ISSN 2221-5646(Print) 2523-4641(Online)
Visnyk of V.N.Karazin Kharkiv National University
Ser. "Mathematics, Applied Mathematics
and Mechanics"
2024, Vol. 100, p. 48–60
DOI: 10.26565/2221-5646-2024-100-03
УДК 517.93

Вісник Харківського національного
університету імені В.Н. Каразіна
Серія "Математика, прикладна
математика і механіка"
2024, Том 100, с. 48–60

V. I. Korobov
D.Sc. in physics and mathematics, Prof.
Head Dep. of Applied Mathematics
V. N. Karazin Kharkiv National University
4 Svobody Sq., Kharkiv, Ukraine, 61022
valeriikorobov@gmail.com *http://orcid.org/0000-0001-8421-1718*

O. S. Vozniak
MS in applied mathematics
PhD applied mathematics student
V. N. Karazin Kharkiv National University
Svobody Sq., 4, Kharkiv, Ukraine, 61022
o.vozniak0@gmail.com *http://orcid.org/0000-0001-9729-0742*

Time-optimal control on a subspace for the two and three-dimensional system

This article is devoted to the problem of the time-optimal control onto a subspace for the linear system $\dot{x_1} = u, \dot{x}_i = x_{i-1}, i = \overline{2, n}$ with $|u| \leq 1$ in the case of $n = 2$ and $n = 3$. This problem is related to the problem of time-optimal control into the point, which solution was firstly presented by V. I. Korobov and G. M. Sklyar and is based on the moment min-problem. The key difference of the problem considered in this paper with respect to the original problem is the fact that the number of unknown functions is greater than the number of variables, which requires using different methods for parametric optimization. As in the problem of time-optimal control into the point, we construct the optimal solution in the form of the piecewise function with $u = \pm 1$ and $n - 1$ switching points, which is optimal according to the Pontryagin Maximum Principle. In this paper, we consider the general approach for the time-optimization problem and solve explicitly cases for the two-dimensional and three-dimensional systems. We give the solution for the system with $n = 2$ system onto a subspace $G : \{(x_1, x_2) : x_2 = kx_1\}$ for all values of k using the moment min-problem and the optimization methods. We show that for some values of parameter k the system may not have switching points at all. For the three-dimensional system, we consider the problem of time-optimal control onto a plane $x_3 = k_1x_1 + k_2x_2$ and obtain the number of switching points depending on the values of k_1 and k_2. We construct phase trajectories and present the equations for the optimal time Θ for different cases. Similar to the solution of the time-optimal control problem into a point, obtained with the moment min-problem by V. I. Korobov and G. M. Sklyar, the time-optimal control may have $n - 1$ or fewer points of discontinuity.

Keywords: **controllability; moment min-problem; time-optimal control; variable end point problem**

2020 Mathematics Subject Classification: 93C05; 93B05; 49J15.

1. Introduction

Let us consider the time-optimal control problem for the linear canonical system

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1, \\ \dot{x}_3 = x_2, \\ \quad \dots \\ \dot{x}_n = x_{n-1}, \end{cases} \tag{1}$$

$$|u| \leq 1 \ x(0) = x_0, \ x(T) \in G = \{x : Hx = 0\}, \tag{2}$$

where $H \in \mathbb{R}^{k \times n}$ and $k < n$. We search for the time-optimal control $u(t)$ in the form of a piecewise function with $u(t) = \pm 1$, and $n-1$ points of discontinuity, which is optimal according to the Pontryagin Maximum Principle.

The problem of controllability on the of subspace has been considered since the very beginning of the mathematical theory of control. In 1976 the article [8] V. I. Korobov, A. V. Lutsenko, E. N. Podolskyi obtained the criterion and the explicit form for control $u(x) = Qx$ for the stabilization problem on the subspace G, and in 1977 [9] and 1981 [10] the controllability criteria were also obtained. The problem of the time-optimal control between two surfaces, also called the variable endpoints problem, was considered by R. V. Gamkrelidze in the paper [1] and by L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, E. F. Mishechenko in the monograph [2]. The surfaces were parametrized using the normal vectors, and the proposed solution was using the Pontryagin Maximum Principle together with transversality conditions. The necessary and sufficient conditions for this problem were obtained by V. G. Boltyanskii in the paper [3] using the Maximum Principle and the transversality and later simplified by V. Jankovic in the paper [4].

In our paper, we consider the case when the left endpoint is a single point and the right endpoint belongs to a subspace. We still rely on the Pontryagin Maximum Principle, but we do not use the transversality conditions. Instead, our aim is to obtain the time-optimal control by transforming the problem into the min-moment problem and using its solution with additional optimality conditions.

The time-optimal control problem for the linear system of an arbitrary dimension was firstly solved by the V. I. Korobov and G. M. Sklyar in the paper [6]. They showed that the time-optimal control problem can be transformed into the Markov power moment problem

$$\int_0^{\Theta} t^{k-1} u(t) dt = s_k, \ k = \overline{1, n}, \tag{3}$$

on the shortest time interval $[0,\Theta]$, which they called the moment min-problem, and obtained the complete analytical solution of this problem. Let us now describe this method. Here we use the notation and the algorithm used in the paper [7].

