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Korobov's controllability function method via orthogonal polynomials on $[0, \infty)$

Given a controllable system described by ordinary or partial differential equations and an initial state, the problem of finding a set of bounded positional controls that transfer the initial state to another state, not necessarily an equilibrium point, in finite time is called the synthesis problem.

In the present work, we consider a family of Brunovsky systems of dimension n. A family of bounded positional controls $u_n(x)$ is constructed in order to stabilize a given Brunovsky system in finite time. We employ orthogonal polynomials associated with a function distribution $\sigma(\tau, \theta)$ defined for $\tau \in [0, +\infty)$ and parameter $\theta > 0$. The parameter θ is interpreted as a Korobov's controllability function, $\theta = \theta(x)$, which serves as a Lyapunov-type function. Utilizing $\theta(x)$, we construct the positional control $u_n(x) = u_n(x, \theta(x))$.

Our analysis is based on the foundational work "A general approach to the solution of the bounded control synthesis problem in a controllability problem". Matematicheskii Sbornik, 151(4), 582–606 (1979) by Korobov, V. I, in which the controllability function method was proposed. This method has been applied to solve bounded finite-time stabilization problems in various control scenarios, such as the control of the wave equation, optimal control with mixed cost functions, and other applications.

For the construction of the mentioned positional controls, we employ a member of a family of orthogonal polynomials on $[0, \infty)$. For more details on orthogonal polynomials, we refer to the book "Orthogonal Polynomials". American Mathematical Society, Providence, (1975) by G. Szegö. We also rely on the work "On matrix Hurwitz type polynomials and their interrelations to Stieltjes positive definite sequences and orthogonal matrix polynomials". Linear Algebra and its Applications, 476, 56–84 (2015) by Choque Rivero, A. E.

The results in the present work extend and develop the findings presented in the conference paper "Bounded finite-time stabilizing controls via orthogonal polynomials". 2018 IEEE International Autumn Meeting on Power, Electronics and Computing (ROPEC), Ixtapa, Mexico. –2018 by Choque-Rivero A. E., Orozco B. d. J. G.

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1. Introduction

We focus on the Brunovsky system, also referred to as the chain integrator or canonical linear system. It is commonly used in control theory to analyze the controllability and feedback stabilizability of both linear and nonlinear systems–after applying an appropriate transformation for the latter; see [1], [2]. The statement of the bounded finite time stabilization problem is the following: Consider the system

$$\begin{cases} \dot{x}_1 = u_n, & |u_n| \leq 1, \\ \dot{x}_k = x_{k-1}, & 2 \leq k \leq n. \end{cases} \tag{1}$$

Let $x := \text{column}(x_1, x_2, \ldots, x_n)$. Given an initial position $x_0 \in \mathbb{R}^n$, find a family of positional bounded controls $u_n = u_n(x)$ such that the trajectory $x(t, x_0)$ of system (1) with $u_n = u_n(x)$ and

$$|u_n(x)| \leq 1 \tag{2}$$

reaches the origin at finite time $T = T(x_0)$, i.e.,

$$\lim_{t \to T(x_0)} x(t, x_0) = 0. \tag{3}$$

Additionally, an estimation of motion time $T(x_0)$ should be found. This problem is also called the synthesis problem.

For our convenience, we reformulate system (1) in a matrix form. To that end, we introduce a following notation. Let p, q and n be natural numbers. Denote by $\mathbf{I}_p$ the $p \times p$ identity matrix and by $0_{p \times q}$ the $p \times q$ zero matrix. System (1) is equivalent to the following equation:

$$\dot{x} = \mathbf{A}_n x + b_n u_n, \tag{4}$$

where

$$\mathbf{A}_n = \begin{pmatrix} 0_{1\times(n-1)} & 0 \\ \mathbf{I}_{n-1} & 0_{(n-1)\times 1} \end{pmatrix}, \; b_n := \begin{pmatrix} 1 \\ 0_{(n-1)\times 1} \end{pmatrix}.$$

Our approach involves constructing the positional control $u = u_n(x)$ based on two key components. The first component is the set of orthogonal polynomials

$p_n(\tau,\theta)$ on $[0,+\infty)$ with respect to τ, depending on the parameter $\theta > 0$. The case of orthogonal polynomials $p_n(\tau,1)$ has been studied in [3] and [4].

The second component is the controllability function (CF) $\theta(x)$ method, which serves as a Lyapunov-type function but with two key distinctions. First, the CF ensures finite-time stabilization of the controllable system (4). Second, the CF can also be applied to nonequilibrium points of linear systems (see [5]).

It is worth noting that another connection between orthogonal polynomials and bounded controls, aimed at solving the admissible control problem, was explored in [6].

The present work continues the analysis performed in [7] and provides the following improvements and clarifications:

- We clarify the definition of the polynomials $p_{r,j}$ as given in (13) and (14).
- We improve the proof of Lemma 2 and provide the proof of Theorem 3.
- In Example 2, we present a more detailed explanation.
- We add Remark 4, which explains the controllability function as the time of motion.
- We include Proposition 2, where we prove that it is not possible to construct a positional control such that the corresponding controllability function represents the time of motion of the system's trajectory.

Finally, in the conclusion, we have proposed an open question regarding the possibility of constructing positional controls using a combination of orthogonal polynomials, such that the corresponding controllability function represents the time of motion.

