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Homogeneous approximations of nonlinear control systems with output and weak algebraic equivalence

In the paper, we consider nonlinear control systems that are linear with respect to controls with output; vector fields defining the system and the output are supposed to be real analytic. Following the algebraic approach, we consider series S of iterated integrals corresponding to such systems. Iterated integrals form a free associative algebra, and all our constructions use its properties. First, we consider the set of all (formal) functions of such series $f(S)$ and define the set N_S of terms of minimal order for all such functions. We introduce the definition of the maximal graded Lie generated left ideal $\mathcal{J}_S^{\max}$ which is orthogonal to the set N_S. We describe the relation between this maximal left ideal and the left ideal $\mathcal{J}_S$ generated by the core Lie subalgebra of the system which realizes the series. Namely, we show that $\mathcal{J}_S \subset \mathcal{J}_S^{\max}$. In particular, this implies that the graded Lie subalgebra that generates the left ideal $\mathcal{J}_S^{\max}$ has a finite codimension. Also, we give the algorithm which reduces the series S to the triangular form and propose the definition of the homogeneous approximation for the series S. Namely, homogeneous approximation is a homogeneous series with components that are terms of minimal order in each component of this triangular form. We prove that the set N_S coincides with the set of all shuffle polynomials of components of a homogeneous approximation. Unlike the case when the output is identical, the homogeneous approximation is not completely defined by the ideal $\mathcal{J}_S^{\max}$. In order to describe this property, we introduce two different concepts of equivalence of series: algebraic equivalence (when two series have the same homogeneous approximation) and weak algebraic equivalence (when two series have the same maximal left ideal and therefore have the same minimal realizing system). We prove that if two series are algebraically equivalent, then they are weakly algebraically equivalent. The examples show that in general the converse is not true.

Keywords: **homogeneous approximation; nonlinear control system; series of iterated integrals; core Lie subalgebra; maximal left ideal.**

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1. Introduction

In the paper, we consider nonlinear control systems with output of the form

$$\dot{x} = \sum_{i=1}^{m} X_i(x)u_i, \quad x(0) = 0, \quad y = h(x), \tag{1}$$

where $X_1(x), \ldots, X_m(x)$ are real analytic vector fields in a neighborhood of the origin in $\mathbb{R}^n$ and $h(x)$ is a real analytic nonzero map from a neighborhood of the origin in $\mathbb{R}^n$ to $\mathbb{R}^p$ such that $h(0) = 0$.

Various problems for such systems including controllability, observability, stability, optimal control were deeply studied during many decades [6]. In particular, differential geometric methods were intensively developed which allowed applying the deep theory related to Lie algebras of vector fields [7]. Another approach based on algebraic and combinatorial tools was proposed by M. Fliess [4] and turned out to be perspective [8]. As the first step, instead of the system (1), the series of iterated integrals is considered. In particular, the algebraic approach was successfully used for studying the problem of homogeneous approximation of nonlinear control systems (1) in the case of identity output $h(x) = x$ [12]. One of the advantages is that the obtained algorithms can be efficiently implemented as computer programs [11]. We recall the main ideas in Section 2. Later, the approach was developed to study homogeneous approximations of systems (1) in the case of one-dimensional output, i.e., when $p = 1$ [1], [2].

In the present paper we consider the general case, when the output can be of arbitrary dimension. The main results are given in Section 3. We propose the definition of a homogeneous approximation of a series of iterated integrals corresponding to the system (1) (Definition 4) and describe the method to construct it (Lemma 2). Further, we introduce two definitions of equivalence for series, namely, algebraic equivalence and weak algebraic equivalence (Definitions 5 and 6), and study their properties (Theorem 1 and Corollary 1). In the case of identity output $h(x) = x$ these two kinds of equivalence coincide.

2. Background

Series of iterated integrals. Let us consider the system (1). The form of the right hand side of the system, namely, linearity in u_i, allows us to express explicitly the output y via controls

$$y(T) = \sum_{k=1}^{\infty} \sum_{1 \le i_1, \ldots, i_k \le m} c_{i_1 \ldots i_k} \eta_{i_1 \ldots i_k}(T, u), \tag{2}$$

where

$$\eta_{i_1\ldots i_k}(T,u)=\int_0^T\int_0^{\tau_1}\cdots\int_0^{\tau_{k-1}}u_{i_1}(\tau_1)u_{i_2}(\tau_2)\cdots u_{i_k}(\tau_k)d\tau_k\cdots d\tau_2 d\tau_1 \qquad (3)$$

