ISSN 2221-5646(Print) 2523-4641(Online)
Visnyk of V.N.Karazin Kharkiv National University
Ser. “Mathematics, Applied Mathematics
and Mechanics”
2025, Vol. 101, p. 5–20
DOI: 10.26565/2221-5646-2025-101-01
УДК 532.59

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університету імені В.Н. Каразіна
Серія "Математика, прикладна
математика і механіка"
2025, Том 101, с. 5–20

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Return condition for oscillating systems

This paper is devoted to the problem of null-controllability for the oscillating linear system $\dot{x}_{2i-1} = x_{2i}, \dot{x}_{2i} = -x_{2i-1} + u$, $i = \overline{1, n}$ under control constraints $u \in [c, 1]$ and $u \in \{c, 1\}, c > 0$. In this case the origin is not an equilibrium point. Null-controllability means the existence of a moment of time T_0 such that, for any time $T \geq T_0$ it is possible to reach the origin precisely at time T. The criterion of controllability to a non-equilibrium point was obtained by V. I. Korobov and a new condition called the return condition on an interval was introduced, which must be satisfied, together with the classical conditions for controllability to an equilibrium point. This condition requires the existence of a time interval $I = [T, T + \alpha]$, $\alpha > 0$, such that a trajectory starting at the origin may return to it at any moment $T \in I$ with some control $u_T(t)$. The objective of this paper is to show that the return conditions are satisfied for the considered oscillatory system, and to obtain an analytical solution for the control which ensures this condition. The considered approach involves constructing a piecewise-constant control using values $u = c$ and $u = 1$. This problem admits multiple solutions, and in our paper we present one involving $2n$ switching points and another with only 2 in the case when $c \leq \frac{1}{2}$. The solution with 2 switching moments is especially interesting since it does not depend on the dimensionality of the system. We also generalize the problem to the case where the eigenvalues are of the form $\lambda_{2k}, \lambda_{2k-1} = \pm i\nu_k$, where ν_k are rational numbers. Additionally, we discuss some partial cases where the eigenvalues are irrational.

Keywords: **return condition on an interval, null-controllability; trigonometric moment-problem; linear control systems**

2020 Mathematics Subject Classification: 93B05; 93C05; 93C15; 93C35

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1. Introduction

In this paper, we consider the problem of null-controllability for the linear control system,

$$\dot{x} = Ax + bu, \quad x \in \mathbb{R}^n \quad u \in \Omega \subset \mathbb{R}, \quad u(t) \in L^1[0, T], \tag{1}$$

with an assumption that the origin is not an equilibrium point. Null-controllability means that there exists a time moment $T_0 \geq 0$ such that, for any $T \geq T_0$, we can select a control $u_T(t)$ such that the origin is reachable precisely at time T. The set S of points that can be transferred to the origin is called the null-controllability set [2]. If $0 \in \operatorname{int} S$, the system is called null-controllable form a neighbourhood, and if $S = \mathbb{R}^n$ it is called globally null-controllable. This problem has been considered in many papers, for example [1, 2, 4, 5]. Typically, the null-controllability into an equilibrium point is archived by steering the system to the origin at some time T, and then selecting the value of control $u = 0$, to stay in the equilibrium indefinitely [1].

However, it is also natural to consider the case when such value of control does not exist – namely, the problem of controllability into a non-equilibrium point. Such a system may occur, for example, after applying a change of variables, when solving the controllability problem into a point different from the origin [3]. This more general case has also been studied in many works [3, 4, 6]. For instance, in the paper [6] it is shown that for controllability into a non-equilibrium point the necessary and sufficient conditions are complete controllability of the reduced system and existence of an internal feedback control.

The criterion used in this work was developed by V. I. Korobov in the paper [3], where a new concept called the return condition on an interval was also introduced. In the present paper, we examine the problem of null-controllability for the following oscillating linear system:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 + u, \\ \dot{x}_3 = 2x_4, \\ \dot{x}_4 = -2x_3 + u, \\ \quad \vdots \\ \dot{x}_{2n-1} = n\, x_{2n}, \\ \dot{x}_{2n} = -n\, x_{2n-1} + u, \end{cases} \tag{2}$$

with control constraints $u \in [c, 1]$, $c > 0$ and $u \in \{c, 1\}$. In the case $n = 1$, this system corresponds to the classic pendulum problem, and for $n > 1$, it represents a system of pendulums, and also serves as a model of first n terms of the decomposition of an oscillating string. Notice that each pair of coordinates is independent of the other, but all pendulums are coupled through the common control u and must reach the origin simultaneously. However, we are not able to choose the control $u = 0$ because of the constraints, and the system leaves origin immediately after reaching it. This is why for null-controllability, we have to check the return condition.

Let us first consider the problem of null-controllability for the system (1) in the case when origin is an equilibrium point. This means that we can select $u = 0$ after reaching the origin. We consider here the geometrical criterion for null-controllability obtained in the paper [2], which is the following:

Theorem 1. *[2] The system (1) is null-controllable from a neighbourhood if and only if there exists $m \geq 0$ for which the following inclusions hold:*

1. $0 \in \operatorname{int}\operatorname{co}L = \operatorname{int}\operatorname{co}\{b\,\Omega, Ab\,\Omega, \ldots, A^m b\,\Omega\}$,

2. $0 \in \operatorname{int}\operatorname{co}L^{-}\operatorname{int}\operatorname{co}\{b\,\Omega, -Ab\,\Omega, \ldots, (-1)^m A^m b\,\Omega\}$.