Orthogonal polynomials on $[0,+\infty)$ with parameter θ

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and let $\sigma(\tau,\theta)$ be a bounded nondecreasing function with respect to τ on $[0,+\infty)$ with parameter $\theta > 0$ such that all moments

$$s_j(\theta) := \int_0^\infty \tau^j d\sigma(\tau,\theta), \quad j \in \mathbb{N}_0 \tag{5}$$

are finite. In the present work, we will restrict ourselves to the case when

$$s_k(\theta) = \frac{c_k}{\theta^k}, \quad k \in \mathbb{N}_0 \tag{6}$$

where c_k is a real number, and θ is a positive parameter. For simplicity, we will typically omit the dependence on θ.

Define the Hankel matrices:

$$\mathbf{H}_{1,j} := \begin{pmatrix} s_0 & s_1 & \dots & s_j \\ s_1 & s_2 & \dots & s_{j+1} \\ \vdots & \vdots & \vdots & \vdots \\ s_j & s_{j+1} & \dots & s_{2j} \end{pmatrix}, \tag{7}$$

$$\mathbf{H}_{2,j} := \begin{pmatrix} s_1 & s_2 & \dots & s_{j+1} \\ s_2 & s_3 & \dots & s_{j+2} \\ \vdots & \vdots & \vdots & \vdots \\ s_{j+1} & s_{j+2} & \dots & s_{2j+1} \end{pmatrix}. \tag{8}$$

Definition 1. *The sequence* $(s_j(\theta))_{j=0}^{2n}$ *(resp.* $(s_j(\theta))_{j=0}^{2n-1}$*) is called a Stieltjes positive definite sequence if* $\mathbf{H}_{1,n}$ *and* $\mathbf{H}_{2,n-1}$ *(resp.* $\mathbf{H}_{1,n-1}$ *and* $\mathbf{H}_{2,n-1}$*) are positive definite matrices.*

In what follows, we consider only Stieltjes positive definite sequences.

Additionally, we introduce some auxiliary matrices:

$$Y_{1,j} := \text{column}\,(s_j, s_{j+1}, \dots, s_{2j-1})\,, \tag{9}$$

and

$$Y_{2,j} := \text{column}\,(s_{j+1}, s_{j+2}, \dots, s_{2j})\,. \tag{10}$$

Let τ be an arbitrary real number. We define

$$E_j(\tau) := \text{column}\left(1, \tau, \dots, \tau^j\right). \tag{11}$$

We define (as in [8, Remark E.5]) the polynomials $p_{r,j}$ for $r = 1, 2$ and $1 \le j \le n$ as follows

$$p_{r,0}(\tau,\theta) := 1, \tag{12}$$

$$p_{1,j}(\tau,\theta) := (-Y_{1,j}^{\mathsf{T}}\mathbf{H}_{1,j-1}^{-1}, 1)E_j(\tau), \tag{13}$$

$$p_{2,j}(\tau,\theta) := (-Y_{2,j}^{\mathsf{T}}\mathbf{H}_{2,j-1}^{-1}, 1)E_j(\tau). \tag{14}$$

Let us recall the definition of a finite family of orthogonal polynomials depending on the parameter θ. Here $Y_{r,j} = Y_{r,j}(\theta)$ and $\mathbf{H}_{r,j} = \mathbf{H}_{r,j}(\theta)$ for $r = 1, 2$.

Definition 2. *Let there be given a sequence of polynomials* $(p_j(\tau,\theta))_{j=0}^{\infty}$ *defined on* $[0, +\infty)$ *with respect to* τ *with parameter* $\theta > 0$

$$p_j(\tau,\theta) = \tau^n + \frac{\widetilde{a}_1}{\theta}\tau^{n-1} + \frac{\widetilde{a}_2}{\theta^2}\tau^{n-2} + \dots + \frac{\widetilde{a}_{j-1}}{\theta^{j-1}}\tau + \frac{\widetilde{a}_j}{\theta^j}. \tag{15}$$

If there exists a bounded nondecreasing distribution $\sigma(\tau,\theta)$ *with respect to* τ *on* $[0, +\infty)$ *with parameter* $\theta > 0$ *such that the sequence of monic polynomials* $(p_j(\tau,\theta))_{j=0}^{\infty}$ *satisfies the relation*

$$\int_0^{\infty} p_j(\tau,\theta)p_k(\tau,\theta)d\sigma(\tau,\theta) = \begin{cases} 0, & j \ne k, \\ d_j(\theta), & j = k, \end{cases} \quad d_j(\theta) > 0,$$

then this sequence is family a of orthogonal polynomials.

Remark 1. *a) The orthogonality of polynomials $p_{r,j}$ for $r = 1, 2$, $j \geq 0$ with $\theta = 1$ defined by (12)-(14) was proved in [9].*
b) The family $\{p_{1,j}(\tau)\}$ (resp. $\{p_{2,j}(\tau)\}$) is orthogonal on $[0, \infty)$ with respect to a nondecreasing distribution $d\sigma(\tau)$ (resp. $\tau d\sigma(\tau)$) [10].

Orthogonal polynomials have been widely applied to practical problems, including signal processing [11] and filter design [12], [13]. Additionally, the zeros of certain families of orthogonal polynomials can be interpreted as the electrostatic energy of a system with a finite number of charges (see [14]).