are iterated integrals and $c_{i_1\ldots i_k}\in\mathbb{R}^p$ are constant coefficients that can be found via values of vector fields $X_i(x)$ and the map $h(x)$ and their derivatives at the origin,

$$c_{i_1\ldots i_k}=X_{i_k}\cdots X_{i_1}h(0). \qquad (4)$$

Here X_i act as differential operators of the first order, $X_i\psi(x)=\psi'(x)X_i(x)$. Suppose we consider admissible controls from a sufficiently wide class, for example, from the unit ball of the space $L_\infty([0,T];\mathbb{R}^m)$

$$B^T=\{u(t)=(u_1(t),\ldots,u_m(t))\in L_\infty([0,T];\mathbb{R}^m):u_1^2(t)+\cdots+u_m^2(t)\le 1 \text{ a. e.}\}. \qquad (5)$$

Then one can show [4] that iterated integrals are linearly independent functionals. Hence, they form a basis of the linear space over $\mathbb{R}$

$$\mathcal{F}_T=\mathrm{Lin}\{\eta_{i_1\ldots i_k}(T,u):k\ge 1,\ 1\le i_1,\ldots,i_k\le m\}.$$

The form of these basis functionals suggest introducing a concatenation operation,

$$\eta_{i_1\ldots i_k}(T,u)\vee\eta_{j_1\ldots j_q}(T,u)=\eta_{i_1\ldots i_k j_1\ldots j_q}(T,u),$$

which turns $\mathcal{F}_T$ into a free associative algebra. This interpretation allows applying algebraic and combinatorial tools for control systems (1). We briefly recall several results used in this paper below.

Abstract free associative algebra. First, let us notice that all algebras $\mathcal{F}_T$ for $T>0$ are isomorphic. Hence, we can consider the unique abstract algebra $\mathcal{F}$ isomorphic to all $\mathcal{F}_T$, and then interpret the series in the right hand side of (2) as a series of elements from $\mathcal{F}$. More specifically, let us introduce m abstract independent elements denoted by $\eta_1,\ldots\eta_m$, and consider all finite sequences of these elements

$$\eta_{i_1\ldots i_k}=\eta_{i_1}\cdots\eta_{i_k}.$$

Then the linear span of $\eta_{i_1\ldots i_k}$ (over $\mathbb{R}$)

$$\mathcal{F}=\mathrm{Lin}\{\eta_{i_1\ldots i_k}:k\ge 1,\ 1\le i_1,\ldots,i_k\le m\}. \qquad (6)$$

with the concatenation operation

$$\eta_{i_1\ldots i_k}\eta_{j_1\ldots j_q}=\eta_{i_1\ldots i_k j_1\ldots j_q}$$

is a free associative algebra isomorphic to any $\mathcal{F}_T$, $T>0$.

For convenience, we use the notation for the set of multi-indices

$$M=\bigcup_{k\ge 1}M_k,\quad M_k=\{I=(i_1,\ldots,i_k):1\le i_1,\ldots,i_k\le m\}.$$

Then, instead of the series of iterated integrals, we consider the formal series

$$S = \sum_{I \in M} c_I \eta_I \tag{7}$$

with coefficients (4). Let us introduce the linear map $c : \mathcal{F} \to \mathbb{R}^p$ defined on basis elements by

$$c(\eta_I) = c_I, \; I \in M.$$

Free Lie algebra and realizability conditions. Let us consider the free Lie algebra $\mathcal{L}$ generated by the same elements $\eta_1, \dots \eta_m$ as $\mathcal{F}$ and by the Lie bracket operation $[\ell_1, \ell_2] = \ell_1 \ell_2 - \ell_2 \ell_1, \; \ell_1, \ell_2 \in \mathcal{L}$. There exists a close relation between the Lie algebra $\mathcal{L}$ and the Lie algebra of vector fields L generated by $X_1(x), \dots, X_m(x)$. More specifically, let us consider the anti-homomorphism of Lie algebras $\varphi : \mathcal{L} \to L$ defined by $\varphi(\eta_i) = X_i(x)$ and such that $\varphi([\ell_1, \ell_2]) = [\varphi(\ell_2), \varphi(\ell_1)]$. Then for any $(i_1, \dots, i_k)$ and any $\ell \in \mathcal{L}$