For example, the linear control system:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = u, \end{cases} \tag{3}$$

with control constraints $u \in [0, 1]$ is not null-controllable. Here

$$bu = \begin{pmatrix} 0 \\ u \end{pmatrix}, Abu = \begin{pmatrix} u \\ 0 \end{pmatrix}, A^2bu = \begin{pmatrix} 0 \\ u \end{pmatrix}, A^3bu = \begin{pmatrix} u \\ 0 \end{pmatrix}, \ldots \tag{4}$$

so

$$L = (bu, Abu) = \begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix}, \tag{5}$$

$$L^{-} = (bu, -Abu) = \begin{pmatrix} 0 & -u \\ u & 0 \end{pmatrix}. \tag{6}$$

For $u \in [0, 1]$, the columns of the matrix L span two line segments in the space $\mathbb{R}^2$, and their convex hull is a triangle with vertices at $(1, 0), (0, 1)$ and $(0, 0)$. Clearly, $0 \notin \operatorname{int}\operatorname{co}L$ and the system is not null-controllable. However, if the control constraints are extended to $u \in [-1, 1]$, the system will be null-controllable since the convex hull for both L and L^{-1} would be a square with vertices $(1, 0), (0, 1), (-1, 0)$ and $(0, -1)$.

Let us now give the definition of the return condition introduced in [3].

Definition 1. *[3] For the system (1) the return condition is satisfied on the interval $I = [T^*, T^* + \alpha]\,, (\alpha > 0, T^* \geq 0)$, if for any $T \in I$, there exists a control $u_T(t)$, such that the solution $\dot{x} = Ax + bu_T(t)$, $x(0) = 0$, satisfies the condition $x(T) = 0$.*

In other words, the return condition means that for any $T \in [T^*, T^* + \alpha]$ we are able to construct a control $u_T(t)$ such that the trajectory starting at the origin return there at time T. It should be noted that the condition $\alpha > 0$, meaning the non-zero length if the interval I, is essential in this definition.

The following theorem extends the null-controllability criterion to the more general case:

Theorem 2. *[3] The system (1) is null-controllable from a neighbourhood if and only if the following conditions hold:*

1. *there exists an interval $I = [T^*, T^* + \alpha]$ such that the return condition is satisfied,*
2. *there exists an $m \geq 0$ such that:*

   $0 \in \operatorname{int} \operatorname{co} L = \operatorname{int} \operatorname{co}\{b\,\Omega, Ab\,\Omega, \ldots, A^m b\,\Omega\}$,

   $0 \in \operatorname{int} \operatorname{co} L^{-} \operatorname{int} \operatorname{co}\{b\,\Omega, -Ab\,\Omega, \ldots, (-1)^m A^m b\,\Omega\}$.

This theorem also includes the previous case when the origin is an equilibrium. Indeed, after reaching the origin at time T, we may select $u = 0$. Then the system remains at rest, satisfying the return condition on any interval $[T, T + \alpha]$, $\alpha > 0$.

2. Return condition for oscillating system

Let us now consider the return condition for the system (2) with control constraints $u \in \{c, 1\}$. For a linear system of the form $\dot{x} = Ax + bu$ for the control that transfers a point x_0 to a point x_1 we can write:

$$x_1 = e^{At}\left(x_0 + \int_0^T e^{-A\tau} b u(\tau)\,\mathrm{d}\tau\right). \tag{7}$$

In the case when $x_0 = x_1 = 0$, this simplifies to:

$$0 = \int_0^T e^{-At} b u(t)\,\mathrm{d}t. \tag{8}$$

We will look for a solution in the form of a piecewise-constant function, with u alternating between the values c and 1:

$$u(t) = \begin{cases} c, & 0 \leq t \leq T_1, \\ 1, & T_1 < t \leq T_2, \\ c, & T_2 < t \leq T_3, \\ \quad\vdots & \\ 1, & T_{k-1} < t \leq T_k, \\ c, & T_k < t \leq T, \end{cases} \tag{9}$$

with switching moments $T_1, \ldots, T_k$. The number of switching moments k is unknown, and multiple solutions may exist. We will be looking for such a control where number of switching moments is even, $k = 2\tilde{k}$, and they are placed symmetrically, that is, $T_i = T - T_{k-i+1}$ for $i = \overline{1, \tilde{k}}$. For the system (2), we have:

$$e^{At} = \begin{pmatrix} \cos t & \sin t & 0 & 0 & \dots & 0 & 0 \\ -\sin t & \cos t & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \cos 2t & \sin 2t & \dots & 0 & 0 \\ 0 & 0 & -\sin 2t & \cos 2t & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \cos nt & \sin nt \\ 0 & 0 & 0 & 0 & \dots & -\sin nt & \cos nt \end{pmatrix}, b = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \quad (10)$$

Substituting into equation (8), we obtain the system

$$\begin{cases} \int_0^T u(t) \sin t \, \mathrm{d}t = 0, \\ \int_0^T u(t) \cos t \, \mathrm{d}t = 0, \\ \quad \vdots \\ \int_0^T u(t) \sin nt \, \mathrm{d}t = 0, \\ \int_0^T u(t) \cos nt \, \mathrm{d}\, t = 0. \end{cases} \quad (11)$$