Example 1. *Let $d\sigma(\tau, \theta) = \tau^\alpha e^{-\tau\theta} d\tau$. For $j \geq 0$ and parameters $\theta > 0$ and $\alpha > -1$, the corresponding moments are given by $s_j(\alpha, \theta) = \frac{\int_0^\infty \tau^j \tau^\alpha e^{-\tau\theta} d\tau}{\int_0^\infty \tau^\alpha e^{-\tau\theta} d\tau}$. The polynomials (12) and (13) constructed from these moments are referred to as monic generalized-type Laguerre polynomials. For $j = 2$ and $j = 3$, we have:*

$$\begin{aligned} p_{1,2}(-\tau, \theta) =& \frac{(\alpha+1)(\alpha+2)}{\theta^2} + \frac{2(\alpha+2)\tau}{\theta} + \tau^2, \\ -p_{1,3}(-\tau, \theta) =& \frac{(\alpha+1)(\alpha+2)(\alpha+3)}{\theta^3} + \frac{3(\alpha+2)(\alpha+3)\tau}{\theta^2} \\ &+ \frac{3(\alpha+3)\tau^2}{\theta} + \tau^3. \end{aligned} \tag{16}$$

Note that $(p_{1,j}(\tau, 1))_{j\geq 0}$ represents the classical Laguerre polynomials. By applying the Routh-Hurwitz criterion, one verifies that $p_{1,2}(-\tau, \theta)$ and $-p_{1,3}(-\tau, \theta)$ are Hurwitz polynomials.

Controllability function $\theta(x)$

The CF method, introduced by V.I. Korobov in 1979 [15], has been applied to a variety of control problems [16], [17], [18], [19], [20], [32]. In [21], the method is used to construct finite-time stabilizing positional controls for wave equations and linear systems with a mixed cost functional. In [22] and [23], a family of bounded positional controls is developed, with the CF $\theta(x)$ representing the exact time of motion for both single-variable and multivariable controls. Additionally, in [24], bounded finite-time stabilizing controls are derived for a family of nonlinear control systems.

Let us now return to the Brunovsky system (4). For the solution of the synthesis problem, the positional control [15], [21]

$$u_n(x) = \sum_{k=1}^{n} \frac{a_{n,k} x_k}{\theta^k(x)}, \tag{17}$$

was proposed. The positional control (17) can be written as

$$u_n(x) = \theta^{-1/2}(x) a_n^\intercal \mathbf{D}_{\theta(x)} x \tag{18}$$

where

$$a_n := \text{column}\,(a_{n,1}, a_{n,2}, \ldots, a_{n,n}) \tag{19}$$

and

$$\mathbf{D}_\theta := \text{diag}\left(\theta^{-\frac{2j-1}{2}}\right)_{j=1}^{n}. \tag{20}$$

Let a_n be such that the matrix $\mathbf{A}_n + b_n a_n^\intercal$ has eigenvalues with negative real part. Thus, for $\mathbf{W}_n$ being a positive definite matrix, the Lyapunov equation

$$\mathbf{F}_n(\mathbf{A}_n + b_n a_n^\intercal) + (\mathbf{A}_n + b_n a_n^\intercal)^\intercal \mathbf{F}_n = -\mathbf{W}_n \tag{21}$$

has a unique solution $\mathbf{F}_n$, such that $\mathbf{F}_n$ is a positive definite matrix; [25, Theorem 3.20].

The proof of the following result is given in [15, Page 542].

Proposition 1. *Let $\mathbf{F}_n$ be a solution to (21), and let*

$$0 < a_0 \leq \frac{1}{2a_n^\intercal \mathbf{F}_n^{-1} a_n}. \tag{22}$$

Furthermore, let

$$\mathbf{H}_n := \text{diag}\left(-\frac{2j-1}{2}\right)_{j=1}^{n}, \tag{23}$$

and

$$\widetilde{\mathbf{F}}_n := \mathbf{F}_n - \mathbf{H}_n\mathbf{F}_n - \mathbf{F}_n\mathbf{H}_n. \tag{24}$$

Thus, for a positive definite matrix $\widetilde{\mathbf{F}}_n$, the following equation

$$2a_0\theta = x^\intercal \mathbf{D}_\theta \mathbf{F}_n \mathbf{D}_\theta x \tag{25}$$

has a unique positive solution $\theta = \theta(x)$.

Recall that the matrix $\widetilde{\mathbf{F}}_n$ (24) appears when taking the derivative with respect to time on both the left- and right-hand sides of (25). Specifically, after taking the derivative of the left-hand side of (25) one obtains the quadratic form $\frac{1}{\theta}x^\intercal \mathbf{D}_\theta \widetilde{\mathbf{F}}_n \mathbf{D}_\theta x\,\dot{\theta}$, while on the right-hand side one obtains $\frac{1}{\theta}x^\intercal\left(\mathbf{D}_\theta \mathbf{F}_n(\mathbf{A}_n + b_n a_n^\intercal) + (\mathbf{A}_n + b_n a_n^\intercal)^\intercal \mathbf{F}_n\right)\mathbf{D}_\theta x$.

The special cases when $\det \widetilde{\mathbf{F}}_n = 0$ and when $\widetilde{\mathbf{F}}_n$ is an indefinite matrix were studied in [26] and [27]. In both these works, it was assumed that $\mathbf{F}_n$ is positive definite. Note that when $\det \widetilde{\mathbf{F}}_n = 0$ or $\widetilde{\mathbf{F}}_n$ is an indefinite matrix, Equation (25) does not have a unique, simple positive solution $\theta(x)$. In contrast, when $\widetilde{\mathbf{F}}_n$ is a positive definite matrix, Equation (25) has a unique positive solution [15].

The value $\theta(x_0)$ is the root the function

$$\Psi(\theta, x_0) = 2a_0\theta - x_0^\intercal \mathbf{D}_\theta \mathbf{F}_n \mathbf{D}_\theta x_0. \tag{26}$$

Note that $\theta^{2n-1}\Psi(\theta, x_0)$ is a polynomial of degree $2n$ on the variable θ; see [27, Equality (7.2)].