$$c(\eta_{i_1 \dots i_k} \ell) = \varphi(\ell) X_{i_k} \cdots X_{i_1} h(0).$$

This property explains why a series of the form (7) defined by system (1) satisfies some additional conditions that require relations between coefficients. We recall the result [6]. The series (7) is called *realizable* if there exist real analytic vector fields $X_1, \dots, X_m$ and a real analytic map h such that equalities (4) are satisfied for any $I = (i_1, \dots, i_k) \in M$. Obviously, a realizable series should satisfy the following growth condition: there exist $C, C_1 > 0$ such that

$$\|c_I\| \le C_1 |I|! C^{|I|} \; \text{ for any } I \in M \tag{8}$$

where $|I|$ denotes the length of the multi-index I. For any $\ell \in \mathcal{L}$, let us denote by $F_c(\ell)$ the series

$$F_c(\ell) = \sum_{I \in M \cup \{\varnothing\}} c(\eta_I \ell) \eta_I$$

assuming $\eta_\varnothing = 1$ and introduce the *Lie rank of the series* S as

$$\rho_L(c) = \dim \{F_c(\ell) : \ell \in \mathcal{L}\}.$$

The following realizability theorem [6] holds: the series (7) satisfying the growth condition (8) is realizable if and only if its Lie rank is finite, $\rho_L(c) < \infty$. In this case, $n = \rho_L(c)$ is the minimal dimension of the system that realizes the series; we call such a system *a minimal realization* of the series.

Iterated integrals and grading in abstract algebras. Let us turn to iterated integrals (3). If the control belongs to the set (5), then obviously $|\eta_I(T, u)| \le \frac{1}{k!} T^k$, where $k = |I|$. Hence, locally, when T is small, the main role is played by terms of the series S containing integrals of minimal length. Algebraically, we express this property introducing the grading in the algebra $\mathcal{F}$

$$\mathcal{F} = \sum_{k=1}^{\infty} \mathcal{F}^k, \quad \mathcal{F}^k = \mathrm{Lin}\{\eta_I : |I| = k\},$$

and the corresponding grading in the Lie algebra $\mathcal{L}$

$$\mathcal{L} = \sum_{k=1}^{\infty} \mathcal{L}^k, \quad \mathcal{L}^k = \mathcal{L} \cap \mathcal{F}^k.$$

If $a \in \mathcal{F}^k$, we say that a is homogeneous and has the order k and write $\mathrm{ord}(a) = k$.

In [5], [11], [12], [13], [14], the particular case of systems (1) was considered where $h(x) = x$ was an identity output. In this case, the output coincides with the trajectory of the system. For such systems, the concept of homogeneous approximation was studied by use of algebraic approach. We recall the main constructions.

Core Lie subalgebra of the system and its graded left ideal. Suppose the series (7) with n-dimensional coefficients satisfies the growth condition (8), the Rashevsky-Chow condition

$$c(\mathcal{L}) = \mathbb{R}^n, \tag{9}$$

and the realizability condition of the following form:

$$\text{if} \ \ c(\ell) = 0 \ \ \text{for some} \ \ \ell \in \mathcal{L}, \ \ \text{then} \ \ c(a\ell) = 0 \ \ \text{for any} \ \ a \in \mathcal{F}. \tag{10}$$

Then it is realizable and the minimal realizing system (1) such that $h(x) = x$ has the dimension n. Let us introduce the subspaces

$$\mathcal{P}^k = \{\ell \in \mathcal{L}^k : c(\ell) \in c(\mathcal{L}^1 + \cdots + \mathcal{L}^{k-1})\}, \ k \geq 0,$$

and the Lie subalgebra [5]

$$\mathcal{L}_{X_1,\ldots,X_m} = \sum_{k=1}^{\infty} \mathcal{P}^k,$$

which is called *the core Lie subalgebra* of the system (1). One can show that its codimension in $\mathcal{L}$ equals n.

Now, we choose any homogeneous elements $\ell_1, \ldots, \ell_n \in \mathcal{L}$ such that

$$\mathcal{L}_{X_1,\ldots,X_m} + \mathrm{Lin}\{\ell_1, \ldots, \ell_n\} = \mathcal{L};$$

for convenience we assume that $\mathrm{ord}(\ell_i) \leq \mathrm{ord}(\ell_j)$ if $i < j$. Besides, we choose any homogeneous basis of $\mathcal{L}_{X_1,\ldots,X_m}$ and denote it by $\{\ell_i\}_{i=n+1}^{\infty}$. Thus, $\{\ell_i\}_{i=1}^{\infty}$ is a homogeneous basis of the Lie algebra $\mathcal{L}$.