Thus, we obtain a trigonometric moment problem [7], for which we need the solution $u_T(t)$ to exist for each $T \in [T^*, T^* + \alpha]$. This system can also be written in exponential form as

$$\begin{cases} \int_0^T u(t) e^{i\,t} \mathrm{d}t = 0, \\ \int_0^T u(t) e^{i\,t} \mathrm{d}t = 0, \\ \quad \vdots \\ \int_0^T u(t) e^{n\,i\,t} \mathrm{d}t = 0. \end{cases} \quad (12)$$

By substituting the control (9) into the system (11), we obtain:

$$\begin{cases} c \sin T_1 + (\sin T_2 - \sin T_1) + \cdots + c\,(\sin T - \sin T_k) = 0, \\ c\,\cos T_1 - c + \cdots + c\,(\cos T - \cos T_k) = 0, \\ \quad \vdots \\ \frac{c}{n} \sin n\,T_1 + \frac{1}{n}(\sin n\,T_2 - \sin n\,T_1) + \cdots + \frac{c}{n}(\sin n\,T - \sin n\,T_k) = 0, \\ \frac{c}{n} \cos n\,T_1 - \frac{c}{n} + \cdots + \frac{c}{n}(\cos n\,T - \cos n\,T_k) = 0. \end{cases} \quad (13)$$

For $c = \frac{1}{2}$, we present two solutions: one with $2n$ switching moments, and another, which has only 2 for any dimension n.

3. Control with $2n$ switching moments

First, we consider the case when $k = 2n$, which is equal to the dimension of system (2). Let us first notice that for $T = 2\pi$ the system (2) has a solution

$u = c = const$. We now explore how to construct a solution on a slightly larger time interval by introducing short segments during which $u = 1$. More precisely, we claim that for $c = \frac{1}{2}$ for any T on the time interval $[2\pi, 2\pi+\alpha]$, where $\alpha \in [0, \frac{2\pi}{n+1}]$, the origin can be reached using the following control:

$$u_n(t) = \begin{cases} \frac{1}{2}, & 0 \le t \le \frac{2\pi}{n+1}, \\ 1, & \frac{2\pi}{n+1} < t \le \frac{2\pi}{n+1} + \alpha, \\ \frac{1}{2}, & \frac{2\pi}{n+1} + a < t \le 2\,\frac{2\pi}{n+1}, \\ 1, & 2\,\frac{2\pi}{n+1} < t \le 2\,\frac{2\pi}{n+1} + \alpha, \\ \vdots & \\ 1, & n\,\frac{2\pi}{n+1} < t \le n\,\frac{2\pi}{n+1} + \alpha, \\ \frac{1}{2}, & n\,\frac{2\pi}{n+1} + a < t \le 2\pi + \alpha. \end{cases} \tag{14}$$

Proof. For the control (14) the switching moments are:

$$T_1 = \frac{2\pi}{n+1}, T_2 = \frac{2\pi}{n+1} + \alpha, \ldots, T_{2n-1} = \frac{2\pi n}{n+1}, T = 2\pi + \alpha. \tag{15}$$

The system (13) consists of independent pairs of equalities that need to be proven for $m \le n$, and, in the case $c = \frac{1}{2}$, they take the form:

$$\begin{cases} \frac{1}{2m}\sin mT_1 + \frac{1}{m}(\sin mT_2 - \sin nT_1) + \cdots + \frac{1}{2m}(\sin mT - \sin nT_{2n}) = 0, \\ \frac{1}{2m}\cos mT_1 - \frac{1}{2m} + \cdots + \frac{1}{2m}(\cos mT - \cos nT_{2n}) = 0, \end{cases} \tag{16}$$

For the switching moments T_i from (15), this becomes

$$\begin{cases} \sum_{k=1}^{n} \left(\sin\left(\frac{2\pi mk}{n+1} + a\right) - \sin\left(\frac{2\pi mk}{n+1}\right)\right) + \sin(2\pi + \alpha) = 0, \\ \sum_{k=1}^{n} \left(\cos\left(\frac{2\pi mk}{n+1} + a\right) - \cos\left(\frac{2\pi mk}{n+1}\right)\right) + \cos(2\pi + \alpha) - 1 = 0, \end{cases} \tag{17}$$

or,

$$\begin{cases} \sum_{k=0}^{n} \left(\sin\left(\frac{2\pi mk}{n+1} + \alpha\right) - \sin\left(\frac{2\pi mk}{n+1}\right)\right) = 0, \\ \sum_{k=0}^{n} \left(\cos\left(\frac{2\pi mk}{n+1} + \alpha\right) - \cos\left(\frac{2\pi mk}{n+1}\right)\right) = 0, \end{cases} \tag{18}$$

This is equivalent to

$$\begin{cases} \sum_{k=0}^{n} \sin\left(\frac{a}{2}\right)\cos\left(\frac{2\pi mk}{n+1} + \frac{\alpha}{2}\right) = 0, \\ \sum_{k=0}^{n} \sin\left(\frac{a}{2}\right)\sin\left(\frac{2\pi mk}{n+1} + \frac{\alpha}{2}\right) = 0, \end{cases} \tag{19}$$

or factoring out the common term:

$$\begin{cases} \sum_{k=0}^{n} \cos\left(\frac{2\pi mk}{n+1} + \frac{\alpha}{2}\right) = 0, \\ \sum_{k=0}^{n} \sin\left(\frac{2\pi mk}{n+1} + \frac{\alpha}{2}\right) = 0. \end{cases} \tag{20}$$

