^*(a)$*,* $P_{2,j}^*(b)$*, and* $Q_{2,j}^*(b)$*, respectively.*

The matrix polynomials introduced below will be employed in Lemma 6 of Section 5. They were first defined in [28], and their orthogonality will be examined later in Proposition 1.

Definition 3. *Let* $(s_k)_{k=0}^{2j+1}$ *be a sequence that is Hausdorff positive definite on* $[a,b]$*. Let* $u_{r,j}$*,* $\Sigma_{r,j}$ *for* $r=3,4$*,* R_j*, and* v_j *be defined by (16), (17), (20), (7), and (9), respectively. We define for all* $z\in\mathbb{C}$ *and* $1\le j\le n$

$$P_{3,0}^*(\bar z) := I_q, \quad Q_{3,0}^*(\bar z) := s_0, \quad P_{4,0}^*(\bar z) := I_q, \quad Q_{4,0}^*(\bar z) := -s_0, \tag{26}$$

$$P_{3,j}^*(b,\bar z) := v_j^* R_j^*(\bar z)\Sigma_{3,j}, \tag{27}$$

$$Q_{3,j}^*(b,\bar z) := u_{3,j}^* R_j^*(\bar z)\Sigma_{3,j}, \tag{28}$$

$$P_{4,j}^*(a,\bar z) := v_j^* R_j^*(\bar z)\Sigma_{4,j}, \tag{29}$$

$$Q_{4,j}^*(a,\bar z) := u_{4,j}^* R_j^*(\bar z)\Sigma_{4,j}. \tag{30}$$

The matrix polynomials $Q_{3,j}$ *and* $Q_{4,j}$ *are polynomials of the second kind with respect to* $P_{3,j}$ *and* $P_{4,j}$*.*

As in the Remark 2, we will often omit the parameters a and b in the polynomials $P_{3,j}^*$, $Q_{3,j}^*$, $P_{4,j}^*$, and $Q_{4,j}^*$.

3. Main algebraic identities

Here we present the identities that will express the Nevanlinna matrix of the THMM problem through OMP in Section 5.

The following definition below is based on [7, Equations (2.9)–(2.10), (2.19)–(2.20)].

Definition 4. *Let* $(s_k)_{k=0}^m$ *for* $m = 2j$ *(resp.* $m = 2j+1$*) be a sequence that is Hausdorff positive definite on* $[a, b]$. *Let* $H_{r,j}$, $Y_{r,j}$ *for* $r = 1, 2, 3,$ *and* 4 *be defined by (2), (18), and (19). Let* $\widehat{H}_{r,j}$ *denote the Schur complement of the block matrix* $H_{r,j-1}$ *in* $H_{r,j}$*:*

$$\widehat{H}_{r,0} := s_0^{(r)}, \quad \widehat{H}_{r,j} := s_{2j}^{(r)} - Y_{r,j}^* H_{r,j-1}^{-1} Y_{r,j}, \tag{31}$$

for $r = 1, 3, 4$ *(resp.* $r = 2$*) and for* $1 \le j \le n$ *(resp.* $1 \le j \le n-1$*).*

In Lemmas 1, 2, and 3, we introduce a collection of auxiliary identities for the block Hankel matrices and the block matrices introduced in (7)–(12), as well as in Definition 4.

Lemma 1. *Let* $(s_k)_{k=0}^m$ *for* $m = 2j$ *(resp.* $m = 2j+1$*) be a sequence that is Hausdorff positive definite on* $[a, b]$. *Let* T_j *and* $L_{2,j}$ *be defined as in (8) and (10). Let* $H_{r,j}$, $\widehat{H}_{r,j}$, *and* $\Sigma_{r,j}$ *for* $r = 1, 2, 3,$ *and* 4 *be defined by (2), (31), and (20). Therefore, the following identities hold:*

$$H_{r,j}^{-1} = \begin{pmatrix} H_{r,j-1}^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \Sigma_{r,j} \widehat{H}_{r,j}^{-1} \Sigma_{r,j}^*, \tag{32}$$

$$T_j H_{r,j} \Sigma_{r,j} = 0, \tag{33}$$

$$L_{2,j}^* H_{r,j} \Sigma_{r,j} = 0, \tag{34}$$

for $r = 1, 3, 4$ *(resp.* $r = 2$*) and for* $1 \le j \le n$ *(resp.* $1 \le j \le n-1$*).*

Proof. The identity (32) can be proven as in [3, pages 935–936]. The identities (33) and (34) are proven in [17, Corollary 2.2]. □

Lemma 2. *Let* T_j, v_j, u_j, $u_{1,j}$, $L_{1,j}$, $L_{2,j}$, $\widetilde{H}_{1,j}$, $\widetilde{H}_{2,j}$, *and* $H_{1,j}$ *be defined by (8), (9), (13), (14), (10), (12), and (2) for* $r = 1$, *respectively. Thus, for* $0 \le j \le n$ *(resp.* $1 \le j \le n$*) the following identities are valid:*

$$T_j \widetilde{H}_{1,j} - H_{1,j} + v_j v_{j+1}^* H_{1,j+1} L_{2,j+1} = 0, \tag{35}$$

$$T_j \widetilde{H}_{2,j} - \widetilde{H}_{1,j} + v_j v_{j+1}^* \widetilde{H}_{1,j+1} L_{2,j+1} = 0, \tag{36}$$

$$L_{1,j}^* H_{1,j} - L_{2,j}^* \widetilde{H}_{1,j} = 0, \tag{37}$$

$$L_{2,j}^* H_{1,j} + u_{j-1} v_j^* - \widetilde{H}_{1,j-1} L_{1,j}^* = 0, \tag{38}$$

$$T_j u_j - u_{1,j} = 0, \tag{39}$$

$$u_j^* - u_{j+1}^* L_{2,j+1} = 0, \tag{40}$$

$$u_j^* + v_j^* H_{1,j} = 0, \tag{41}$$

$$L_{1,j} L_{1,j}^* - T_j T_j^* = 0, \tag{42}$$

$$L_{2,j} T_{j-1}^* - T_j^* L_{2,j} = 0, \tag{43}$$

$$u_{1,j}^* + v_j^* H_{1,j} T_j^* = 0, \tag{44}$$

$$v_j - L_{2,j+1}^* v_{j+1} = 0, \tag{45}$$

$$L_{2,j} L_{1,j}^* - T_j^* = 0. \tag{46}$$

Proof. The identities can be derived through direct calculations. □

Lemma 3. *Let $w, z \in \mathbb{C}$. Let R_j, T_j, v_j, u_j, $u_{1,j}$, $u_{2,j-1}$, $u_{3,j}$, $u_{4,j}$, $L_{1,j}$, $L_{2,j}$, and $H_{1,j}$ be defined by (7), (8), (9), (13), (14), (15), (16), (17), (10), and (2) for $r = 1$, respectively. Therefore, for $0 \le j \le n$ (resp. $1 \le j \le n$) the following identities are valid:*