For the proof of the next result, see [15, Pages 545-547].

Theorem 1. *The positional control (17) transfers any initial point $x_0 \in \mathbb{R}^n$ to the origin along the trajectory of the system $\dot{x} = \mathbf{A}_n x + b_n u_n(x)$ in time*

$$T(x_0) \le \frac{\theta(x_0)}{\beta} \tag{27}$$

and satisfies the restriction $|u_n(x)| \le 1$. Here $-\beta$ is the largest eigenvalue of the matrix pencil $\mathbf{W}_n + \lambda\widetilde{\mathbf{F}}_n$.

Note that the control $u_n(x)$ and the estimation of the time $T(x_0)$ depend on the selection of a_n and $\mathbf{W}_n$.

2. Positional controls through orthogonal polynomials

Relation between orthogonal polynomials and Hurwitz polynomials An explicit relation between Hurwitz polynomials and orthogonal polynomials was considered in [9], [28], [33] and [34]. The mentioned relation indicates that every Hurwitz polynomial $p_n(\tau)$ is represented by $h_n(\tau^2) + \tau g_n(\tau^2)$, where h_n and g_n are orthogonal polynomials on $[0, +\infty)$ or their second kind polynomials.

Recall that the polynomials

$$q_{1,j}(\tau) := \int_0^\infty \frac{p_{1,j}(\tau) - p_{1,j}(t)}{\tau - t} d\sigma(t), \tag{28}$$

$$q_{2,j}(\tau) := \int_0^\infty \frac{\tau p_{2,j}(\tau) - t p_{2,j}(t)}{\tau - t} d\sigma(t), \tag{29}$$

for $j \ge 0$ are called second-kind polynomials associated with the polynomials $p_{1,j}$ and $p_{2,j}$, as in (13) and(14); [8, Remark E.4] and [8, Lemma E.11]. The distribution $d\sigma(t)$ appearing in (28) and (29) is the same as the distribution mentioned in part (b) of Remark 1.

For the convenience of the reader, we reproduce Theorem 3.3.1 of [4] by G. Szegö on the locations of the zeros of orthogonal polynomials.

Theorem 2. *The zeros of the orthogonal polynomials $p_n(\tau)$, associated with the distribution $d\alpha(\tau)$ on the interval $[a, b]$, are real and distinct and are located in the interior of the interval $[a, b]$.*

Note that on page 1 of [4], it is stated that the notation $[a, b]$ is also used to denote the interval $[0, \infty)$.

In this paper, we highlight another relationship between orthogonal polynomials and Hurwitz polynomials.

Lemma 1. *Let $(s_j(\theta))_{j=0}^{2n}$ (resp. $(s_j(\theta))_{j=0}^{2n-1}$) be a Stieltjes positive sequence. Let the polynomials $p_{r,j}(\tau, \theta)$ for $r = 1, 2$, with parameter $\theta > 0$ be as in (13) and (14). Thus, the polynomials*

$$(-1)^j p_{r,j}(-\tau, \theta) \tag{30}$$

for $1 \le j \le n$ are Hurwitz or stable polynomials.

The proof of this lemma is based on Theorem 2, which ensures that the roots of polynomials of the form (15), such as $p_{r,n}(\tau,\theta)$ as defined in (13) and (14), belong to the interval $[0,+\infty)$ for a fixed θ. It remains to 'correct' the coefficients of $p_{r,n}(\tau,\theta)$, which, according to the necessary condition for Hurwitz polynomials, should be positive numbers. To this end, we change the independent variable τ to $-\tau$ in (13), (14) and multiply this polynomial by $(-1)^n$.

Remark 2. *a) From Theorem 2 and Lemma 1, it readily follows that, for fixed $\theta > 0$, the roots of the polynomial $(-1)^n p_{r,n}(-\tau,\theta)$ belong to $(-\infty,0]$.*
b) Since the roots of $(-1)^n p_{r,n}(-\tau,\theta)$ are distinct (see Theorem 2), the set of polynomials $\{(-1)^n p_{r,n}(-\tau,\theta)\}$ is a subset of Hurwitz polynomials with real negative roots.

Remark 3. *For fixed $\theta > 0$, the polynomial $f_2(t,\theta) = t^2 + \frac{2t}{\theta} + \frac{2}{\theta^2} = \left(t+\frac{1-i}{\theta}\right)\left(t+\frac{1+i}{\theta}\right)$ is a Hurwitz polynomial of the form (15). The roots of f_2 are complex numbers. Thus, taking into account Theorem 2 and Lemma 1, we conclude that Hurwitz polynomials with complex roots are not included in the set of polynomials defined by (30).*

Bounded finite-time stabilizing controls We propose the bounded positional control $u_n(x)$ that stabilizes the system (4) based on the orthogonal polynomials (13) – (14).