Applying the angle addition formula, this leads to:

$$\begin{cases} \sum\limits_{k=0}^{n} \left(\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{2\pi mk}{n+1}\right) + \sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{2\pi mk}{n+1}\right)\right) = 0, \\ \sum\limits_{k=0}^{n} \left(\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{2\pi mk}{n+1}\right) + \cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{2\pi mk}{n+1}\right)\right) = 0. \end{cases} \tag{21}$$

Then, it is enough to show that:

$$\begin{cases} \sum\limits_{k=0}^{n} \cos\left(\frac{2\pi mk}{n+1}\right) = 0, \\ \sum\limits_{k=0}^{n} \sin\left(\frac{2\pi mk}{n+1}\right) = 0 \end{cases} \tag{22}$$

or

$$\sum_{k=0}^{n} \left(\cos\left(\frac{2\pi mk}{n+1}\right) + i\sin\left(\frac{2\pi mk}{n+1}\right)\right) = 0, \tag{23}$$

which is equivalent to

$$\sum_{k=0}^{n} \left(\cos\left(\frac{2\pi k}{n+1}\right) + i\sin\left(\frac{2\pi k}{n+1}\right)\right)^m = 0. \tag{24}$$

This identity is true because the numbers $\cos\left(\frac{2\pi k}{n+1}\right) + i\sin\left(\frac{2\pi k}{n+1}\right)$ are the roots x_{k+1} of the polynomial

$$x^{n+1} - 1 = 0, \tag{25}$$

and by the Vieta's formulas:

$$\begin{cases} x_1 + x_2 + \cdots + x_{n+1} = 0, \\ x_1x_2 + \cdots + x_nx_{n+1} = 0, \\ \quad\vdots \\ x_1 \ldots x_n + \cdots + x_2 \ldots x_{n+1} = 0 \\ x_1x_2 \ldots x_{n+1} = -1. \end{cases} \implies \begin{cases} x_1 + ... + x_{n+1} = 0, \\ x_1^2 + ... + x_{n+1}^2 = 0, \\ \quad\vdots \\ x_1^n + ... + x_{n+1}^n = 0, \\ x_1^{n+1} + ... + x_{n+1}^{n+1} = n+1. \end{cases} \tag{26}$$

Thus, the equality (24), and consequently the original system (16), holds for all $m \le n$, and the proof is complete.

Example 1. *The Figures 1 and 2 illustrate the control and individual trajectories in the case of* $n = 4$ *and time* $T = 2\pi + 0.5$.

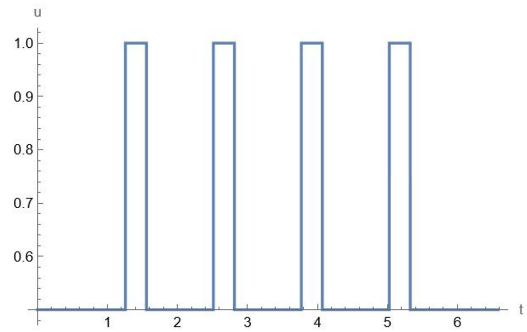

Fig. 1. The graph of control
Рис. 1. Графік керування

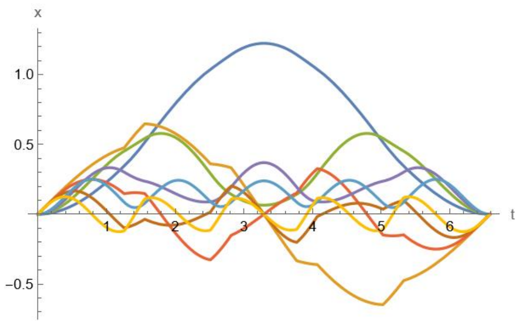

Fig. 2. Individual trajectories
Рис. 2. Індивідуальні траєкторії

The trajectories for coordinates pairs (x_1, x_2) *(blue) and* (x_3, x_4) *(red) are shown on the Figure 3, and for pairs* (x_5, x_6) *(purple) and* (x_7, x_8) *(green) on the Figure 4. If we had constant control* $u = c$ *the trajectories would be circles. However, for the piecewise control considered here, we have some symmetrical trajectories along the arcs of circles.*

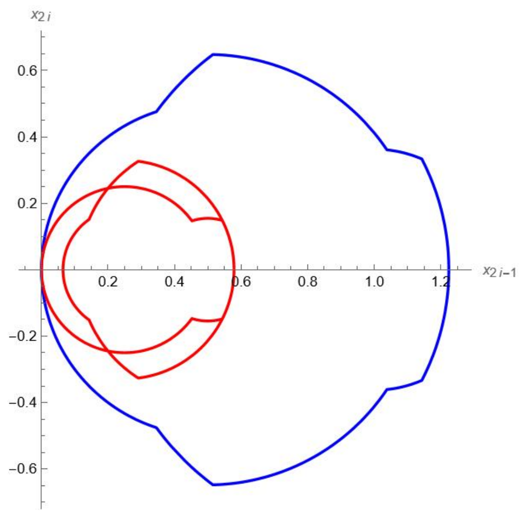

Fig. 3. Pairwise trajectories, $x_1 - x_4$
Рис. 3. Попарні траєкторії, $x_1 - x_4$