$$R_j(z) - R_j(w) = (z-w)R_j(z)T_jR_j(w), \tag{47}$$
$$zR_j(z) - wR_j(w) = (z-w)R_j(z)R_j(w), \tag{48}$$
$$R_j^{*^{-1}}(\bar{z}) + (z-a)T_j^* - R_j^{*^{-1}}(a) = 0, \tag{49}$$
$$u_{1,j}^*R_j^*(\bar{z})(T_j^*L_{1,j} - L_{1,j}T_{j-1}^*) = 0, \tag{50}$$
$$u_{3,j} = -R_j^{-1}(b)u_j, \quad u_{4,j} = R_j^{-1}(a)u_j, \tag{51}$$
$$R_j^{*^{-1}}(\bar{z})L_{2,j}R_{j-1}^*(\bar{z}) - L_{2,j} = 0, \tag{52}$$
$$(u_{2,j-1}^* + bs_0v_{j-1}^*)R_{j-1}^*(b) - v_j^*H_{1,j}(L_{1,j} - aL_{2,j}) = 0. \tag{53}$$

Proof. The identities from (47) to (52) can be verified through direct calculation. A similar identity to (53) is established in [2, Proposition 3.4]. □

We recall the linear relations between the block Hankel matrices $H_{r,j}$ and the auxiliary matrices $\widetilde{H}_{1,j}$, $\widetilde{H}_{2,j}$. These relations were introduced in [7, Equations (1.5)–(1.6)].

Remark 3. *Let $\widetilde{H}_{1,j}$, $\widetilde{H}_{2,j}$, and $H_{r,j}$ be defined by (12) and (2) for $r = 1, 2, 3,$ and 4. Thus, for $0 \le j \le n$ the following identities hold:*

$$H_{2,j} = -abH_{1,j} + (a+b)\widetilde{H}_{1,j} - \widetilde{H}_{2,j}, \tag{54}$$
$$H_{3,j} = bH_{1,j} - \widetilde{H}_{1,j}, \tag{55}$$
$$H_{4,j} = -aH_{1,j} + \widetilde{H}_{1,j}. \tag{56}$$

In the following lemma, we obtain new coupling identities concerning the block matrices that were introduced in (7)–(12), as well as the block Hankel matrices introduced in (2).

Lemma 4. *Let $z \in \mathbb{C}$. Let R_j, T_j, v_j, u_j, $u_{1,j}$, $u_{2,j-1}$, $L_{1,j}$, $L_{2,j}$, and $H_{r,j}$ be defined by (7), (8), (9), (13), (14), (15), (10), and (2) for $r = 1, 2, 3,$ and 4,* *respectively. For $0 \leq j \leq n$ (resp. $1 \leq j \leq n$) the following identities hold:*

$$-R_j^{*^{-1}}(\bar{z}) + (b-z)T_j^* + R_j^{*^{-1}}(b) = 0, \tag{57}$$
$$-v_{j+1}^* L_{2,j+1} H_{1,j} + v_{j+1}^* H_{1,j+1} L_{2,j+1} = 0, \tag{58}$$
$$H_{1,j-1} L_{1,j}^* - L_{2,j}^* H_{1,j} T_j^* = 0, \tag{59}$$
$$u_{2,j-1}^* T_{j-1}^* + s_0 v_{j-1}^* + u_{1,j}^*(L_{1,j}(I - bT_{j-1}^*) - a(I - bT_j^*)L_{2,j}) = 0, \tag{60}$$
$$R_{j-1}^*(b) L_{1,j}^* R_j^{*^{-1}}(b) - L_{1,j}^* = 0, \tag{61}$$
$$u_{j-1} v_j^* + H_{3,j-1} L_{1,j}^* + L_{2,j}^* H_{1,j} R_j^{*^{-1}}(b) = 0, \tag{62}$$
$$u_j v_{j+1}^* R_{j+1}^*(b) + H_{3,j} R_j^*(b) L_{1,j+1}^* + L_{2,j+1}^* H_{1,j+1} = 0, \tag{63}$$
$$-v_j v_{j+1}^* H_{1,j+1}(L_{1,j+1} - aL_{2,j+1}) + (I - bT_j)H_{4,j} + T_j H_{2,j} = 0, \tag{64}$$
$$(u_{2,j}^* + zs_0 v_j^*)R_j^*(\bar{z}) - (u_{2,j}^* + bs_0 v_j^*)R_j^*(b) + (z-b)u_{1,j+1}^* R_{j+1}^*(\bar{z}) \cdot (L_{1,j+1} - aL_{2,j+1}) = 0. \tag{65}$$

Proof. The identities (57)–(61) are established by straightforward computation. Moreover, (62) is obtained from (55), (38), and (59). By combining (61) with (62), we have (63). Identity (64) follows from (56), (54), (37), (35), and (36).

Let $\Delta_{(65)}$ be the left-hand side of (65). By using (47) and (48), we have for all $z \in \mathbb{C}$

$$\begin{aligned}
\Delta_{(65)} &= u_{2,j-1}^*(R_{j-1}^*(b) - R_{j-1}^*(\bar{z})) + s_0 v_{j-1}^*(bR_{j-1}^*(b) - zR_{j-1}^*(\bar{z})) \\
&\quad - (z-b)u_{1,j}^* R_j^*(\bar{z})(L_{1,j} - aL_{2,j}) \\
&= (b-z)\left[(u_{2,j-1}^* T_{j-1}^* + s_0 v_{j-1}^*)R_{j-1}^*(\bar{z})R_{j-1}^*(b) + u_{1,j}^* R_j^*(\bar{z})(L_{1,j} - aL_{2,j})\right. \\
&= (b-z)\left(-u_{1,j}^*[L_{1,j}(I - bT_{j-1}^*) - a(I - bT_j^*)L_{2,j}]R_{j-1}^*(\bar{z})R_{j-1}^*(b)\right. \\
&\quad \left.+ u_{1,j}^* R_j^*(\bar{z})(L_{1,j} - aL_{2,j})\right) \\
&= (b-z)u_{1,j}^*\left(-[L_{1,j} - aL_{2,j}]R_{j-1}^*(\bar{z}) + R_j^*(\bar{z})(L_{1,j} - aL_{2,j})\right) \\
&= z(b-z)u_{1,j}^* R_j^*(\bar{z})\left[T_j^*(L_{1,j} - aL_{2,j}) - (L_{1,j} - aL_{2,j})T_{j-1}^*\right] R_{j-1}^*(\bar{z}) \\
&= z(b-z)u_{1,j}^* R_j^*(\bar{z})(T_j^* L_{1,j} - L_{1,j} T_{j-1}^*)R_{j-1}^*(\bar{z}) \\
&= 0.
\end{aligned}$$

The third equality follows from (60), whereas (52) was used in the fourth equality. The fifth equality follows from (43), whereas (50) was used in the last equality. □

4. Orthogonality of matrix polynomials

The proposition below presents the orthogonality of the matrix polynomials $P_{r,j}$ for $r = 1, 2, 3$, and 4 on the interval $[a, b] \subset \mathbb{R}$. This result is partially adapted from [7, Proposition 2.5] where we restrict attention to parts a) and b).