Remark 4. *For fixed θ, let $a_n^\intercal(\theta) = \theta^{-1/2} a_n^\intercal \mathbf{D}_\theta$. Substituting the positional control (17) in (4) for fixed θ, we have the system*

$$\dot{x} = (\mathbf{A}_n + b_n a_n^\intercal(\theta))x. \tag{31}$$

The characteristic polynomial $\widetilde{p}_n$ of system (31) has the form

$$\begin{aligned}
&\widetilde{p}_n(\tau,\theta) = \det(\tau I_n - \mathbf{A}_n - b_n a_n^\intercal(\theta))\\
&= \Big(\tau^n - \frac{a_{n,n}}{\theta^n}\tau^{n-1} - \frac{a_{n,n-1}}{\theta^{n-1}}\tau^{n-2} - \ldots - \frac{a_{n,2}}{\theta^2}\tau - \frac{a_{n,1}}{\theta}\Big)\\
&= \Big(1, -\Big(\frac{a_{n,n}}{\theta^n}, \frac{a_{n,n-1}}{\theta^{n-1}}, \ldots, \frac{a_{n,2}}{\theta^2}, \frac{a_{n,1}}{\theta}\Big)\Big)\begin{pmatrix}\tau^n\\ \tau^{n-1}\\ \ldots\\ \tau\\ 1\end{pmatrix}\\
&= \tau^n - \Big(\frac{a_{n,1}}{\theta}, \frac{a_{n,2}}{\theta^2}, \ldots, \frac{a_{n,n-1}}{\theta^{n-1}}, \frac{a_{n,n}}{\theta^n}\Big)\begin{pmatrix}1\\ \ldots\\ \tau^{n-2}\\ \tau^{n-1}\end{pmatrix}.
\end{aligned} \tag{32}$$

The following lemma enables us to determine the control coefficients with the help of orthogonal polynomials.

Lemma 2. *Let $(s_j(\theta))_{j=0}^{2n}$ (resp. $(s_j(\theta))_{j=0}^{2n-1}$) be a Stieltjes positive definite sequence. For $r = 1, 2$, let $Y_{r,n}$, $\mathbf{H}_{r,n-1}$ and $p_{r,n}$ be as in (7), (8), (9), (10), (13) and (14), respectively. Define $a_{n,i}$ by*

$$\left(\frac{a_{n,1}}{\theta}, \frac{a_{n,2}}{\theta^2}, \ldots, \frac{a_{n,n}}{\theta^n}\right) = (-1)^n Y_{r,n}^{\intercal} \mathbf{H}_{r,n-1}^{-1} \mathbf{J}_{n-1}, \tag{33}$$

where $a_{n,j}$ are negative numbers and

$$\mathbf{J}_n := \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & -1 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & (-1)^{n-1} & \ldots & 0 & 0 \\ (-1)^n & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

Then, the characteristic polynomial $\widetilde{p}_n(\tau, \theta)$ of the matrix $\mathbf{A}_n + b_n a_n^{\intercal}(\theta)$ is given by $\widetilde{p}_n(\tau, \theta) = (-1)^n p_{r,n}(-\tau, \theta)$.

Proof. Let $r = 1$. Using (13) and (11), we have:

$$\begin{aligned}(-1)^n p_{1,n}(-\tau, \theta) =& (-1)^n (-Y_{1,n}^{\intercal} \mathbf{H}_{1,n-1}^{-1}, 1) E_n(-\tau) \\ =& (-1)^n (-Y_{1,n}^{\intercal} \mathbf{H}_{1,n-1}^{-1}, 1) \begin{pmatrix} 1 \\ \ldots \\ (-1)^{n-1}\tau^{n-1} \\ (-1)^n \tau^n \end{pmatrix} \\ =& \tau^n + (-1)^{n+1} Y_{1,n}^{\intercal} \mathbf{H}_{1,n-1}^{-1} \mathbf{J}_{n-1} \begin{pmatrix} 1 \\ \ldots \\ \tau^{n-2} \\ \tau^{n-1} \end{pmatrix}. \end{aligned} \tag{34}$$

From (32) and (34), we obtain (33). The proof for $r = 2$ is similar. □

Now, we formulate one of the main results of the present work.

Theorem 3. *For $r = 1, 2$, let $p_{r,n}(\tau, \theta)$ be a polynomial defined on $[0, +\infty)$ with respect to τ with parameter $\theta > 0$ as in (13) and (14). Let $a_{n,j}$ be defined as in (33), and let $\mathbf{F}_n$ be a solution of (21). Furthermore, let $\theta(x)$ be the solution of equation (25) with a_0 as in (22). Thus, the positional control of the form (17) solves the synthesis problem for system (4).*

Proof. Taking into account Lemma 2, the definition of $p_{r,j}$ for $r = 1, 2$ as in (13) and (14) and Equality (33), we see that positional control as in (17) satisfies all assumptions of Theorem 1. □

Due to the representation of the characteristic polynomial of $\mathbf{A}_n + b_n a_n^{\intercal}(\theta)$ for fixed θ as in (32) and the representation of the orthogonal polynomial as in (34), in combination with Lemma 1, we see that a_n, chosen via (33), guarantees that the matrix $\mathbf{A}_n + b_n a_n^{\intercal}(\theta)$ is Hurwitz.

From Theorem 1, we know that every member of the family of orthogonal polynomials, as defined in (13) and (14), generates a positional control that solves the synthesis problem for the canonical system.

Remark 5. *[21] To construct the graphic of the trajectory $x(t)$, as well as the CF $\theta(x)$ and the bounded control $u_n(x)$ both on the trajectory $x = x(t)$ one proceeds to solve the following Cauchy problem:*

$$\begin{aligned}
\dot{x}_1 &= \sum_{k=1}^{n} \frac{a_{n,k} x_k}{\theta^k(x)}, \\
\dot{x}_k &= x_{k-1}, \quad k = 2, \dots n, \\
\dot{\theta} &= -\frac{x^{\mathsf{T}} \mathbf{D}_\theta \mathbf{W}_n \mathbf{D}_\theta \, x}{x^{\mathsf{T}} \mathbf{D}_\theta \widetilde{\mathbf{F}}_n \mathbf{D}_\theta \, x}, \\
x(0) &= x_0, \quad \theta(0) = \theta_0,
\end{aligned}$$

where x_0 is the given initial position and θ_0 is the solution of (25) for $x = x_0$.