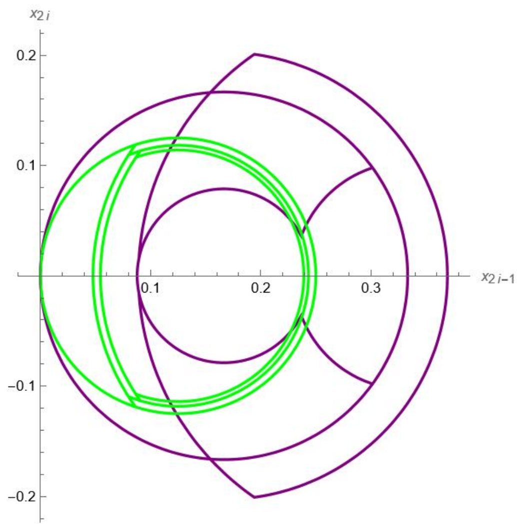

Fig. 4. Pairwise trajectories, $x_5 - x_8$
Рис. 4. Попарні траєкторії, $x_5 - x_8$

Remark 1. *Since the equations (15) are independent for different values of* n*, the control* $u_n(t)$ *is also a solution for any system* $x_{2j-1} = k_j x_{2j}, x_{2j} = -k_j x_{2j-1} + u$, $k_j \in \mathbb{N}$, $j = \overline{1, N}$ *if we select* $n = \max k_j$.

Remark 2. *I we modify any pair of equations to the form* $x_{2j-1} = -k_j x_{2j}$, $x_{2j} = k_j x_{2j-1} + u$ *then the corresponding* 2×2 *block in the matrix exponent changes to*

$$\begin{pmatrix} \cos k\,t & -\sin k\,t \\ \sin k\,t & \cos k\,t \end{pmatrix} \tag{27}$$

changing the corresponding pair of equations to

$$\begin{cases} -\int_0^T u(t)\sin t\,\mathrm{d}t = 0,\\ \int_0^T u(t)\cos t\,\mathrm{d}t = 0, \end{cases} \tag{28}$$

which has the same solution. Therefore, the remark 1 can be extended to the case where $k_j \in \mathbb{Z}\setminus\{0\}$.

Remark 3. *The solution also remains valid for the control constraints of the form $u \in \left\{\frac{d}{2}, d\right\}$, since the constant d simply appears as the multiplicative factor outside the integrals in the equation (16).*

Let us now consider the system with simple purely imaginary eigenvalues $\lambda_{2k-1,2k} = \pm i\,\nu_k$:

$$\begin{cases} \dot{x}_1 = \nu_1 x_2,\\ \dot{x}_2 = -\nu_1 x_1 + u,\\ \dot{x}_3 = \nu_2 x_4,\\ \dot{x}_4 = -\nu_2 x_3 + u,\\ \quad\vdots\\ \dot{x}_{2n-1} = \nu_n x_{2n},\\ \dot{x}_{2n} = -\nu_n x_{2n-1} + u, \end{cases} \tag{29}$$

where $\nu_k = \frac{n_k}{d_k}$, $n_k \in \mathbb{Z}\setminus\{0\}, d_k \in \mathbb{N}$, and $\nu_i \neq \nu_j$ for $i,j = \overline{1,n}, i \neq j$. Let us introduce the rescaled time variable $\tau = \frac{t}{D}$, where $D = \prod_{i=1}^n d_i$. Then for the $k-$th pair of equations, we have:

$$\begin{gathered} \frac{\mathrm{d}x_{2k-1}}{\mathrm{d}\tau} = \frac{\mathrm{d}\,x_{2k-1}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = n_k \prod_{j=1, j\neq k}^{n} (d_j)\, x_{2k} = N_k x_{2k},\\ \frac{\mathrm{d}x_{2k}}{\mathrm{d}\tau} = \frac{\mathrm{d}\,x_{2k}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = -n_i \prod_{j=1, j\neq k}^{n} (d_j)\, x_{2k-1} + Du = -N_k x_{2k-1} + Du, \end{gathered} \tag{30}$$

where $N_k \in \mathbb{Z}\setminus\{0\}$. We may now rescale the control as $v = Du$, $v \in \left\{\frac{D}{2}, D\right\}$, and using the previous remarks, construct the control for the transformed system:

$$\begin{cases} \dot{x}_1 = N_1 x_2,\\ \dot{x}_2 = -N_1 x_1 + v,\\ \dot{x}_3 = N_2 x_4,\\ \dot{x}_4 = -N_2 x_3 + v,\\ \quad\vdots\\ \dot{x}_{2n-1} = N_n\, x_{2n},\\ \dot{x}_{2n} = -N_n\, x_{2n-1} + v. \end{cases} \tag{31}$$

Remark 4. *The result also holds when $\nu_k \in \mathbb{R}$, but there exists a common number $\nu \in \mathbb{R}$ such that $\frac{\nu_k}{\nu} \in \mathbb{Q}$ for all $i = 1, \ldots, n$.*

Let us now consider the general system

$$\dot{x} = Ax + bu, u \in \left\{\frac{d}{2}, d\right\}, \tag{32}$$

where the matrix $A \in \mathbb{R}^{2n\times 2n}$ has simple purely imaginary eigenvalues $\lambda_{2k-1,2k} = \pm i\nu_k$, $k = \overline{1,n}$, and $\nu_k \in \mathbb{Q}$. In this case, there exists an invertible matrix P, such that

$$A = PNP^{-1}, \tag{33}$$

where N is a matrix of the system (29). Then we can write

$$e^{At} = Pe^{Nt}P^{-1}, \tag{34}$$

where

$$e^{Nt} = \begin{pmatrix} \cos\nu_1 t & \sin\nu_1 t & 0 & 0 & \ldots & 0 & 0 \\ -\sin\nu_1 t & \cos\nu_1 t & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \cos\nu_2 t & \sin\nu_2 t & \ldots & 0 & 0 \\ 0 & 0 & -\sin\nu_2 t & \cos\nu_2 t & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \cos\nu_n t & \sin\nu_n t \\ 0 & 0 & 0 & 0 & \ldots & -\sin\nu_n t & \cos\nu_n t \end{pmatrix}, \tag{35}$$