Proposition 1. *Let $\widehat{H}_{r,j}$ for $r = 1, 2, 3$, and 4 be as in (31). Let $P_{r,j}$ and $Q_{r,j}$ for $r = 1, 2, 3$, and 4 be the matrix polynomials introduced in Definitions 2 and 3.*

a) *The polynomials* $P_{1,j}$ *and* $P_{2,j}$ *are OMP on* $[a,b]$ *with respect to* $\sigma(dt)$ *and* $(b-t)(t-a)\sigma(dt)$, *respectively. More precisely,*

$$\int_{[a,b]} P_{r,j}(t)((b-t)(t-a))^{r-1} d\sigma(t) P^*_{r,\ell}(t) = \begin{cases} 0_q & j \neq \ell \\ \widehat{H}_{r,j} & j = \ell, \end{cases} \quad r = 1, 2.$$

b) *The polynomials* $P_{3,j}$ *and* $P_{4,j}$ *are OMP on* $[a,b]$ *with respect to* $(b-t)\sigma(dt)$ *and* $(t-a)\sigma(dt)$, *respectively. Specifically,*

$$\int_{[a,b]} P_{3,j}(t)(b-t) d\sigma(t) P^*_{3,\ell}(t) = \begin{cases} 0_q & j \neq \ell \\ \widehat{H}_{3,j} & j = \ell, \end{cases}$$

$$\int_{[a,b]} P_{4,j}(t)(t-a) d\sigma(t) P^*_{4,\ell}(t) = \begin{cases} 0_q & j \neq \ell \\ \widehat{H}_{4,j} & j = \ell. \end{cases}$$

Proof. Part *a*) is proven in [3, Section 4]. Part *b*) is proven in [28, Theorems 2.12 and 2.13]. □

5. The Nevanlinna matrix of the THMM problem at point $z = b$ via OMP

This section focuses on the representation of the Nevanlinna matrix $\widehat{V}^{(2j+1)}$ associated with the THMM problem through OMP regarding to point $z = b$, for an even number of moments and specifically when $m = 2j + 1$ in the sequence $(s_k)_{k=0}^m$.

Let us reproduce the Nevanlinna matrix $\widehat{V}^{(2j+1)}(z)$ for the case of an even number of moments obtained in [15, Definition 4.1].

Definition 5. *Let* $(s_k)_{k=0}^{2j+1}$ *be a sequence that is Hausdorff positive definite on* $[a,b]$. *Let* $H_{r,j}$ *for* $r = 3, 4$, R_j, v_j, $u_{3,j}$, *and* $u_{4,j}$ *be defined as in (2) for* $r = 3, 4$, *(7), (9), (17), and (16), respectively. The* $2q \times 2q$ *matrix polynomial*

$$\widehat{V}^{(2j+1)}(z) := \begin{pmatrix} \widehat{\alpha}^{(2j+1)}(z) & \widehat{\beta}^{(2j+1)}(z) \\ \widehat{\gamma}^{(2j+1)}(z) & \widehat{\delta}^{(2j+1)}(z) \end{pmatrix}, \quad 0 \leq j \leq n, \tag{66}$$

with

$$\widehat{\alpha}^{(2j+1)}(z) := I + (b-z) u^*_{4,j} R^*_j(\bar{z}) H^{-1}_{4,j} R_j(b) v_j, \tag{67}$$

$$\widehat{\gamma}^{(2j+1)}(z) := (b-z)(z-a) v^*_j R^*_j(\bar{z}) H^{-1}_{4,j} R_j(b) v_j, \tag{68}$$

$$\widehat{\beta}^{(2j+1)}(z) := -u^*_{3,j} R^*_j(\bar{z}) H^{-1}_{3,j} R_j(b) u_{3,j}, \tag{69}$$

$$\widehat{\delta}^{(2j+1)}(z) := I - (b-z) v^*_j R^*_j(\bar{z}) H^{-1}_{3,j} R_j(b) u_{3,j}, \tag{70}$$

is called the Nevanlinna matrix of the THMM problem with respect to point $z = b$ *in the case of an even number of moments.*

In the analysis that follows, which includes Definition 5, we omit the explicit dependence on the parameters a and b in the notation for the matrix-valued functions $\widehat{\alpha}^{(2j+1)}$, $\widehat{\beta}^{(2j+1)}$, $\widehat{\gamma}^{(2j+1)}$, $\widehat{\delta}^{(2j+1)}$, as well as in the Nevanlinna matrix $\widehat{V}^{(2j+1)}$.

The lemma stated below is vital in deriving the results presented in Lemma 6.

Lemma 5. *Let $\widehat{H}_{r,j}$, for $r = 3,4$ be as in (31). Let $\widehat{\alpha}^{(2j+1)}$, $\widehat{\beta}^{(2j+1)}$, $\widehat{\gamma}^{(2j+1)}$, and $\widehat{\delta}^{(2j+1)}$ be as in (67), (69), (68), and (70), respectively. Furthermore, let $P_{r,j}$ and $Q_{r,j}$ for $r = 3,4$ be the OMP and their polynomials of the second kind in Definition 3. Therefore, for all $z \in \mathbb{C}$ and $1 \leq j \leq n$, the following identities hold:*

$$\widehat{\alpha}^{(2j+1)}(z) - \widehat{\alpha}^{(2(j-1)+1)}(z) = (b-z)Q^*_{4,j}(\bar{z})\widehat{H}^{-1}_{4,j}P_{4,j}(b), \tag{71}$$

$$\widehat{\beta}^{(2j+1)}(z) - \widehat{\beta}^{(2(j-1)+1)}(z) = -Q^*_{3,j}(\bar{z})\widehat{H}^{-1}_{3,j}Q_{3,j}(b), \tag{72}$$

$$\widehat{\gamma}^{(2j+1)}(z) - \widehat{\gamma}^{(2(j-1)+1)}(z) = (b-z)(z-a)P^*_{4,j}(\bar{z})\widehat{H}^{-1}_{4,j}P_{4,j}(b), \tag{73}$$

$$\widehat{\delta}^{(2j+1)}(z) - \widehat{\delta}^{(2(j-1)+1)}(z) = -(b-z)P^*_{3,j}(\bar{z})\widehat{H}^{-1}_{3,j}Q_{3,j}(b). \tag{74}$$

Proof. The equalities follow from the technique used in [2, Proposition 2.1]. □

Each entry of the Nevanlinna matrix $\widehat{V}^{(2j+1)}$, as defined in Definition 5, can be represented in an additive form.