Example 2. *Consider the polynomial (16). By using (33), (21) and (22), we compute $\mathbf{F}_2$, $\widetilde{\mathbf{F}}_2$ and $2a_0$ with $\mathbf{W}_2 = \mathbf{I}_2$. For $\alpha > -1$, we have*

$$\mathbf{F}_2 = \begin{pmatrix} \frac{\alpha^2+3\alpha+3}{4(\alpha+1)(\alpha+2)^2} & \frac{1}{2(\alpha+1)(\alpha+2)} \\ \frac{1}{2(\alpha+1)(\alpha+2)} & \frac{\alpha^3+4\alpha^2+10\alpha+11}{4(\alpha+1)(\alpha+2)} \end{pmatrix}, \tag{35}$$

$$\widetilde{\mathbf{F}}_2 = \begin{pmatrix} \frac{\alpha^2+3\alpha+3}{2(\alpha+1)(\alpha+2)^2} & \frac{3}{2(\alpha+1)(\alpha+2)} \\ \frac{3}{2(\alpha+1)(\alpha+2)} & \frac{\alpha^3+4\alpha^2+10\alpha+11}{(\alpha+1)(\alpha+2)} \end{pmatrix}, \tag{36}$$

$$2a_0 = \frac{\alpha^4 + 6\alpha^3 + 19\alpha^2 + 34\alpha + 25}{4(\alpha+2)^3 \left(5\alpha^4 + 29\alpha^3 + 74\alpha^2 + 109\alpha + 75\right)}. \tag{37}$$

For the value of $2a_0$ (37), we have selected the equality in (22).

For $\alpha = -1/2$, polynomial (16) has the form $p_{1,2}(-\tau, \theta) = \tau^2 + \frac{3}{\theta}\tau + \frac{3}{4\theta^2}$ and

$$\mathbf{F}_2 = \begin{pmatrix} \frac{7}{18} & \frac{2}{3} \\ \frac{2}{3} & \frac{55}{24} \end{pmatrix}, \ \widetilde{\mathbf{F}}_2 = \begin{pmatrix} \frac{7}{9} & 2 \\ 2 & \frac{55}{6} \end{pmatrix}, \quad 2a_0 = 386/15417.$$

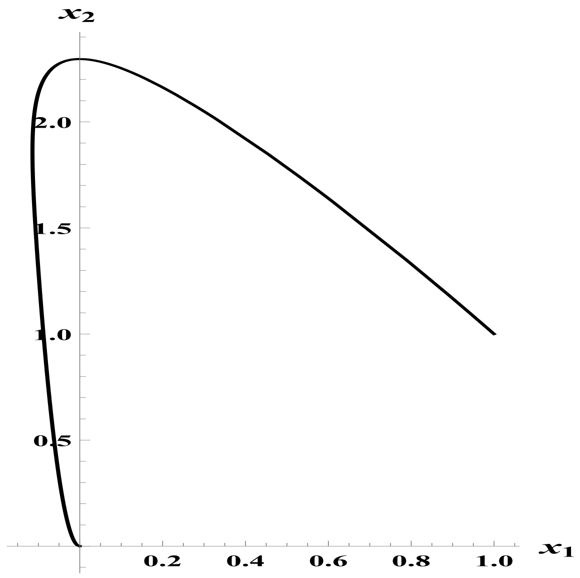
The equation (25) has the form

$$-\frac{386}{15417}\theta + \frac{7x_1^2}{18\theta} + \frac{4x_2x_1}{3\theta^2} + \frac{55x_2^2}{24\theta^3} = 0. \tag{38}$$

Let $x_0 = (1,1)$ be the initial position. The unique positive solution of (38) for $x_0 = (1,1)$ is equal to $\theta_0 = 5.35449$. The largest eigenvalue of matrix $\mathbf{I}_2 + \lambda\widetilde{\mathbf{F}}_2$ is equal to -0.10396. By employing (27), we obtain that the time of movement from $x_0 = (1,1)$ to origin satisfies the following inequality $T(x_0) \leq 51.5053$. The corresponding positional control has the form

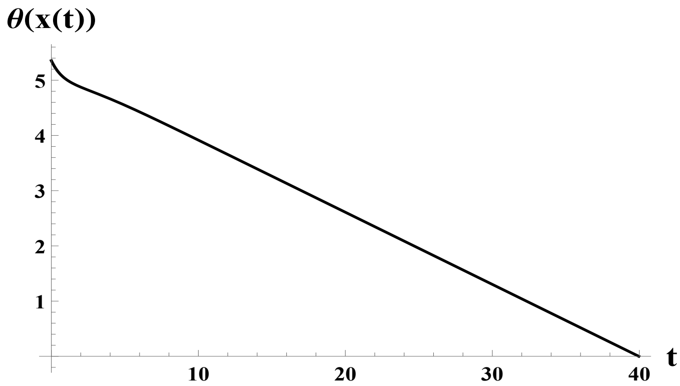
$$u_2(x_1, x_2) = -\frac{3x_1}{4\theta(x_1,x_2)} - \frac{3x_2}{\theta^2(x_1,x_2)}$$

where $\theta(x_1, x_2)$ is the unique positive solution of (38).
The following figure represents the phase portrait for the initial position $x_0 = (1, 1)$.