Therefore, each element of the matrix exponent e^{At} is a linear combination of functions $\sin\nu_i t$, $\cos\nu_i t$ and in each equation of the system (9) left-hand side is a linear combination of terms $\int_0^T u(t)\sin\nu_i t$, $\int_0^T u(t)\cos\nu_i t$. Because solving the problem for the system 29 implies that all these terms are equal to zero, the exact same control would also solve the problem for system (32).

Example 2. *Let us consider the system*

$$\begin{cases} \dot{x}_1 = \frac{1}{2}x_2, \\ \dot{x}_2 = -\frac{1}{2}x_1 + u, \\ \dot{x}_3 = 3x_2 + \frac{1}{3}x_4 + u, \\ \dot{x}_4 = x_1 - \frac{1}{3}x_3 + u, \end{cases} \qquad u \in \left\{\frac{1}{2}, 1\right\}. \tag{36}$$

It has eigenvalues $\lambda_{1,2} = \pm\frac{1}{2}i$, $\lambda_{3,4} = \pm\frac{1}{3}i$. In the previous notation we have $D = 2\cdot 3 = 6$, $\tau = \frac{t}{6}, v = 6u$, and a new system of equations

$$\begin{cases} \dot{x}_1 = 3x_2, \\ \dot{x}_2 = -3x_1 + v, \\ \dot{x}_3 = 18x_2 + 2x_4 + v, \\ \dot{x}_4 = 6x_1 - 2x_3 + v, \end{cases} \tag{37}$$

with $\lambda_{1,2} = \pm 3i$, $\lambda_{3,4} = \pm 2i$. *Using the remark 1 we select* $n = 3$ *and obtain control with* 6 *switching moments:* $T_1 = \frac{2\pi}{4}, T_2 = \frac{2\pi}{4} + \alpha$, $T_3 = 2 \cdot \frac{2\pi}{4}, T_4 = 2 \cdot \frac{2\pi}{4} + \alpha$, $T_5 = 3 \cdot \frac{2\pi}{4}, T_6 = 3 \cdot \frac{2\pi}{4} + \alpha$. *Since these switching moments are for the rescaled time variable* τ, *multiplying by* $D = 6$ *gives the switching moments for the original time* t, *and the control is:*

$$u(t) = \begin{cases} \frac{1}{2}, & 0 \le t \le 3\pi, \\ 1, & 3\pi < t \le 3\pi + \beta, \\ \frac{1}{2}, & 3\pi + \beta < t \le 6\pi, \\ 1, & 6\pi < t \le 6\pi + \beta, \\ \frac{1}{2}, & 6\pi + \beta < t \le 9\pi, \\ 1, & 9\pi < t \le 9\pi + \beta, \\ \frac{1}{2}, & 9\pi + \beta < t \le 12\pi + \beta, \end{cases} \tag{38}$$

where $\beta = 6\alpha$. *The figure below illustrates the pairwise trajectories in the case when* $\beta = 1$ *for the pairs of variables* (x_1, x_2) *and* (x_3, x_4). *Since the subsystem for pair* (x_1, x_2) *has the same form as in system (2), its trajectory (blue) is still circular. However, the trajectory for the pair* (x_3, x_4) *(red) is no longer circular or symmetrical.*

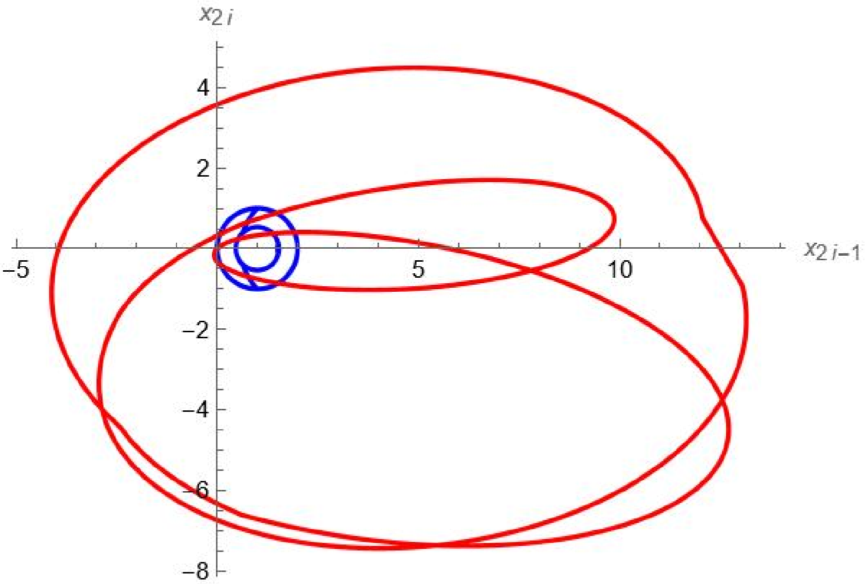

Fig. 5. Pairwise trajectories for the system (36)
Рис. 5. Попарні траєкторії для системи (36)