Lemma 6. *Under the same conditions as in Lemma 5, for all $z \in \mathbb{C}$ and $0 \leq j \leq n$, the following identities hold:*

$$\widehat{\alpha}^{(2j+1)}(z) = I + (b-z)\sum_{k=0}^{j} Q^*_{4,k}(a,\bar{z})\widehat{H}^{-1}_{4,k}P_{4,k}(a,b), \tag{75}$$

$$\widehat{\beta}^{(2j+1)}(z) = -\sum_{k=0}^{j} Q^*_{3,k}(b,\bar{z})\widehat{H}^{-1}_{3,k}Q_{3,k}(b,b), \tag{76}$$

$$\widehat{\gamma}^{(2j+1)}(z) = (b-z)(z-a)\sum_{k=0}^{j} P^*_{4,k}(a,\bar{z})\widehat{H}^{-1}_{4,k}P_{4,k}(a,b), \tag{77}$$

$$\widehat{\delta}^{(2j+1)}(z) = I - (b-z)\sum_{k=0}^{j} P^*_{3,k}(b,\bar{z})\widehat{H}^{-1}_{3,k}Q_{3,k}(b,b). \tag{78}$$

Proof. We prove (75). From (67), we obtain

$$
\begin{aligned}
\widehat{\alpha}^{(2j+1)}(z) &= I_q + (b-z)u_{4,j}^* R_j^*(\bar{z}) H_{4,j}^{-1} R_j(b) v_j \\
&= I_q + (b-z)\begin{pmatrix} u_{4,j-1}^* & -s_j + a s_{j-1} \end{pmatrix} \begin{pmatrix} R_{j-1}^*(\bar{z}) & \begin{matrix} z^j I \\ \vdots \\ zI \end{matrix} \\ 0 & I \end{pmatrix} \begin{pmatrix} H_{4,j-1}^{-1} & 0 \\ 0 & 0 \end{pmatrix} \\
&\quad \begin{pmatrix} R_{j-1}(b) & 0 \\ b^j I \quad \dots \quad bI & I \end{pmatrix} \begin{pmatrix} v_{j-1} \\ 0 \end{pmatrix} \\
&\quad + (b-z) u_{4,j}^* R_j^*(\bar{z}) \Sigma_{4,j} \widehat{H}_{4,j}^{-1} \Sigma_{4,j}^* R_j(b) v_j \\
&= \widehat{\alpha}^{(2(j-1)+1)}(z) + (b-z) Q_{4,j}^*(a,\bar{z}) \widehat{H}_{4,j}^{-1} P_{4,j}(a,b) \\
&= \widehat{\alpha}^{(2(j-2)+1)}(z) + (b-z)[Q_{4,j-1}^*(a,\bar{z}) \widehat{H}_{4,j-1}^{-1} P_{4,j-1}(a,b) \\
&\quad + Q_{4,j}^*(a,\bar{z}) \widehat{H}_{4,j}^{-1} P_{4,j}(a,b)] \\
&= \widehat{\alpha}^{(1)}(z) + (b-z) \sum_{k=1}^{j} Q_{4,k}^*(a,\bar{z}) \widehat{H}_{4,k}^{-1} P_{4,k}(a,b) \\
&= I + (b-z) \sum_{k=0}^{j} Q_{4,k}^*(a,\bar{z}) \widehat{H}_{4,k}^{-1} P_{4,k}(a,b).
\end{aligned}
$$

In the second equality, we apply (32). The third equality follows from (29) and (30). To derive the fourth equality, we consider the third equality evaluated at $j-1$ and invoke (71). By repeating this procedure recursively for $j-2, j-3, \dots, 0$, we obtain the penultimate equality. Finally, the last equality is deduced using (67), (17), (7), (2), and (9) for $j = 0$, together with (26) and (31).

A similar line of reasoning establishes the identities (76)–(78). □

The polynomials given in (22)–(25) are connected to the structure of the Nevanlinna matrix $\widehat{V}^{(2j+1)}$ as in Definition 5.

Lemma 7. *Let $\widehat{\alpha}^{(2j+1)}$, $\widehat{\beta}^{(2j+1)}$, $\widehat{\gamma}^{(2j+1)}$, and $\widehat{\delta}^{(2j+1)}$ be as in (67), (69), (68), and (70), respectively. Let $P_{r,j}$ and $Q_{r,j}$ for $r = 1, 2$ be the OMP and their polynomials of the second kind introduced in Definition 2. Thus, for all $z \in \mathbb{C}$ and $0 \le j \le n$, the following identities hold:*

$$\widehat{\alpha}^{(2j+1)}(z) Q_{2,j}^*(a,b,b) - Q_{2,j}^*(a,b,\bar{z}) = 0, \tag{79}$$

$$\widehat{\beta}^{(2j+1)}(z) P_{1,j+1}^*(b) + Q_{1,j+1}^*(\bar{z}) = 0, \tag{80}$$

$$\widehat{\gamma}^{(2j+1)}(z) Q_{2,j}^*(a,b,b) + (b-z)(z-a) P_{2,j}^*(a,b,\bar{z}) = 0, \tag{81}$$

$$\widehat{\delta}^{(2j+1)}(z) P_{1,j+1}^*(b) - P_{1,j+1}^*(\bar{z}) = 0. \tag{82}$$

Proof. We prove (79). From (67) and (25), we have

$$
\begin{aligned}
&\widehat{\alpha}^{(2j+1)}(z)Q_{2,j}^*(a,b,b) - Q_{2,j}^*(a,b,\bar{z})\\
&= -[(u_{2,j}^* + zs_0v_j^*)R_j^*(b) + (b-z)u_{4,j}^*R_j^*(\bar{z})H_{4,j}^{-1}R_j(b)v_j(u_{2,j}^* + zs_0v_j^*)R_j^*(b)\\
&\quad - (u_{2,j}^* + zs_0v_j^*)R_j^*(\bar{z})]\Sigma_{2,j}\\
&= -[v_{j+1}^*H_{1,j+1}(L_{1,j+1} - aL_{2,j+1}) - (b-z)v_{j+1}^*H_{1,j+1}L_{2,j+1}R_j^{*^{-1}}(a)R_j^*(\bar{z})H_{4,j}^{-1}\\
&\quad \cdot R_j(b)v_jv_{j+1}^*H_{1,j+1}(L_{1,j+1} - aL_{2,j+1}) - v_{j+1}^*H_{1,j+1}R_{j+1}^{*^{-1}}(b)R_{j+1}^*(\bar{z})\\
&\quad \cdot (L_{1,j+1} - aL_{2,j+1})]\Sigma_{2,j}\\
&= -(b-z)v_{j+1}^*H_{1,j+1}R_{j+1}^*(\bar{z})[T_{j+1}^* - L_{2,j+1}R_j^{*^{-1}}(a)H_{4,j}^{-1}R_j(b)v_jv_{j+1}^*H_{1,j+1}]\\
&\quad \cdot (L_{1,j+1} - aL_{2,j+1})\Sigma_{2,j}\\
&= -(b-z)v_{j+1}^*H_{1,j+1}R_{j+1}^*(\bar{z})L_{2,j+1}R_j^{*^{-1}}(a)H_{4,j}^{-1}R_j(b)[-v_jv_{j+1}^*H_{1,j+1}\\
&\quad \cdot (L_{1,j+1} - aL_{2,j+1}) + (I - bT_j)H_{4,j}]\Sigma_{2,j}\\
&= -(b-z)v_{j+1}^*H_{1,j+1}R_{j+1}^*(\bar{z})L_{2,j+1}R_j^{*^{-1}}(a)H_{4,j}^{-1}R_j(b)T_jH_{2,j}\Sigma_{2,j}\\
&= 0.
\end{aligned}
$$

In the second equality, we use (53), (51), (41), (45), (58), (65), (44), and (49). In the third equality, we use (52) and (57). The fourth equality is obtained with (46) and (42). The penultimate equality follows from (64), and the last equality follows from (33) for $r = 2$.