Consider the system (1) with $|u| \leq 1$, the initial point x_0 and the end point x_T. The aim is to construct the time-optimal control $u(t)$ and to find the minimum time Θ. The system (1) is a linear system

$$\dot{x} = Ax + bu, \tag{4}$$

with

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \dots \\ 0 \end{pmatrix}. \tag{5}$$

The trajectory $x(t)$ for the system (4) is given by the equation

$$x(t) = e^{At}\left(x_0 + \int_0^t e^{-A\tau} bu(\tau)d\tau\right). \tag{6}$$

hence for $x_T = x(\Theta)$ we have that that

$$\int_0^\Theta e^{-At} bu(t)dt = e^{-A\Theta}x_T - x_0, \tag{7}$$

where

$$e^{-At} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ -t & 1 & \dots & 0 \\ \frac{t^2}{2} & -t & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \frac{(-t)^{n-1}}{(n-1)!} & \frac{(-t)^{n-2}}{(n-2)!} & \dots & 1 \end{pmatrix}. \tag{8}$$

From the equation (7) we obtain the Markov power moment problem:

$$\int_0^\Theta t^{k-1}u(t)dt = s_k,\ k = \overline{1,n}, \tag{9}$$

where

$$s_k = (-1)^k (k-1)! \left(x_{0,k} + \sum_{j=1}^{k} \frac{(-1)^j \Theta^{j-1} x_{T,k-j+1}}{(j-1)!} \right),\ k = \overline{1,n}. \tag{10}$$

The equation (9) is called the Markov power moment problem. The aim of the moment min-problem of V. I. Korobov and G. M. Sklyar is to find the minimum possible interval $[0,\Theta]$ and the respective solution $u(t)$ such that there exists a solution to the problem (9).

From the Pontryagin Maximum Principle [2] we know that the optimal control $u(t)$ is a piecewise function with $u = \pm 1$ and no more than $n-1$ points $T_1, T_2, \ldots, T_{n-1}$ of the discontinuity. The sign of $u(t)$ on the last time interval is unknown and has to be determined, By integrating the equation (9) we have,

$$\begin{cases} (-1)^n T_1 + (-1)^{n-1} T_2 + (-1)^{n-2} T_3 + \cdots + T_{n-1} = c_1^{\pm}(\Theta, s), \\ (-1)^n T_1^2 + (-1)^{n-1} T_2^2 + (-1)^{n-2} T_3^2 + \cdots + T_{n-1}^2 = c_2^{\pm}(\Theta, s), \\ \cdots \\ (-1)^n T_1^n + (-1)^{n-1} T_2^n + (-1)^{n-2} T_3^n + \cdots + T_{n-1}^n = c_n^{\pm}(\Theta, s), \end{cases} \tag{11}$$

where $c_k^{\pm} = \frac{1}{2}(\Theta^k \mp k s_k)$ and the upper index of c_k means the sign of $u(t)$ on the last time interval. The main idea proposed by V. I. Korobov and G. M. Sklyar, which allowed to solve this system, is the following:

We add the infinite amount of the equations to the system (11) and obtain the system:

$$\begin{cases} (-1)^n T_1 + (-1)^{n-1} T_2 + (-1)^{n-2} T_3 + \cdots + T_{n-1} = c_1^{\pm}(\Theta, s), \\ (-1)^n T_1^2 + (-1)^{n-1} T_2^2 + (-1)^{n-2} T_3^2 + \cdots + T_{n-1}^2 = c_2^{\pm}(\Theta, s), \\ \cdots \\ (-1)^n T_1^n + (-1)^{n-1} T_2^n + (-1)^{n-2} T_3^n + \cdots + T_{n-1}^n = c_n^{\pm}(\Theta, s), \\ (-1)^{n+1} T_1^{n+1} + (-1)^{n} T_2^{n+1} + (-1)^{n-1} T_3^{n+1} + \cdots + T_{n-1}^{n+1} = c_{n+1}^{\pm}(\Theta, s), \\ \cdots \end{cases} \tag{12}$$

Now we divide the $k-th$ equation by kz^k, and sum the equations on the left and on the right side of the system (12). We obtain the following equality:

$$\ln R(z) = -\sum_{k=1}^{\infty} \frac{c_k^{\pm}(\Theta, s)}{kz^k}, \tag{13}$$

where, if $n = 2m+1$ then

$$R(z) = \frac{\prod_{j=1}^{m} \left(1 - \frac{T_{2j}}{z}\right)}{\prod_{j=1}^{m} \left(1 - \frac{T_{2j-1}}{z}\right)} = \frac{z^m + a_1 z^{m-1} + \cdots + a_m}{z^m + b_1 z^{m-1} + \cdots + b_m}, \tag{14}$$

.And if $n = 2m$ then

$$R(z) = \frac{\prod_{j=1}^{m} \left(1 - \frac{T_{2j-1}}{z}\right)}{\prod_{j=1}^{m} \left(1 - \frac{T_{2j}}{z}\right)} = \frac{z^m + a_1 z^{m-1} + \cdots + a_m}{z(z^{m-1} + b_1 z^{m-2} + \cdots + b_{m-1})}, \tag{15}$$

For the equation (14) the roots of the numerator represent are even switching points, and the roots of the denominator are the odd switching points. And for the equation (15) roots of the numerator represent the odd switching points, and the roots of the denominator are the even switching points.