4. Control with 2 switching moments

The assumption of symmetry also allows us to construct a control with only two switching points in the case $c = \frac{1}{2}$. To do this, we write the system in exponential form as in (12) and consider the control:

$$u(t) = \begin{cases} \frac{1}{2}, & 0 \le t \le T_1, \\ 1, & T_1 < t \le T_2, \\ \frac{1}{2}, & T_2 < t \le T. \end{cases} \tag{39}$$

Let us set $T - T_2 = T_1$. Then, by making the substitution $e^{T_1} = x, e^T = s$, we obtain $e^{T_2} = \frac{s}{x}$. This leads to the following the system of equations for the variables x and s:

$$\begin{cases} -1 - x + \frac{s}{x} + s = 0, \\ -1 - x^2 + \frac{s^2}{x^2} + s^2 = 0, \\ \quad \vdots \\ -1 - x^n + \frac{s^n}{x^n} + s^n = 0, = 0. \end{cases} \tag{40}$$

This system has the solutions $x = s$. We choose s such that: $|s| = 1$. Since $x = e^{T_1}$ and $s = e^T$, both T_1 and T are defined up to the shift 2π. Taking $T = 2\pi + T_1$, we obtain a valid solution to the problem.

Remark 5. *The solution $u(t)$ does not depend on the dimensionality of the system and remains valid for any value of n.*

Remark 6. *An alternative construction of the solution is possible under the constraint $u \in \{c, 1-c, 1\}$, where $0 < c < 1$. In this case the switching points are kept symmetric with respect to the midpoint of the interval, but the values of the control are chosen asymmetrically:*

$$u(t) = \begin{cases} c, & 0 \le t \le T_1, \\ 1, & T_1 < t \le T_2, \\ 1 - c, & T_2 < t \le T. \end{cases} \tag{41}$$

Since all the remarks for the control with $2n$ switching moments remain valid, the we have the following theorem:

Theorem 3. *For the linear system*

$$\dot{x} = Ax + bu, \tag{42}$$

with control constraints $u \in [c, 1]$, where $c \le \frac{1}{2}$ and matrix $A \in \mathbb{R}^{2n \times 2n}$ has simple purely imaginary eigenvalues $\lambda_{2k-1,2k} = \pm i\nu_k$, $k = 1, \dots, n$, the return condition is satisfied if $\nu_k \in \mathbb{Q}$ or $\nu_k \in \mathbb{R} \setminus \mathbb{Q}$, but there exists a positive real number k such that $\frac{\nu_k}{k}$ is rational for all $k = \overline{1, n}$.

Remark 7. *In both cases, the control constraints must be of the form $u \in [c, 1]$, $c \le \frac{1}{2}$ or $u \in \left\{\frac{1}{2}, 1\right\}$. The question for an arbitrary c in the general case remains open.*

5. The case of irrational coefficients

The condition of rationality guarantees the existence of a time moment $T > 0$ such that there exists a solution with control $u = const$. For mutually irrational frequencies such a time T does not exist. However, this does not mean that it is impossible to construct a control that returns the system to the point 0. For example, in the case $n = 2$ the following control is possible:

Statement 1. *For the linear system*

$$\dot{x} = Ax + bu, \tag{43}$$

with eigenvalues $\pm\nu_1 i$, $\pm\nu_2 i$ *the control*

$$u(t) = \begin{cases} \frac{1}{2}, & 0 \le t \le \frac{\pi}{\nu_2}, \\ 1, & \frac{\pi}{\nu_2} < t \le \frac{2m_1\pi}{\nu_1}, \\ \frac{1}{2}, & \frac{2m_1\pi}{\nu_1} < t \le \frac{2m_1\pi}{\nu_1} + \frac{\pi}{\nu_1} + \frac{\pi}{\nu_2}, \\ 1, & \frac{2m_1\pi}{\nu_1} + \frac{\pi}{\nu_1} + \frac{\pi}{\nu_2} < t \le \frac{2m_1\pi}{\nu_1} + \frac{\pi}{\nu_2} + \frac{2m_2\pi}{\nu_2}, \\ \frac{1}{2}, & \frac{2m_1\pi}{\nu_1} + \frac{\pi}{\nu_2} + \frac{2m_2\pi}{\nu_2} < t \le \frac{2m_1\pi}{\nu_1} + \frac{\pi}{\nu_1} + \frac{\pi}{\nu_2} + \frac{2m_2\pi}{\nu_2}, \end{cases} \tag{44}$$

with numbers $m_1, m_2 \in \mathbb{N}$ *chosen such that* $\frac{\pi}{\nu_2} < \frac{2m_1\pi}{\nu_1}, \frac{\pi}{\nu_1} < \frac{2m_2\pi}{\nu_2}$, *returns the system to the origin at time* $T = \frac{2m_1\pi}{\nu_1} + \frac{\pi}{\nu_1} + \frac{\pi}{\nu_2} + \frac{2m_2\pi}{\nu_2}$.

The conditions on m_1 and m_2 are present to ensure that the switching moments are selected in the right order.

Example 3. *As an example, Figure 3 shows the illustrates trajectories for the system*

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 + u, \\ \dot{x}_3 = \sqrt{2}x_4, \\ \dot{x}_4 = -\sqrt{2}x_3 + u, \end{cases} \tag{45}$$

with eigenvalues $\pm i$ *and* $\pm i\sqrt{2}$.