We prove (80). From (68), (23), and (22), we have

$$
\begin{aligned}
&\widehat{\beta}^{(2j+1)}(z)P_{1,j+1}^*(b) + Q_{1,j+1}^*(\bar{z})\\
&= -\left(u_{j+1}^*L_{2,j+1}R_j^{*^{-1}}(b)R_j^*(\bar{z})H_{3,j}^{-1}u_jv_{j+1}^*R_{j+1}^*(b) + u_{1,j+1}^*R_{j+1}^*(\bar{z})\right)\Sigma_{1,j+1}\\
&= -\left(u_{j+1}^*R_{j+1}^*(\bar{z})L_{2,j+1}R_j^{*^{-1}}(b)H_{3,j}^{-1}u_jv_{j+1}^*R_{j+1}^*(b) + u_{j+1}^*T_{j+1}^*R_{j+1}^*(\bar{z})\right)\Sigma_{1,j+1}\\
&= -u_{j+1}^*R_{j+1}^*(\bar{z})L_{2,j+1}R_j^{*^{-1}}(b)H_{3,j}^{-1}\left(u_jv_{j+1}^*R_{j+1}^*(b) + H_{3,j}R_j^*(b)L_{1,j+1}^*\right)\Sigma_{1,j+1}\\
&= -u_{j+1}^*R_{j+1}^*(\bar{z})L_{2,j+1}R_j^{*^{-1}}(b)H_{3,j}^{-1}L_{2,j+1}^*H_{1,j+1}\Sigma_{1,j+1}\\
&= 0.
\end{aligned}
$$

In the second equality, we use (51) (40), (52), and (39). The third equality follows from (46). The penultimate equality follows from (63), and the last equality follows from (34) for $r = 1$.

Equalities (81) and (82) follow by employing an analogous method. □

By taking into account Lemmas 6 and 7, we can now formulate the following theorem.

Theorem 1. *Let $\widehat{H}_{r,j}$ for $r = 3, 4$ be as in (31). Furthermore, let $P_{r,j}$ and $Q_{r,j}$ for $r = 1, 2, 3,$ and 4 be the OMP and their polynomials of the second kind in Definitions 2 and 3. Therefore, for all $z \in \mathbb{C}$ and $0 \leq j \leq n$, the following identities hold:*

$$Q^*_{2,j}(a,b,\bar z) = \left(I + (b-z)\sum_{k=0}^{j} Q^*_{4,k}(a,\bar z)\widehat{H}^{-1}_{4,k}P_{4,k}(a,b)\right) Q^*_{2,j}(a,b,b), \tag{83}$$

$$Q^*_{1,j+1}(\bar z) = \left(\sum_{k=0}^{j} Q^*_{3,k}(b,\bar z)\widehat{H}^{-1}_{3,k}Q_{3,k}(b,b)\right) P^*_{1,j+1}(b), \tag{84}$$

$$P^*_{2,j}(a,b,\bar z) = -\left(\sum_{k=0}^{j} P^*_{4,k}(a,\bar z)\widehat{H}^{-1}_{4,k}P_{4,k}(a,b)\right) Q^*_{2,j}(a,b,b), \tag{85}$$

$$P^*_{1,j+1}(\bar z) = \left(I - (b-z)\sum_{k=0}^{j} P^*_{3,k}(b,\bar z)\widehat{H}^{-1}_{3,k}Q_{3,k}(b,b)\right) P^*_{1,j+1}(b). \tag{86}$$

As a consequence, the following corollary establishes a connection between the Schur complements $\widehat{H}_{3,j}$ and $\widehat{H}_{4,j}$ and the matrix polynomials $P^*_{1,j+1}$, $Q_{3,j}$, $Q^*_{2,j}$, and $P_{4,j}$. Additionally, it justifies the existence of the inverses of these matrix polynomials evaluated at the point $z=b$.

Corollary 1. *Under the same assumptions as in Theorem 1, for $0 \le j \le n$ the following equalities hold:*

$$\widehat{H}_{3,j} = Q_{3,j}(b,b)P^*_{1,j+1}(b), \tag{87}$$

$$\widehat{H}_{4,j} = -P_{4,j}(a,b)Q^*_{2,j}(a,b,b). \tag{88}$$

Moreover, the OMP $P^*_{1,j+1}$ and $P_{4,j}$, as well as the polynomials of the second kind $Q_{3,j}$ and $Q^*_{2,j}$, are invertible at the point $z=b$.

Proof. Equalities (87) and (88) readily follow from (86) and (85) by comparing the leading coefficients of the matrix polynomials. Since $\widehat{H}_{3,j}$ is invertible, we have that $Q_{3,j}(b,b)P^*_{1,j+1}(b)$ is invertible. Regarding determinanats, $\det(Q_{3,j}(b,b)P^*_{1,j+1}(b)) \neq 0$. This implies $\det(Q_{3,j}(b,b)) \neq 0$ and $\det(P^*_{1,j+1}(b)) \neq 0$. Therefore, $Q_{3,j}(b,b)$ and $P^*_{1,j+1}(b)$ are invertible. Similarly, we conclude that $Q^*_{2,j}(a,b,b)$ and $P_{4,j}(a,b)$ are invertible. □

By combining Lemma 7 and Corollary 1, we derive a novel representation of the Nevanlinna matrix $\widehat{V}^{(2j+1)}$ that is associated with the THMM problem at point $z=b$ and that corresponds to an even number of moments.

Theorem 2. *Let $\widehat{V}^{(2j+1)}$ be the Nevanlinna matrix given by Definition 5. Let $P_{r,j}$ and $Q_{r,j}$ for $r=1,2$ be the OMP and their polynomials of the second kind as in Definition 2. Thus, for all $z \in \mathbb{C}$ and $0 \le j \le n$, the following equality holds:*

$$\widehat{V}^{(2j+1)}(a,b,z) = \begin{pmatrix} Q^*_{2,j}(a,b,\bar z)Q^{*^{-1}}_{2,j}(a,b,b) & -Q^*_{1,j+1}(\bar z)P^{*^{-1}}_{1,j+1}(b) \\ -(b-z)(z-a)P^*_{2,j}(a,b,\bar z)Q^{*^{-1}}_{2,j}(a,b,b) & P^*_{1,j+1}(\bar z)P^{*^{-1}}_{1,j+1}(b) \end{pmatrix}. \tag{89}$$

6. Derivation of identities related to orthogonal matrix polynomials

By considering the result in Corollary 1, we apply it to derive identities at points $z = a$ and $z = b$, involving the OMP introduced in Definitions 2 and 3.