Now we write $R(z)$ as a series

$$R(z) = 1 - \sum_{k=1}^{\infty} \frac{\gamma_k}{z^k}, \tag{16}$$

and get the equation

$$\ln\left(1 - \sum_{k=1}^{\infty} \frac{\gamma_k}{z^k}\right) = -\sum_{k=1}^{\infty} \frac{c_k^{\pm}(\Theta, s)}{kz^k}. \tag{17}$$

The equations for γ_k can be written explicitly. By differentiating (17) by $\frac{1}{z}$ and comparing terms for same degrees of $\frac{1}{z}$ we get the equations:

$$\gamma_k^{\pm}(\Theta, s) = \frac{(-1)^k}{k!} \begin{vmatrix} c_1^{\pm} & 1 & 0 & \dots & 0 \\ c_2^{\pm} & c_1^{\pm} & 2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ c_{k-1}^{\pm} & c_{k-2}^{\pm} & c_{k-3}^{\pm} & \dots & k-1 \\ c_k^{\pm} & c_{k-1}^{\pm} & c_{k-2}^{\pm} & \dots & c_1^{\pm} \end{vmatrix}. \tag{18}$$

Now we have two sets of coefficients γ_i^+ and γ_i^-, corresponding to the positive and negative sign of the control of $u(t)$ on the last time interval. Then in the paper [6] by V. I. Korobov and G. M. Sklyar it is shown that the optimal time $\Theta = \min(\Theta^+, \Theta^-)$, where Θ^+ and Θ^- are the maximum positive roots of the equations:

$$\begin{vmatrix} \gamma_1^{\pm} & \gamma_2^{\pm} & \dots & \gamma_{m+1}^{\pm} \\ \gamma_2^{\pm} & \gamma_3^{\pm} & \dots & \gamma_{m+2}^{\pm} \\ \dots & \dots & \dots & \dots \\ \gamma_{m+1}^{\pm} & \gamma_{m+2}^{\pm} & \dots & \gamma_{2m+1}^{\pm} \end{vmatrix} = 0, \tag{19}$$

if $n = 2m + 1$ and

$$\begin{vmatrix} \gamma_2^{\pm} & \gamma_3^{\pm} & \dots & \gamma_{m+1}^{\pm} \\ \gamma_3^{\pm} & \gamma_4^{\pm} & \dots & \gamma_{m+2}^{\pm} \\ \dots & \dots & \dots & \dots \\ \gamma_{m+1}^{\pm} & \gamma_{m+2}^{\pm} & \dots & \gamma_{2m}^{\pm} \end{vmatrix} = 0, \tag{20}$$

if $n = 2m$ respectively.

If $n = 2m + 1$ then the odd and even switching points are found respectively from the equations

$$\begin{vmatrix} \gamma_1^{\pm} & \gamma_2^{\pm} & \dots & \gamma_{m+1}^{\pm} \\ \gamma_2^{\pm} & \gamma_3^{\pm} & \dots & \gamma_{m+2}^{\pm} \\ \dots & \dots & \dots & \dots \\ \gamma_m^{\pm} & \gamma_{m+1}^{\pm} & \dots & \gamma_{2m}^{\pm} \\ 1 & T & \dots & T^m \end{vmatrix} = 0; \begin{vmatrix} \tilde{\gamma}_1^{\pm} & \tilde{\gamma}_2^{\pm} & \dots & \tilde{\gamma}_{m+1}^{\pm} \\ \tilde{\gamma}_2^{\pm} & \tilde{\gamma}_3^{\pm} & \dots & \tilde{\gamma}_{m+2}^{\pm} \\ \dots & \dots & \dots & \dots \\ \tilde{\gamma}_m^{\pm} & \tilde{\gamma}_{m+1}^{\pm} & \dots & \tilde{\gamma}_{2m}^{\pm} \\ 1 & T & \dots & T^m \end{vmatrix} = 0, \tag{21}$$

where

$$\tilde{\gamma}_k^{\pm}(\Theta, s) = \frac{(-1)^k}{k!} \begin{vmatrix} c_1^{\pm} & -1 & 0 & \dots & 0 \\ c_2^{\pm} & c_1^{\pm} & -2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ c_{k-1}^{\pm} & c_{k-2}^{\pm} & c_{k-3}^{\pm} & \dots & -(k-1) \\ c_k^{\pm} & c_{k-1}^{\pm} & c_{k-2}^{\pm} & \dots & c_1^{\pm} \end{vmatrix}, \tag{22}$$

and if $n = 2m$ then from the equations

$$\begin{vmatrix} \tilde{\gamma}_1^{\pm} & \tilde{\gamma}_2^{\pm} & \dots & \tilde{\gamma}_{m+1}^{\pm} \\ \tilde{\gamma}_2^{\pm} & \tilde{\gamma}_3^{\pm} & \dots & \tilde{\gamma}_{m+2}^{\pm} \\ \dots & \dots & \dots & \dots \\ \tilde{\gamma}_m^{\pm} & \tilde{\gamma}_{m+1}^{\pm} & \dots & \tilde{\gamma}_{2m}^{\pm} \\ 1 & T & \dots & T^m \end{vmatrix} = 0; \begin{vmatrix} \gamma_2^{\pm} & \gamma_3^{\pm} & \dots & \gamma_{m+1}^{\pm} \\ \gamma_3^{\pm} & \gamma_4^{\pm} & \dots & \gamma_{m+2}^{\pm} \\ \dots & \dots & \dots & \dots \\ \gamma_m^{\pm} & \gamma_{m+1}^{\pm} & \dots & \gamma_{2m}^{\pm} \\ 1 & T & \dots & T^{m-1} \end{vmatrix} = 0. \tag{23}$$