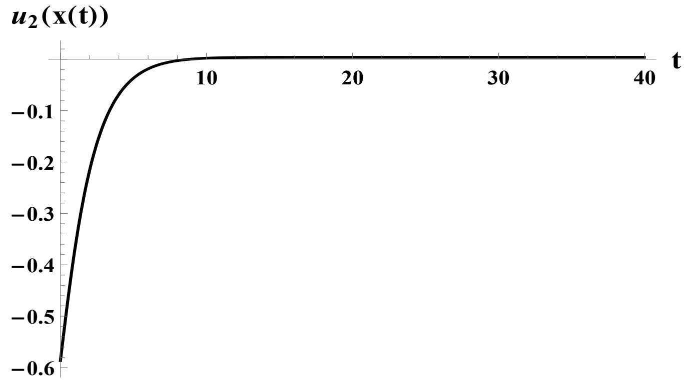
Pic. 1. Phase portrait for the initial position $x_0 = (1, 1)$.
Рис. 1. Фазовий портрет для початкової позиції $x_0 = (1, 1)$

The graphic for the CF on the trajectory $\theta(x(t))$ has the following form.



Pic. 2. The graphic for the CF on the trajectory $\theta(x(t))$.
Рис. 2. Графік для CF на траєкторії $\theta(x(t))$.

Finally, the graphic of the position control on the trajectory $u_2(x(t))$ is the following.



Pic. 3. The graphic of the position control on the trajectory $u_2(x(t))$.
Рис. 3. Графік позиційного керування на траєкторії $u_2(x(t))$.

2. The CF as motion time is unobtainable via orthogonal polynomials

In this section, for $n = 2$ and for $n = 3$, we demonstrate that it is not possible to construct positional controls $u(x, \theta(x))$ as in (17) through orthogonal polynomials $p_{1,j}$ (resp $p_{2,j}$) as in (13) (resp. (14)). To this end, in the next remarks, we reproduce important results concerning the derivative of the CF with respect to time $\dot{\theta}$ and the estimation of time of motion $T(x_0)$.

Remark 6. *In [29, Theorem 1] and [30, Theorem 1] while studying finite-time stability or finite-time output feedback control, a slight modification of the fundamental inequality*

$$\dot{\theta}(x) \leq -\beta\theta^{1-\frac{1}{\alpha}}(x) \tag{39}$$

and the time estimation

$$T(x_0) \leq \frac{\alpha}{\beta}\theta^{1/\alpha}(x_0) \tag{40}$$

were used. Here α and β are positive numbers. Inequalities (39) and (40) appear as part of [15, Theorem 1], which gives sufficient conditions for a bounded multidimensional control $u(x, \theta(x))$ to stabilize at finite time the control system $\dot{x} = f(x, u)$, where f satisfies a Lipschitz condition in a subset of the space (x, u).

The following remark recalls the case when $\theta(x_0)$ is exactly the time of motion from x_0 to the origin. For the canonical system (4), this occurs if

$$\mathbf{W}_n = \widetilde{\mathbf{F}}_n. \tag{41}$$

Remark 7. *a) If inequality (39) becomes an equality with $\alpha = 1$ and $\beta = 1$, we have*

$$\dot{\theta} = -1. \tag{42}$$

This case was studied in [31], [22], [23], [26], [27]. Integrating (42) with respect to time t from 0 to t, we obtain

$$\theta(x(0)) - \theta(x(t)) = t. \tag{43}$$

Let $T(x_0)$ be motion time as in (40) and (3). Due condition 1) of [15, Theorem 1], we have

$$\theta(x(T(x_0))) = 0. \tag{44}$$

For $t = T(x_0)$ in (42), taking into account (44), we have that

$$\theta(x(0)) = T(x_0). \tag{45}$$

b) The importance of (45) is evident, namely,

$$\theta_0 = \theta(x(0)) \tag{46}$$

represents exactly the time of motion from the initial state x_0 to the origin.

Let $n = 2$ in (4). From [26, Equality (1.10)], for $a_{2,2} < -4$, we know that

$$u_2(x) = -\frac{3x_1}{\theta(x_1, x_2)} + \frac{a_{2,2}x_2}{\theta^2(x_1, x_2)} \tag{47}$$

is the positional control which solves the synthesis problem for the canonical system when the controllability function is the motion time. The control (47) is constructed for the case when $\mathbf{F}_2$ is a positive definite matrix, while $\widetilde{\mathbf{F}}_2$ can be a positive definite matrix, or an indefinite matrix, or satisfy $\det \widetilde{\mathbf{F}}_2 = 0$.

Proposition 2. *For $n = 2$ and $n = 3$, the controllability function as motion time is not obtainable via orthogonal polynomials on $[0, \infty)$.*

Proof. Let $n = 2$. Inserting the control (47) in (4) and computing the characteristic polynomial of matrix

$$\widetilde{\mathbf{A}}_2 = \begin{pmatrix} -\frac{3}{\theta} & \frac{a_{2,2}}{\theta^2} \\ 1 & 0 \end{pmatrix} \tag{48}$$

for fixed positive θ with $a_{2,2} < -4$, we obtain

$$p_2(-\tau, \theta) = \tau^2 + \frac{3\tau}{\theta} - \frac{a_{2,2}}{\theta^2} \tag{49}$$

The roots of polynomial (49) have the form $\frac{1}{2\theta}\left(-3 \pm 2\sqrt{a_{2,2} + 9/4}\right)$. Thus, for $a_{2,2} < -4$, we have that the roots of polynomial (49) are complex numbers. On the other hand, the roots of polynomial $p_{r,2}(-\tau, \theta)$ for $\theta > 0$ and $r = 1, 2$ are real; see [4]. Consequently, one cannot construct a controllability function as a motion time with the help of the orthogonal polynomial $p_{r,2}(-\tau, \theta)$ for $r = 1, 2$.