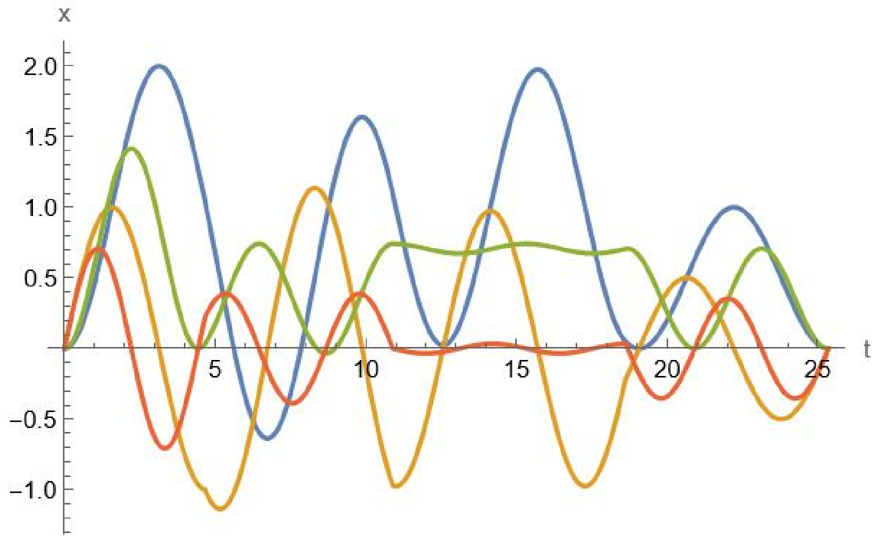

Fig. 6. Individual trajectories for the system (45)
Рис. 6. Індивідуальні траєкторії для системи (45)

Conclusions

In this paper, we considered the problem of null-controllability and the return condition on an interval. For the controllability it was earlier assumed the existence of rest point for the system (1), that is $u = 0 \in \Omega$, implying that $x = 0$ is the rest point.

In the paper, we did not assume the existence of rest point for the system (1) ($u = 0 \notin \Omega$). In such cases the return condition on an interval can be used to establish local controllability. We showed that the return condition on the interval is satisfied for the linear system $\dot{x} = Ax + bu$, where matrix A has simple purely imaginary eigenvalues $\lambda_{2k-1,2k} = \pm i\nu_k$, $k = 1, \ldots, n$, with $\nu_k \in \mathbb{Q}$. We presented two ways of constructing controls that solve this problem under control constraints $u \in \left\{\frac{1}{2}, 1\right\}$.

Also, an example of control construction is discussed for the case when eigenvalue ratios are irrational. However, this case, and the case of control constraints $u = c, 1, c \neq \frac{1}{2}$ require further study.

Conflicts of Interest: The authors declare no conflict of interest.

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Article history: Received: 5 May 2025; Accepted: 9 June 2025.

How to cite this article:

V. I. Korobov, O. S. Vozniak, Return condition for oscillating systems, Visnyk of V. N. Karazin Kharkiv National University. Ser. Mathematics, Applied Mathematics and Mechanics, Vol. 101, 2025, p. 5–20. DOI: 10.26565/2221-5646-2025-101-01

Умова повертання для коливальних систем

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Дана стаття присвячена задачі нуль-керованості для коливальної лінійної системи вигляду $\dot{x}_{2i-1} = x_{2i}, \dot{x}_{2i} = -x_{2i-1} + u$, $i = \overline{1, n}$ з обмеженнями на керування $u \in [c, 1]$ або $u \in \{c, 1\}, c > 0$. У цьому випадку початок координат не є точкою рівноваги. Нуль-керованість означає, що існує такий момент часу T_0, що для будь-якого часу $T \geq T_0$ ми можемо побудувати керування яке досягає початку координат саме в момент часу T. Критерій керованості в точку, що не є рівноважною, був запропонований В. І. Коробовим, і включає нову умову, яка називається умовою повертання на інтервалі, яка повинна виконуватися разом з класичними умовами керованості в точку рівноваги. Ця умова означає, що існує проміжок часу $I = [T, T+\alpha]$, $\alpha > 0$, для якого траєкторія з початком в точці 0 може повернутись назад в будь-який момент часу $T \in I$ за деякого керування $u_T(t)$.

Метою даної роботи є показати, що умови повертання виконуються для розглянутої коливальної системи з даними обмеженнями на керування, і отримати аналітичний розв'язок для керування, що задовольняє цю умову. Розглянутий підхід використовує побудову кусково-сталого керування зі значеннями $u = c$ і $u = 1$. Ця

задача має неєдиний розв'язок, і в нашій статті ми представляємо один розв'язок з $2n$ точками перемикання та інший лише з двома, у випадку, коли $c \leq \frac{1}{2}$. Розв'язок з 2 моментами перемикання є особливо цікавим, оскільки не залежить від розмірності системи. Ми також узагальнюємо задачу на випадок, коли власні значення мають вигляд $\lambda_{2k}, \lambda_{2k-1} = \pm i\nu_k$, де ν_k — раціональні числа. Ми також розглядаємо деякі часткові випадки коли власні значення є ірраціональними.
Ключові слова: **умова повертання на інтервалі, нуль-керованість; тригонометрична проблема моментів; лінійні керовані системи**

Історія статті: отримана: 5 травня 2025; прийнята: 9 червня 2025.