We establish identities regarding to point $z = a$ that involve OMP. To support this, we must reproduce the Nevanlinna matrix $U^{(2j+1)}$ in terms of OMP, as obtained in [7, Theorem 3.8]. Note that both matrices are formulated with respect to point $z = a$. For more detail, see Theorem 3.

Furthermore, we derive identities at point $z = b$ for the OMP $P_{1,j}$ and $P_{2,j}$, together with their polynomials of the second kind $Q_{1,j}$ and $Q_{2,j}$. To substantiate these findings, expressing the inverse of the Nevanlinna matrix $\widehat{V}^{(2j+1)}$ in terms of these polynomials is necessary. See Theorem 4 for a precise statement.

We proceed to reproduce results analogous to Corollary 1, which were originally established in [7, Corollary 3.10] in the context of the evaluation point $z = a$.

Lemma 8. *Let $\widehat{H}_{r,j}$ for $r = 3, 4$ be as in (31). Let $P_{1,j}$ and $Q_{2,j}$ be introduced in Definition 2, as well as, let $P_{3,j}$ and $Q_{4,j}$ be introduced in Definition 3. Thus, for $0 \leq j \leq n$, the following identities are satisfied:*

$$\widehat{H}_{3,j} = P_{3,j}(b, a) Q^*_{2,j}(a, b, a), \tag{90}$$

$$\widehat{H}_{4,j} = Q_{4,j}(a, a) P^*_{1,j+1}(a). \tag{91}$$

In the proposition below, we establish new identities at points $z = a$ and $z = b$ by involving the OMP.

Proposition 2. *Let $P_{r,j}$ and $Q_{r,j}$ for $r = 1, 2, 3,$ and 4 be the OMP and their polynomials of the second kind in Definitions 2 and 3. Therefore, for $0 \leq j \leq n$, the following identities are satisfied:*

$$Q_{3,j}(b, b) P^*_{1,j+1}(b) - P_{3,j}(b, a) Q^*_{2,j}(a, b, a) = 0, \tag{92}$$

$$P_{4,j}(a, b) Q^*_{2,j}(a, b, b) + Q_{4,j}(a, a) P^*_{1,j+1}(a) = 0. \tag{93}$$

Proof. Identities (92), and (93) are proven by using the identities obtained in (87), (88), (90), and (91).

We next revisit the formulation of the Nevanlinna matrix $U^{(2j+1)}$ associated with the THMM problem at point $z = a$. This representation, originally from [7, Theorem 3.8], expresses $U^{(2j+1)}$ in terms of the OMP and their corresponding polynomials of the second kind.

Definition 6. *Let $P_{r,j}$ and $Q_{r,j}$ for $r = 1, 2$ be the OMP and their polynomials of the second kind in Definition 2. We introduce the Nevanlinna matrix of the THMM problem with respect to point $z = a$ in the case of an even number of moments for all $z \in \mathbb{C}$ and $0 \leq j \leq n$:*

$$U^{(2j+1)}(a, b, z) := \begin{pmatrix} Q^*_{2,j}(a, b, \bar{z}) Q^{*^{-1}}_{2,j}(a, b, a) & -Q^*_{1,j+1}(\bar{z}) P^{*^{-1}}_{1,j+1}(a) \\ -(z - a)(b - z) P^*_{2,j}(a, b, \bar{z}) Q^{*^{-1}}_{2,j}(a, b, a) & P^*_{1,j+1}(\bar{z}) P^{*^{-1}}_{1,j+1}(a) \end{pmatrix}. \tag{94}$$

Remark 4. *As established in [12, Proposition 6.10], the Nevanlinna matrix $U^{(2j+1)}$ is invertible for all $z \in \mathbb{C}$.*

The next remark explicitly represents the inverse of $U^{(2j+1)}$ with OMP and their corresponding polynomials of the second kind.

Remark 5. *The inverse of (94) is*

$$U^{(2j+1)^{-1}}(a,b,z) = \begin{pmatrix} P_{1,j+1}^{-1}(a)P_{1,j+1}(z) & P_{1,j+1}^{-1}(a)Q_{1,j+1}(z) \\ (z-a)(b-z)Q_{2,j}^{-1}(a,b,a)P_{2,j}(a,b,z) & Q_{2,j}^{-1}(a,b,a)Q_{2,j}(a,b,z) \end{pmatrix}. \tag{95}$$

In the following theorem, we derive identities at point $z = a$ by incorporating the OMP $P_{1,j}$ and $P_{2,j}$, together with their polynomials of the second kind $Q_{1,j}$ and $Q_{2,j}$.

Theorem 3. *Let $P_{r,j}$ and $Q_{r,j}$, for $r = 1, 2$ be the OMP and their polynomials of the second kind in Definition 2. Therefore, for all $z \in \mathbb{C}$ and $0 \le j \le n$, the following identities are fulfilled:*

$$Q_{2,j}^*(\bar{z})Q_{2,j}^{*^{-1}}(a)P_{1,j+1}^{-1}(a)P_{1,j+1}(z) - (b-z)(z-a) \cdot Q_{1,j+1}^*(\bar{z})P_{1,j+1}^{*^{-1}}(a)Q_{2,j}^{-1}(a)P_{2,j}(z) = I_q, \tag{96}$$

$$Q_{2,j}^*(\bar{z})Q_{2,j}^{*^{-1}}(a)P_{1,j+1}^{-1}(a)Q_{1,j+1}(z) - Q_{1,j+1}^*(\bar{z})P_{1,j+1}^{*^{-1}}(a)Q_{2,j}^{-1}(a)Q_{2,j}(z) = 0, \tag{97}$$

$$(b-z)(z-a)[P_{1,j+1}^*(\bar{z})P_{1,j+1}^{*^{-1}}(a)Q_{2,j}^{-1}(a)P_{2,j}(z) - P_{2,j}^*(\bar{z})Q_{2,j}^{*^{-1}}(a)P_{1,j+1}^{-1}(a)P_{1,j+1}(z)] = 0, \tag{98}$$

$$P_{1,j+1}^*(\bar{z})P_{1,j+1}^{*^{-1}}(a)Q_{2,j}^{-1}(a)Q_{2,j}(z) - (b-z)(z-a) \cdot P_{2,j}^*(\bar{z})Q_{2,j}^{*^{-1}}(a)P_{1,j+1}^{-1}(a)Q_{1,j+1}(z) = I_q. \tag{99}$$

Proof. From (94), (95), and the following equality

$$U^{(2j+1)}(a,b,z)U^{(2j+1)^{-1}}(a,b,z) = \begin{pmatrix} I_q & 0 \\ 0 & I_q \end{pmatrix},$$

we obtain the identities (96)–(99). □

The following observation justifies the invertibility of the Nevanlinna matrix $\widehat{V}^{(2j+1)}$.