2. Time-optimal control onto a subspace

Recall that we consider the problem (1)-(2). If the endpoint $x_T \in G$ was known, we would have the time-optimal control problem with fixed ends. Thus, as in the fixed endpoints problem the time optimal control $u(t)$ is a piecewise function with $u(t) = \pm 1$ maximum of $n-1$ points of discontinuity $T_1, T_2, \dots, T_{n-1}$. Let us multiply from the left both sides of the equation (6) by matrix H. Since $x_T \in G$ we obtain:

$$0 = He^{A\Theta}\left(x_0 + \int_0^{\Theta} e^{-At} bu(t)dt\right), \tag{24}$$

or

$$\begin{cases} \left(h_{1n} + ... + \dfrac{h_{11}\Theta^{n-1}}{(n-1)!}\right)\left(x_{0n} + \displaystyle\int_0^{\Theta} udt\right) + ... + h_{11}\left(x_{01} + \displaystyle\int_0^{\Theta} \dfrac{(-t)^{n-1}}{(n-1)!} udt\right) = 0, \\ \left(h_{2n} + ... + \dfrac{h_{21}\Theta^{n-1}}{(n-1)!}\right)\left(x_{0n} + \displaystyle\int_0^{\Theta} udt\right) + ... + h_{21}\left(x_{01} + \displaystyle\int_0^{\Theta} \dfrac{(-t)^{n-1}}{(n-1)!} udt\right) = 0, \\ \dots \\ \left(h_{kn} + ... + \dfrac{h_{k1}\Theta^{n-1}}{(n-1)!}\right)\left(x_{0n} + \displaystyle\int_0^{\Theta} udt\right) + ... + h_{k1}\left(x_{01} + \displaystyle\int_0^{\Theta} \dfrac{(-t)^{n-1}}{(n-1)!} udt\right) = 0. \end{cases}$$

This is a system of k equations for n unknown terms $\int_0^{\Theta} t^k u(t)dt$. Its solution comes down to the solution of moment min-problem of V. I. Korobov and G. M. Sklyar, but in this case the number of equations is less than the number of variables.

Let us parametrize $x_T \in G$ as

$$x_T = K \cdot \alpha = \begin{pmatrix} k_{11}\alpha_1 + k_{12}\alpha_2 + ... + k_{1k}\alpha_k \\ k_{21}\alpha_1 + k_{22}\alpha_2 + ... + k_{2k}\alpha_k \\ \dots \\ k_{n1}\alpha_1 + k_{n2}\alpha_2 + ... + k_{nk}\alpha_k \end{pmatrix}, \tag{25}$$

where $K \in \mathbb{R}^{n\times k}$ is a known matrix, and $\alpha \in \mathbb{R}^k$ is a vector of parameters. If G is a hyperplane and is not a coordinate plane then equation (25) can be simplified to

$$x_T = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \dots \\ \alpha_{n-1} \\ k_1\alpha_1 + k_2\alpha_2 + ... + k_{n-1}\alpha_{n-1} \end{pmatrix}, \tag{26}$$

and we get the moment min-problem:

$$\begin{cases} \int\limits_0^\Theta t^{k-1}u(t)dt = (-1)^k(k-1)!\left(x_{0,k} + \sum\limits_{j=1}^{k} \frac{(-1)^j\Theta^{j-1}\alpha_{k-j+1}}{(j-1)!}\right), \ k = \overline{1, n-1}, \\ \int\limits_0^\Theta t^{n-1}u(t)dt = (-1)^n(n-1)!\left(x_{0,n} + \sum\limits_{j=1}^{n-1}\left(-k_j\alpha_j + \frac{(-1)^j\Theta^{j-1}\alpha_{n-j+1}}{(j-1)!}\right)\right), \end{cases} \tag{27}$$

Then, as in the original problem we have the system:

$$\begin{cases} (-1)^n T_1 + (-1)^{n-1}T_2 + (-1)^{n-2}T_3 + \dots + T_{n-1} = c_1^{\pm}(\Theta, \alpha_1, s), \\ (-1)^n T_1^2 + (-1)^{n-1}T_2^2 + (-1)^{n-2}T_3^2 + \dots + T_{n-1}^2 = c_2^{\pm}(\Theta, \alpha_1, \alpha_2, s), \\ \dots \\ (-1)^n T_1^n + (-1)^{n-1}T_2^n + (-1)^{n-2}T_3^n + \dots + T_{n-1}^n = c_n^{\pm}(\Theta, s, \alpha_1, ..., \alpha_{n-1}), \end{cases} \tag{28}$$