This allows us to use the Poincaré-Birkhoff-Witt Theorem [10] which says that a basis of the associative algebra $\mathcal{F}$ can be obtained by use of the basis of the Lie algebra $\mathcal{L}$. Namely, the set

$$\{\ell_{i_1}^{q_1} \cdots \ell_{i_k}^{q_k} : k \geq 1, \ 1 \leq i_1 < \cdots < i_k, \ q_1, \ldots, q_k \geq 1\} \tag{11}$$

is a homogeneous basis of $\mathcal{F}$, where we denote $\ell^q = \ell \cdots \ell$ (q times). Having in mind the realizability condition (10), we introduce the *graded left ideal* generated by the core Lie subalgebra,

$$\mathcal{J}_{X_1,\ldots,X_m} = \mathrm{Lin}\{a\ell : a \in \mathcal{F} + \mathbb{R}, \ell \in \mathcal{L}_{X_1,\ldots,X_m}\}.$$

It can be shown that if $a \in \mathcal{J}_{X_1,\dots,X_m} \cap \mathcal{F}^k$, then $c(a) \in c(\mathcal{F}^1 + \cdots + \mathcal{F}^{k-1})$. Roughly speaking, this means that elements from the left ideal $\mathcal{J}_{X_1,\dots,X_m}$ cannot be leading terms in the series (7) corresponding to the system with respect to the grading in $\mathcal{F}$.

Dual basis and homogeneous approximation of the system. It turns out that the left ideal $\mathcal{J}_{X_1,\dots,X_m}$ can be described in another way. Let us introduce the inner product in $\mathcal{F}$ assuming that the basis consisting of elements η_I is orthonormal. Also, introduce the *shuffle product* in $\mathcal{F}$ by the recursive rule

$$\eta_i \sqcup\!\sqcup\, \eta_j = \eta_{ij} + \eta_{ji},$$
$$\eta_i \sqcup\!\sqcup\, \eta_{j_1\dots j_k} = \eta_{j_1\dots j_k} \sqcup\!\sqcup\, \eta_i = \eta_i \eta_{j_1\dots j_k} + \eta_{j_1}(\eta_i \sqcup\!\sqcup\, \eta_{j_2\dots j_k}),\ \ k \ge 2,$$
$$\eta_{i_1\dots i_s} \sqcup\!\sqcup\, \eta_{j_1\dots j_k} = \eta_{i_1}(\eta_{i_2\dots i_s} \sqcup\!\sqcup\, \eta_{j_1\dots j_k}) + \eta_{j_1}(\eta_{i_1\dots i_s} \sqcup\!\sqcup\, \eta_{j_2\dots j_k}),\ \ s,k \ge 2.$$

This operation is justified by the following relation with multiplication of iterated integrals,

$$\eta_{i_1\dots i_s}(T,u)\, \eta_{j_1\dots j_k}(T,u) = (\eta_{i_1\dots i_s} \sqcup\!\sqcup\, \eta_{j_1\dots j_k})(T,u),$$

where in the left hand side there is the (usual) product of two functionals and in the right hand side we find the shuffle product in $\mathcal{F}$ and then substitute iterated integrals instead of the corresponding elements of $\mathcal{F}$.

Then, the dual (with respect to the inner product) basis for the basis (11) has the form [9]

$$d_{i_1\dots i_k}^{q_1\dots q_k} = \frac{1}{q_1!\cdots q_k!} d_{i_1}^{\sqcup\!\sqcup q_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, d_{i_k}^{\sqcup\!\sqcup q_k},$$

where $d^{\sqcup\!\sqcup q} = d \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, d$ (q times); for brevity we use the notation $d_i = d_i^1$. More specifically, d_i are orthogonal to all elements of the basis (11) except ℓ_i and the inner product of d_i and ℓ_i equals 1. Moreover, due to the special choice of the basis $\{\ell_i\}_{i=1}^{\infty}$, the set

$$\{d_1^{\sqcup\!\sqcup q_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, d_n^{\sqcup\!\sqcup q_n} : q_1 + \cdots + q_n \ge 1\}$$

forms a basis of the orthogonal complement $\mathcal{J}^{\perp}_{X_1,\dots,X_m}$ to the left ideal $\mathcal{J}_{X_1,\dots,X_m}$ [12].