Remark 6. *Let $\widehat{V}^{(2j+1)}$ denote the Nevanlinna matrix introduced in Definition 5. According to Remark 4, the matrix $U^{(2j+1)}$ is invertible for all $z \in \mathbb{C}$. Furthermore, Remark 8 asserts that the constant matrix $\mathfrak{D}^{(2j+1)}$ is also invertible. Consequently, by the explicit relation given in (6), it follows that $\widehat{V}^{(2j+1)}$ is invertible for all $z \in \mathbb{C}$.*

The following remark explicitly expresses the inverses of the Nevanlinna matrix $\widehat{V}^{(2j+1)}$, which is formulated in terms of OMP and their associated polynomials of the second kind.

Remark 7. *Let $\widehat{V}^{(2j+1)}$ be the Nevanlinna matrix given by (89). Furthermore, let $P_{r,j}$ and $Q_{r,j}$, for $r = 1, 2$ be the OMP and their polynomials of the second kind in Definition 2. Thus, for all $z \in \mathbb{C}$ and $0 \leq j \leq n$, the following equality holds :*

$$\widehat{V}^{(2j+1)^{-1}}(a, b, z) = \begin{pmatrix} P^{-1}_{1,j+1}(b)P_{1,j+1}(z) & P^{-1}_{1,j+1}(b)Q_{1,j+1}(z) \\ (b-z)(z-a)Q^{-1}_{2,j}(a,b,b)P_{2,j}(a,b,z) & Q^{-1}_{2,j}(a,b,b)Q_{2,j}(a,b,z) \end{pmatrix}. \quad (100)$$

In the theorem below, we derive identities at point $z = b$ with the OMP $P_{1,j}$ and $P_{2,j}$, together with their polynomials of the second kind $Q_{1,j}$ and $Q_{2,j}$.

Theorem 4. *Let $P_{r,j}$ and $Q_{r,j}$, for $r = 1, 2$ be the OMP and their polynomials of the second kind in Definition 2. Therefore, for all $z \in \mathbb{C}$ and $0 \leq j \leq n$, the following identities are fulfilled:*

$$Q^*_{2,j}(\bar{z})Q^{*^{-1}}_{2,j}(b)P^{-1}_{1,j+1}(b)P_{1,j+1}(z) - (b-z)(z-a) \cdot Q^*_{1,j+1}(\bar{z})P^{*^{-1}}_{1,j+1}(b)Q^{-1}_{2,j}(b)P_{2,j}(z) = I_q, \quad (101)$$

$$Q^*_{2,j}(\bar{z})Q^{*^{-1}}_{2,j}(b)P^{-1}_{1,j+1}(b)Q_{1,j+1}(z) - Q^*_{1,j+1}(\bar{z})P^{*^{-1}}_{1,j+1}(b)Q^{-1}_{2,j}(b)Q_{2,j}(z) = 0, \quad (102)$$

$$(b-z)(z-a)[P^*_{1,j+1}(\bar{z})P^{*^{-1}}_{1,j+1}(b)Q^{-1}_{2,j}(b)P_{2,j}(z) - P^*_{2,j}(\bar{z})Q^{*^{-1}}_{2,j}(b)P^{-1}_{1,j+1}(b)P_{1,j+1}(z)] = 0, \quad (103)$$

$$P^*_{1,j+1}(\bar{z})P^{*^{-1}}_{1,j+1}(b)Q^{-1}_{2,j}(b)Q_{2,j}(z) - (b-z)(z-a) \cdot P^*_{2,j}(\bar{z})Q^{*^{-1}}_{2,j}(b)P^{-1}_{1,j+1}(b)Q_{1,j+1}(z) = I_q. \quad (104)$$

Proof. By using (89), (100), and the equality

$$\widehat{V}^{(2j+1)}(a,b,z)\widehat{V}^{(2j+1)^{-1}}(a,b,z) = \begin{pmatrix} I_q & 0 \\ 0 & I_q \end{pmatrix},$$

the identities (101)–(104) follow. □

7. Explicit relationships between Nevanlinna matrices via OMP

For $m = 2j + 1$, the explicit relationship (6) between the Nevanlinna matrices of the THMM problem regarding to points $z = a$ and $z = b$ was established in [15, Theorem 4.3].

By using the Nevanlinna matrix from Theorem 2, together with Definition 6, we show the relation (6) in terms of OMP. Furthermore, we introduce and we reformulate the constant matrix $\mathfrak{D}^{(2j+1)}$, that was originally obtained in [15, Theorem 4.3] also with OMP.

Definition 7. *Let* $(s_k)_{k=0}^{2j+1}$ *be a Hausdorff positive definite sequence on* $[a,b]$*. Let* T_j*,* R_j*,* v_j*,* u_j*,* $u_{4,j}$*, and* $u_{3,j}$ *be as in (8), (7), (9), (13), (17), and (16). We introduce the following matrix:*

$$\mathfrak{D}^{(2j+1)} := \begin{pmatrix} a_{11}^{(2j+1)} & 0 \\ 0 & a_{22}^{(2j+1)} \end{pmatrix}, \quad 0 \leq j \leq n \tag{105}$$

with

$$a_{11}^{(2j+1)} := I + (b-a)u_{4,j}^* R_j^*(a) H_{4,j}^{-1} R_j(b) v_j, \tag{106}$$

$$a_{22}^{(2j+1)} := I - (b-a)v_j^* R_j^*(a) H_{3,j}^{-1} R_j(b) u_{3,j}. \tag{107}$$

Remark 8. *As established in [15, Lemma 4.4], the constant matrix* $\mathfrak{D}^{(2j+1)}$ *is invertible.*

We now present a theorem that indicates an explicit relationship between the Nevanlinna matrices of the THMM problem evaluated at points $z = a$ and $z = b$ and expressed with OMP for an even number of moments.