The idea of this paper is to obtain the equation

$$F(\Theta, \alpha_1, \alpha_2, \dots, \alpha_{n-1}) = 0. \tag{29}$$

for the time Θ from the system (28) as a function of variables $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$. This can be done using the solution of V. I. Korobov and G. M. Sklyar for the moment min-problem. Because we have the time-optimization problem, the necessary optimality conditions must hold:

$$\frac{\partial F}{\partial \alpha_k} = 0, \ k = \overline{1, n-1}. \tag{30}$$

These conditions, together with the equation (29) obtain n equations for determining Θ. The solution for this system of equations may be not unique, and we must consider all the solutions. Then for each solution we determine the switching points $T_1, ...T_{n-1}$. If some of the switching points are negative this means that the solution is not valid. It also should be noted that the optimal number of switching points may be less than $n-1$ and these cases must be considered separately.

Now we consider the cases of two-dimensional and three-dimensional systems and show that for some subspaces the optimal control has less than $n-1$ points of discontinuity for any initial point x_0.

3. Two and three dimensional cases

Let us consider the 2-dimensional case

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1, \end{cases} \quad |u| \leq 1, \quad x_\Theta = \begin{pmatrix} \alpha \\ k\alpha \end{pmatrix}, \, k \in \mathbb{R}, \tag{31}$$

Let $u = -1$ on the first interval. We consider the sign on the first interval, and not on the last, as in the original problem, because we don't know the number of switching points in advance. The equation (11) has the form:

$$\begin{cases} 2T_1 - \Theta = x_{0,1} - \alpha, \\ 2T_1^2 - \Theta^2 = -2(x_{0,2} - k\alpha + \alpha\Theta), \end{cases} \tag{32}$$

here

$$c_1 = \frac{1}{2}\left(x_{0,1} - \alpha + \Theta\right); \; c_2 = \frac{1}{2}\left(-2x_{0,2} + 2k\alpha - 2\alpha\Theta + \Theta^2\right), \tag{33}$$

and

$$\begin{gathered} \gamma_1 = \frac{1}{2}\left(x_{0,1} - \alpha + \Theta\right), \\ \gamma_2 = \frac{1}{2}\left(-\frac{x_{0,1}^2}{4} - x_{0,2} + k\alpha + \frac{x_{0,1}\alpha}{2} - \frac{\alpha^2}{4} - \frac{x_{0,1}\Theta}{2} - \frac{\alpha\Theta}{2} + \frac{\Theta^2}{4}\right). \end{gathered} \tag{34}$$

Then the equations for Θ, T_1, and condition $\frac{\partial F}{\partial \alpha} = 0$ have the form:

$$\begin{cases} -\frac{x_{0,1}^2}{4} - x_{0,2} + k\alpha + \frac{x_{0,1}\alpha}{2} - \frac{\alpha^2}{4} - \frac{x_{0,1}\Theta}{2} - \frac{\alpha\Theta}{2} + \frac{\Theta^2}{4} = 0, \\ -T_1 + \frac{x_{0,1}}{2} - \frac{\alpha}{2} + \frac{\Theta}{2} = 0, \\ k + \frac{x_{0,1}}{2} - \frac{\alpha}{2} - \frac{\Theta}{2} = 0. \end{cases} \tag{35}$$

From last two equations we get that.

$$T_1 = \Theta - k, \alpha = 2k + x_{0,1} - \Theta. \tag{36}$$

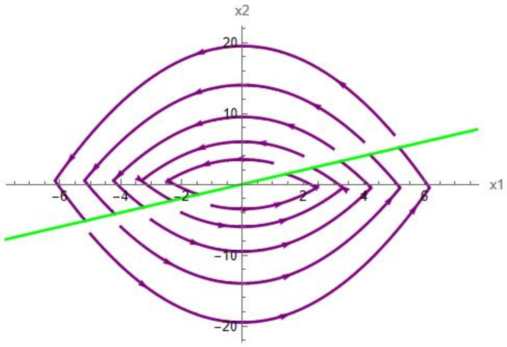
This means that if $k > 0$, that is the line has a positive slope, the control has one switching point $T = \Theta - k$, and if $k \leq 0$ then there will no switching points, since $T > \Theta$ is inadmissible, and we have to set $T_1 = \Theta$. The same holds for $u = 1$ on the first interval, and the optimal time Θ:

$$\Theta = \max\left(k + x_{0,1} + \sqrt{-k^2 + x_{0,1}^2 + 2x_{0,2}}, \; k - x_{0,1} + \sqrt{-k^2 + x_{0,1}^2 - 2x_{0,2}}\right) \tag{37}$$

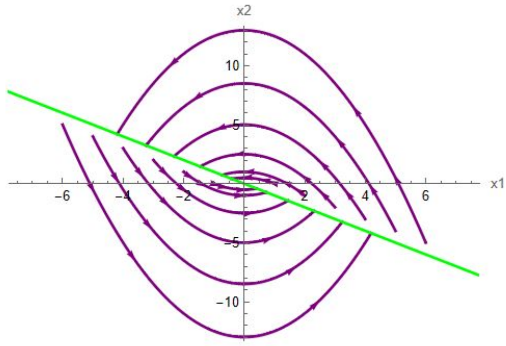
for one switching point, or

$$\Theta = \max\left(k + x_{0,1} + \sqrt{k^2 + x_{0,1}^2 + 2x_{0,2}}, \; k - x_{0,1} + \sqrt{k^2 + x_{0,1}^2 - 2x_{0,2}}\right) \tag{38}$$

for no switching points. The phase trajectories for lines $x_1 = x_2$ and $x_1 = -x_2$ are shown on the pictures 1 and 2 respectively.