For $n = 3$, the positional control for the case when the controllability function $\theta(x)$ is the time of motion when $\mathbf{F}_3$ is a positive definite matrix, has the form [27]

$$u_3(x) = -\frac{6x_1}{\theta} + \frac{(a_{3,3} - 30)\,x_2}{3\theta^2} + \frac{a_{3,3}x_3}{\theta^3}, \quad a_{3,3} < -30. \tag{50}$$

Substituting control (50) in (4), the corresponding characteristic polynomial for fixed positive θ is the following

$$-p_3(-\tau,\theta) = \tau^3 + \frac{6\tau^2}{\theta} - \frac{(a_{3,3}-30)\,\tau}{3\theta^2} - \frac{a_{3,3}x_3}{\theta^3}. \tag{51}$$

Taking the derivative of $-p_3(-\tau,\theta)$ with respect to τ, we have

$$-p_3'(-\tau,\theta) = 3\tau^2 + \frac{12\tau}{\theta} - \frac{a_{3,3}-30}{3\theta^2}. \tag{52}$$

The roots of $-p_3'(-\tau,\theta)$ are $\frac{1}{3\theta}\left(-6 \pm \sqrt{a_{3,3}+6}\right)$, for $a_{3,3} < -30$. Both of them are complex numbers. Hence, for every fixed θ, the polynomial $-p_3(-\tau,\theta)$ has no critical points. Thus, $-p_3(-\tau,\theta)$ is monotone nondecreasing function. Due to Bolzano's theorem, $-p_3(-\tau,\theta)$ has one real root. The remaining roots are complex numbers. On the other hand, polynomials $-p_{r,3}(-\tau,\theta)$ have only negative roots for fixed θ. Consequently, we have a similar result as for $n = 2$. □

Conclusion

We have proved that every orthogonal polynomial on $[0,+\infty)$ described by (13) and (14) is a Hurwitz polynomial after performing two simple algebraic operations: a) replacing the independent variable τ by $-\tau$ and b) multiplying the polynomial by $(-1)^n$. This result follows from the properties of the roots of orthogonal polynomials and the definition of a Hurwitz polynomial. Using these orthogonal polynomials, we constructed bounded finite-time stabilizing controls.

We also demonstrated that no orthogonal polynomial can generate a positional control such that the controllability function represents the system's motion time.

An interesting open question remains: Is it possible to construct positional controls for system (1) using a combination of orthogonal polynomials on $[0,\infty)$, such that the controllability function represents the motion time or satisfies the equality $\dot{\theta} = -C$, where C is a positive constant? Further investigation of this problem would be worthwhile.

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Метод функції керованості Коробова за допомогою ортогональних поліномів на $[0, \infty)$

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Дано керовану систему, описану звичайними диференціальними рівняннями або диференціальними рівняннями із частинними похідними, та початковий стан. Задача знаходження множини обмежених позиційних керувань, які переводять початковий стан у деякий інший стан (не обов'язково точку рівноваги) за скінченний час, називається задачею синтезу.

У даній роботі розглядається сімейство систем у формі Бруновського розмірності n. Для стабілізації заданої системи у формі Бруновського за скінченний час побудовано сімейство обмежених позиційних керувань $u_n(x)$. Ми використовуємо ортогональні поліноми, що асоційовані з функціональним розподілом $\sigma(\tau, \theta)$, визначеним для $\tau \in [0, +\infty)$ і параметра $\theta > 0$. Параметр θ інтерпретується як функція керованості Коробова, $\theta = \theta(x)$, яка слугує функцією типу Ляпунова. Використовуючи $\theta(x)$, ми будуємо позиційне керування: $u_n(x) = u_n(x, \theta(x))$.

Наш аналіз базується на фундаментальній роботі: "A general approach to the solution of the bounded control synthesis problem in a controllability problem", Matematicheskii Sbornik, 151(4), 582-606 (1979) авторства В. І. Коробова, у якій було запропоновано метод функції керованості. Цей метод було застосовано для розв'язання задач стабілізації обмеженим керуванням за скінченний час у різних сценаріях керування, таких як керування хвильовим рівнянням, оптимальне керування зі змішаними критерієм якості та інші застосування.

Для побудови згаданих позиційних керувань ми використовуємо члени сімейства ортогональних на $[0, \infty)$ поліномів. Детальнішу інформацію про ортогональні поліноми можна знайти у книзі: "Orthogonal Polynomials", American Mathematical Society, Providence, (1975) авторства G. Szegö. Ми також спираємося на роботу: "On matrix Hurwitz type polynomials and their interrelations to Stieltjes positive definite sequences and orthogonal matrix polynomials", Linear Algebra and its Applications, 476, 56-84 (2015) авторства A. E. Choque Rivero.

Результати, представлені у цій роботі, розширюють і розвивають напрацювання, викладені у конференційній доповіді: "Bounded finite-time stabilizing controls via orthogonal polynomials", 2018 IEEE International Autumn Meeting on Power, Electronics and Computing (ROPEC), Ixtapa, Mexico, 2018, авторства A. E. Choque Rivero, B. d. J. G. Orozco.

Ключові слова: **обмежене керування; ортогональні поліноми; стабілізація за скінченний час; функція керованості; канонічна система.**

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