Theorem 5. *Let* $P_{1,j}$ *be the orthogonal matrix polynomial, and let* $Q_{2,j}$ *be the polynomial of the second kind introduced in Definition 2. Moreover, let* $U^{(2j+1)}$ *and* $\widehat{V}^{(2j+1)}$ *be the Nevanlinna matrices as in (94) and (89), respectively. If the elements* $a_{11}^{(2j+1)}$ *and* $a_{22}^{(2j+1)}$ *of the matrix* $\mathfrak{D}^{(2j+1)}$ *from Definition 7 are written as*

$$a_{11}^{(2j+1)} = P_{1,j+1}^{-1}(a) P_{1,j+1}(b), \tag{108}$$

$$a_{22}^{(2j+1)} = Q_{2,j}^{-1}(a) Q_{2,j}(b), \tag{109}$$

then for all $z \in \mathbb{C}$ *the following equality is valid:*

$$U^{(2j+1)}(z)\mathfrak{D}^{(2j+1)} - \widehat{V}^{(2j+1)}(z) = 0. \tag{110}$$

Proof. We prove that the left-hand side of (110) vanishes as follows:

$$\begin{aligned} & U^{(2j+1)}(z)\mathfrak{D}^{(2j+1)} - \widehat{V}^{(2j+1)}(z) \\ &= \begin{pmatrix} -Q_{2,j}^*(\bar{z}) Q_{2,j}^{*^{-1}}(b) & Q_{1,j+1}^*(\bar{z}) P_{1,j+1}^{*^{-1}}(b) \\ (z-a)(b-z) P_{2,j}^*(\bar{z}) Q_{2,j}^{*^{-1}}(b) & -P_{1,j+1}^*(\bar{z}) P_{1,j+1}^{*^{-1}}(b) \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} -Q_{2,j}^*(b) Q_{2,j}^{*^{-1}}(a) P_{1,j+1}^{-1}(a) P_{1,j+1}(b) + I_q & 0 \\ 0 & -P_{1,j+1}^*(b) P_{1,j+1}^{*^{-1}}(a) Q_{2,j}^{-1}(a) Q_{2,j}(b) + I_q \end{pmatrix} \\ &= 0. \end{aligned}$$

The last equality is obtained by applying the identities (96) and (99) at $z = b$. □

The elements $a_{22}^{(2j+1)}$ and $a_{22}^{(2j+1)}$ as in (108) and (109) are written at point $z = b$. Thus, we consider the following remark.

Remark 9. *By using (108) and (109), the matrix* $\mathfrak{D}^{(2j+1)}$ *from Definition 7 admits the following representation in terms of OMP with respect to point* $z = b$*:*

$$\mathfrak{D}^{(2j+1)} = \begin{pmatrix} P_{1,j+1}^{-1}(a)\, P_{1,j+1}(b) & 0 \\ 0 & Q_{2,j}^{-1}(a)\, Q_{2,j}(b) \end{pmatrix}. \tag{111}$$

The following remark shows that the elements $a_{11}^{(2j+1)}$ and $a_{22}^{(2j+1)}$ can also be represented with OMP regarding to point $z = a$.

Remark 10. *Let* $P_{1,j}$ *be the orthogonal matrix polynomial, and let* $Q_{2,j}$ *be the polynomial of the second kind introduced in Definition 2. Let* $a_{11}^{(2j+1)}$ *and* $a_{22}^{(2j+1)}$ *be as in (106) and (107), respectively. If the elements* $a_{11}^{(2j+1)}$ *and* $a_{22}^{(2j+1)}$ *of the matrix* $\mathfrak{D}^{(2j+1)}$ *from Definition 7 are written as*

$$a_{11}^{(2j+1)} = Q_{2,j}^{*}(a) Q_{2,j}^{*^{-1}}(b), \tag{112}$$

$$a_{22}^{(2j+1)} = P_{1,j+1}^{*}(a) P_{1,j+1}^{*^{-1}}(b), \tag{113}$$

then Equality (110) is also satisfied for all $z \in \mathbb{C}$*.*

8. Conclusion

We have expressed the Nevanlinna (resolvent) matrix associated with the truncated Hausdorff matrix moment (THMM) problem on the interval $[a, b]$ in terms of orthogonal matrix polynomials (OMP) and their corresponding polynomials of the second kind, at point b. Alongside this representation, we obtained new identities involving OMP and established an explicit connection between the Nevanlinna matrices of the THMM problem at points a and b, which was directly formulated through OMP.

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Матриця Неванлінни усіченої задачі моментів матриці Хаусдорфа через ортогональні матричні поліноми на [a,b] для випадку парної кількості моментів

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Об'єднана аспірантура з математичних наук Національного автономного університету Мексики та Університету Мічоакана-де-Сан-Ніколас-де-Ідальго, Інститут фізики та математики,

Будівля С-3, 58060, Морелія, Мічоакан, Мексика

Скалярна проблема моментів була вперше запропонована Т. Й. Стілтьєсом у роботі: “Recherches sur les fractions continues”, Annals of the Faculty of Sciences of Toulouse 8, 1–122, (1895). Він сформулював її наступним чином: маючи моменти порядку k ($k = 0, 1, 2, \ldots$), знайти додатний розподіл маси на півосі $[0, +\infty)$.

Дослідження матричної та операторної проблем моментів було розпочато М. Г. Крейном у його основоположній роботі “Fundamental aspects of the representation theory of Hermitian operators with deficiency index (m, m)”, Translations of the American Mathematical Society, Series II, 97, 75–143, (1949).

Дана стаття пов'язана з усіченою проблемою моментів Хаусдорфа (англ. THMM): усіченою матричною проблемою моментів Хаусдорфа на компактному інтервалі $[a, b]$ на відміну від проблеми моментів Стілтьєса на $[0, +\infty)$ та проблеми

моментів Гамбургера на $(-\infty, +\infty)$. Наш підхід спирається на метод В. П. Потапова, у якому задача інтерполяції та проблема моментів переформульовуються як еквівалентні матричні нерівності і вводяться допоміжні матриці, що задовольняють властивість $\widetilde{J}_q$-внутрішньої функції класу Потапова разом із системою пар стовпців.

Реалізація методу починається з побудови матриць Ганкеля на основі заданих моментів. Якщо ці матриці є додатно напіввизначеними, то THMM проблема є розв'язною. У випадку строгої додатної визначеності, який називають невиродженим, ми перетворюємо відповідні матричні нерівності, щоб отримати матрицю Неванлінни (або резольвенту) THMM проблеми, яка характеризує її розв'язки.

Цей підхід було широко застосовано, зокрема в роботі A. E. Choque Rivero, Yu. M. Dyukarev, B. Fritzsche та B. Kirstein: "A truncated matricial moment problem on a finite interval", Interpolation, Schur Functions and Moment Problems, Operator Theory: Advances and Applications, Birkhäuser , Basel, 165, 121–173, (2006).

Основний результат цієї роботи полягає у представленні матриці Неванлінни THMM проблеми у термінах ортогональних матричних поліномів (англ. OMP) і пов'язаних з ними поліномів другого роду в точці b. Зауважимо, що аналогічне представлення в точці a було отримано раніше в роботі A. E. Choque Rivero, "From the Potapov to the Krein–Nudel'man representation of the resolvent matrix of the truncated Hausdorff matrix moment problem", Bulletin of the Mexican Mathematical Society, 21(2), 233–259 (2015).

Крім того, ми встановлюємо нові тотожності, що стосуються OMP. і переформульовуємо явний зв'язок між матрицями Неванлінни THMM проблеми в точках a та b за допомогою OMP.

Ключові слова: **усічена матрична проблема моментів Хаусдорфа; матриця Неванлінни; ортогональні матричні поліноми.**

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