Pic. 1. Trajectories for $G : x_1 = x_2$
Рис. 1. Траєкторії для $G : x_1 = x_2$



Pic. 2. Trajectories for $G : x_1 = -x_2$
Рис. 2. Траєкторії для $G : x_1 = -x_2$

If $k = \pm\infty$ then $G = \{x : x_1 = 0\}$ and the problem is equivalent to $\dot{x}_1 = u$. It has the solution $u \equiv 1$ if $x_{0,1} < 0$ and $u \equiv -1$ if $x_{0,1} > 0$, without switching points.

Let us now consider the case with $n = 3$:

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1, \\ \dot{x}_3 = x_2, \end{cases} \quad |u| \le 1, \quad x_\Theta = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ k_1\alpha_1 + k_2\alpha_2 \end{pmatrix}, \; k_1, k_2 \in \mathbb{R}. \tag{39}$$

We want to determine how many switching points does the optimal trajectory $u(t)$ have depending on the values of k_1 and k_2 and obtain the equations for Θ. Let $u = -1$ on the first interval. Then the equation 11 has the form:

$$\begin{cases} 2T_1 - 2T_2 + \Theta = x_{0,1} - \alpha_1, \\ 2T_1^2 - 2T_2^2 + \Theta^2 = -2(x_{0,2} - \alpha_2 + \alpha_1\Theta), \\ 2T_1^3 - 2T_2^3 + \Theta^3 = 6\left(x_{0,3} - k_1\alpha_1 - k_2\alpha_2 + \alpha_2\Theta - \frac{\alpha_1\Theta^2}{2}\right). \end{cases} \tag{40}$$

From the first two equations we obtain

$$\begin{aligned} T_1 &= \frac{x_{0,1}^2 - 4x_{0,2} - 2x_{0,1}\alpha_1 + \alpha_1^2 + 4\alpha_2 - 2x_{0,1}\Theta - 2\alpha_1\Theta - \Theta^2}{4(x_{0,1} - \alpha_1 - \Theta)}, \\ T_2 &= \frac{-x_{0,1}^2 - 4x_{0,2} + 2x_{0,1}\alpha_1 - \alpha_1^2 + 4\alpha_2 + 2x_{0,1}\Theta - 6\alpha_1\Theta - 3\Theta^2}{4(x_{0,1} - \alpha_1 - \Theta)}, \end{aligned} \tag{41}$$

and by substituting in the third equation we get

$$\begin{aligned} F = x_{0,3} - k_1\alpha_1 - k_2\alpha_2 + \alpha_2\Theta - \frac{\alpha_1\Theta^2}{2} - \Bigg(& 32(x_{0,1} - \alpha_1 - \Theta)^3\Theta^3 + \\ + \left(x_{0,1}^2 - 4x_{0,2} + \alpha_1^2 + 4\alpha_2 - 2\alpha_1\Theta - \Theta^2 - 2x_{0,1}(\alpha_1 + \Theta)\right)^3 & + \\ + \frac{\left(x_{0,1}^2 - 4x_{0,2} + \alpha_1^2 + 4\alpha_2 - 2\alpha_1\Theta - \Theta^2 - 2x_{0,1}(\alpha_1 + \Theta)\right)^3}{192(x_{0,1} - \alpha_1 - \Theta)^3} \Bigg) & = 0. \end{aligned} \tag{42}$$

After solving $\frac{\partial F}{\partial \alpha_1} = 0, \frac{\partial F}{\partial \alpha_2} = 0$ for α_1, α_2 and substituting them into equation (41) we get

$$T_1 = -k_2 - \sqrt{2k_1 + k_2^2} + \Theta,\ T_2 = -k_2 + \sqrt{2k_1 + k_2^2} + \Theta. \qquad (43)$$

To show that the this is a minimum point we calculate the second derivatives and check the sufficient condition:

$$\frac{\partial^2 F}{\partial \alpha_1^2}\frac{\partial^2 F}{\partial \alpha_2^2} - \frac{\partial^2 F}{\partial \alpha_1 \alpha_2} = \frac{1}{16} > 0 \implies \text{ local minimum }. \qquad (44)$$

This means that for the plane $x_3 = k_1 x_1 + k_2 x_2$ if both $-k_2 - \sqrt{2k_1 + k_2^2} < 0$ and $-k_2 + \sqrt{2k_1 + k_2^2} < 0$, that is $k_1 < 0$ and $k_2 > \sqrt{-2k_1}$ trajectory can have 2 switching points, if $-k_2 - \sqrt{2k_1 + k_2^2} < 0$ and $-k_2 + \sqrt{2k_1 + k_2^2} > 0$, that is $k_1 > 0$ trajectory can have 1 switching point, and in other cases trajectory has no switching points. Same results hold for $u = 1$ on the first interval.

The picture 3 shows the phase trajectories for the plane $x_3 = -4x_1 + 3x_2$. Here $T_1 = \Theta - 4$, $T_2 = \Theta - 2$. The equations for Θ are

$$8 \mp 24x_{0,1} \pm 18x_{0,2} \mp 6x_{0,3} + (24 \pm 8x_{0,1} \mp 6x_{0,2})\Theta + (-9 - 3x_{0,1})\Theta^2 + \Theta^3 = 0, \quad (45)$$

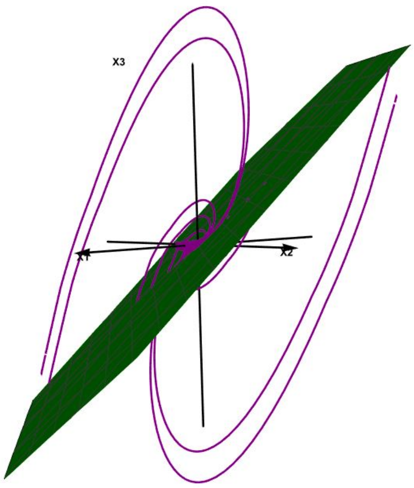
for 2 switching points, where the upper sign is for the trajectory with $u = -1$ on the first interval,

$$-40 \mp 24x_{0,1} \pm 18x_{0,2} \mp 6x_{0,3} + (24 \pm 8x_{0,1} \mp 6x_{0,2})\Theta + (-9 - 3x_{0,1})\Theta^2 + \Theta^3 = 0, \quad (46)$$

for 1 switching point, and

$$\mp 24x_{0,1} \pm 18x_{0,2} \mp 6x_{0,3} + (24 \pm 8x_{0,1} \mp 6x_{0,2})\Theta + (-9 - 3x_{0,1})\Theta^2 + \Theta^3 = 0, \quad (47)$$

for no switching points.



Pic. 3. Phase trajectories for $G : x_3 = -4x_1 + 3x_2$
Рис. 3. Фазові траєкторії для $G : x_3 = -4x_1 + 3x_2$

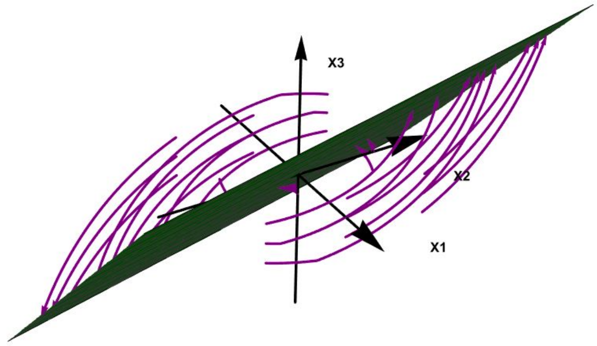
The picture 4 shows the second case, $x_3 = 4x_1 + x_2$, when the trajectory can have maximum 1 switching point. We have $T_1 = \Theta - 4$ and the equations for Θ:

$$160 \pm 24x_{0,1} \pm 6x_{0,2} \mp 6x_{0,3} + (-24 \pm 8x_{0,1} \mp 6x_{0,2})\Theta + (-3 - 3x_{0,1})\Theta^2 + \Theta^3 = 0, \quad (48)$$

for 1 switching point, and

$$\pm 24x_{0,1} \pm 6x_{0,2} \mp 6x_{0,3} + (-24 \pm 8x_{0,1} \mp 6x_{0,2})\Theta + (-3 - 3x_{0,1})\Theta^2 + \Theta^3 = 0, \quad (49)$$

for no switching points.

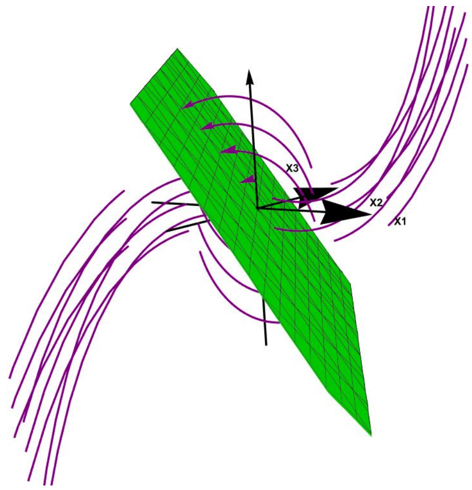


Pic. 4. Phase trajectories for $G : x_3 = 4x_1 + x_2$
Рис. 4. Фазові траєкторії для $G : x_3 = 4x_1 + x_2$

And for the plane $x_3 = -4x_1 - 3x_2$, (Figure 5) we have no switching points and

$$\mp 24x_{0,1} \mp 18x_{0,2} \mp 6x_{0,3} + (24 \mp 18x_{0,1} \mp 6x_{0,2})\Theta + (9 - 3x_{0,1})\Theta^2 + \Theta^3 = 0. \quad (50)$$

The equations for the switching surfaces can be found by considering the cases when equations for different number of switching points give the same time Θ.



Pic. 5. Phase trajectories for $G : x_3 = -4x_1 - 3x_2$
Рис. 5. Фазові траєкторії для $G : x_3 = -4x_1 - 3x_2$

REFERENCES

1. R. V. Gamkrelidze, Theory of processes in linear systems which are optimal with respect to rapidity of action, Izv. Akad. Nauk SSSR, Ser. Mat. – 1958. – **22**. – P. 449–474.

2. L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, E. F. Mishechenko. The Mathematical Theory of Optimal Processes. VIII + 360 S. New York/London 1962. John Wiley & Sons. Preis 90/-. 10.1002/zamm.19630431023

3. V. G. Boltyanskiy. A linear optimal control problem. Differ Uravn. – 1969. – **5**. – P. 783-799.

4. V. Jankovic. The linear optimal control problem with variable endpoints. Publications de l'Institut Mathematique. Nouvelle Serie. – 1989. – **45**(59). – P. 133-142. http://eudml.org/doc/257819.

5. V. I. Korobov. A general approach to the solution of the bounded control synthesis problem in a controllability problem, Math. USSR Sb. – 1980. – **37**, No. 4. – P. 535-557. 10.1070/SM1980v037n04ABEH002094

6. V. I. Korobov, G. M. Sklyar. Time optimality and the power moment problem. Mat. Sb., Nov. Ser. – 1987. – – **134**(176), no. 2(10). – P. 186-206; Math. USSR, Sb. – 1989. – **62**, No. 1. – P. 185-206. 10.1070/SM1989v062n01ABEH003235

7. V.I. Korobov. Time Optimality for Systems with Multidimensional Control and Vector Moment Min-Problem. J Dyn Control Syst. – 2020. – **26**. – P. 525-550. 10.1007/s10883-019-09465-2

8. V. I. Korobov, A.V. Lucenko, E. N. Podolskij. Stabilization of a linear autonomous system with respect to a subspace. 1, Vestnik of Kharkiv State University, Ser. Applied Mathematics and Mechanics. – 1976. – **134**. – P. 114-123, (in Russian).

9. V. I. Korobov, A.V. Lucenko. Controllability of a linear stationary system on a subspace for an unfixed time, Ukrainian Mathematical Journal. – 1977. – **29**, No. 4. – P. 531-534.

10. V. I. Korobov. Controllability criterion for a linear system on a subspace, Vestnik of Kharkiv State University, Ser. Applied Mathematics and Mechanics. – 1981. – **221**. – P. 3-11, (in Russian).

Article history: Received: 5 October 2024; Final form: 11 December 2024
Accepted: 20 December 2024.

How to cite this article:

V. I. Korobov, O. S. Vozniak, Time-optimal control on a subspace for the two and three-dimensional system, Visnyk of V. N. Karazin Kharkiv National University. Ser. Mathematics, Applied Mathematics and Mechanics, Vol. 100, 2024, p. 48–60. DOI: 10.26565/2221-5646-2024-100-03

Оптимальне за часом керування на підпростір для двовимірної та тривимірної системи

В. І. Коробов, О. С. Возняк

Харківський національний університет імені В. Н. Каразіна
майдан Свободи 4, м. Харків, 61022, Україна

Дана стаття присвячена задачі швидкодії на підпростір для лінійної керованої системи $\dot{x_1} = u, \dot{x}_i = x_{i-1}, i = \overline{2,n}$ з $|u| \leq 1$ для випадку $n = 2$ та $n = 3$. Вона пов'язана із задачею оптимального за часом керування в точку, розв'язання якої було вперше представлено В. І. Коробовим та Г. М. Скляром і яке ґрунтується на min-проблемі моментів. Ключовою відмінністю задачі, що розглядається в даній роботі, від вихідної задачі є те, що кількість невідомих функцій перевищує кількість змінних, що вимагає використання інших методів параметричної оптимізації.

Як і в задачі оптимального за часом керування в точку, ми шукаємо оптимальний розв'язок у вигляді кусково-сталої функції $u = \pm 1$ з $n-1$ точками перемикання, який є оптимальним згідно з Принципом Максимуму Понтрягіна. У цій статті ми розглядаємо загальний підхід для розв'язання задачі оптимального за часом керування в точку, та розв'язуємо задачу у явному вигляді для двовимірної та тривимірної системи. У нашій роботі наведено розв'язок задачі оптимального керування для двовимірної системи на підпростір $G : \{(x_1, x_2) : x_2 = kx_1\}$ для всіх значень k з використанням min-проблеми моментів та методів оптимізації функцій. В роботі показано, що для деяких значень параметра k система може взагалі не мати точок перемикання. Для тривимірної системи ми розв'язуємо задачу оптимального керування на площину $x_3 = k_1x_1 + k_2x_2$ і отримуємо кількість точок перемикання залежно від значень k_1 і k_2 а також будуємо траєкторії та отримуємо рівняння для оптимального часу Θ для різних випадків. Подібно до розв'язку задачі оптимального за часом керування в точку, отриманого за допомогою min-проблеми моментів В. І. Коробовим та Г. М. Скляром, оптимальне керування може мати $n-1$ або менше точок перемикання.

Ключові слова: **керованість; min-проблема моментів; оптимальне за часом керування; задача з рухомими кінцями**

Історія статті: отримана: 5 жовтня 2024; останній варіант: 11 грудня 2024
прийнята: 20 грудня 2024.