Finally, one can prove that there exists a change of variables $z = F(x)$ in the system (1) which reduces its series to the form

$$F(S) = \begin{pmatrix} d_1 + \rho_1 \\ \cdots \\ d_n + \rho_n \end{pmatrix},$$

where ρ_i contain terms of order greater than $\mathrm{ord}(d_i)$. Taking into account the sense of grading, we can consider the series

$$\widehat{S} = \begin{pmatrix} d_1 \\ \cdots \\ d_n \end{pmatrix}$$

as a homogeneous approximation of the series S. Moreover, it can be shown that there exists a system with the series $\widehat{S}$; this system is naturally considered as a homogeneous approximation of the system (1). We emphasize that the series $\widehat{S}$, the system corresponding to this series, and the change of variables can be explicitly found and the algebraic framework allows efficient use of numerical computation [11], [13].

3. Main result

Let us consider a series S of the form (7). We assume that it is realizable and $\rho_L(S) = n$. Without loss of generality we assume that each component S_i of the series S is nonzero.

Definition 1. *Denote by r_j the minimal order of terms included to the component S_j of the series* (7),

$$r_j = \min\{k : (c_I)_j \neq 0 \text{ for some } I \in M_k\}, \ j = 1, \dots, p.$$

Define the minimal part of the series S as

$$S_{\min} = \begin{pmatrix} (S_1)_{\min} \\ \cdots \\ (S_p)_{\min} \end{pmatrix},$$

where

$$(S_j)_{\min} = \sum_{|I|=r_j} (c_I)_j \eta_I, \ j = 1, \dots, p.$$

Remark. In the paper [2] we considered one-dimensional series, i.e., the case $p = 1$, where we used the notation $\widehat{S}$ instead of $S_{\min}$ and called it “a homogeneous approximation” of the series S. However, for $p > 1$, such a definition of a homogeneous approximation is not natural, which is shown by the following example.

Example. Consider the series

$$S = \begin{pmatrix} \eta_1 \\ \eta_1 + \eta_{21} + \eta_{211} \end{pmatrix}.$$

In this case $S_{\min} = (\eta_1, \eta_1)^\top$. However, the transformation $F(x) = (x_1, x_2 - x_1)^\top$ reduces S to the form $F(S) = (\eta_1, \eta_{21} + \eta_{211})^\top$, and $(F(S))_{\min} = (\eta_1, \eta_{21})^\top$ has more reasons to be considered as a homogeneous approximation of the series S. Actually, in this case S is realized by the system

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_1 + x_1 u_2 + \tfrac{1}{2} x_1^2 u_2 \end{aligned} \tag{12}$$

with the output $y = h(x) = x$ while $(F(S))_{\min}$ is realized by the homogeneous approximation [12] of the system (12)

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= x_1 u_2 \end{aligned}$$

with the output $y = h(x) = x$.

Below, by a *formal r-dimensional mapping* we mean any formal series of the form
$$f(a_1,\dots,a_p) = \sum_{q_1+\dots+q_p\ge 1} f_{q_1\dots q_p} a_1^{ш q_1} ш \cdots ш a_p^{ш q_p},$$
where $f_{q_1\dots q_p} \in \mathbb{R}^r$. In particular, if $r = 1$, we call f a *formal function.*

If f is a formal function, then $f(S)$ is a series of elements of $\mathcal{F}$ with one-dimensional coefficients. Then $(f(S))_{\min}$ is the sum of elements of the minimal order from this series.

We adopt the following notation. Given a realizable series (7), we denote by $\mathcal{L}_S$ the core Lie subalgebra of a system which is the minimal realization of the series S; by $\mathcal{J}_S$ we denote the graded left ideal generated by $\mathcal{L}_S$.

Lemma 1. *Let S be a realizable series of the form* (7)*. Then for any formal function $f(a_1,\dots,a_p)$*
$$\mathcal{J}_S \subset (f(S))^{\perp}_{\min}.$$

Proof. Let $\operatorname{codim}(\mathcal{L}_S) = \rho_L(c) = n$. Let us consider a realization of S and its (n-dimensional) series $\widetilde{S}$. Then, without loss of generality, we can choose the series $\widetilde{S}$ in the form
$$\widetilde{S}_k = d_k + R_k,\ k = 1,\dots,n,$$
where d_k are elements of the dual basis constructed as described in the previous section, R_k contains terms of order greater than $\operatorname{ord}(d_k)$, and $S = h(\widetilde{S})$, where h is a formal p-dimensional mapping. It is clear that $(f(S))_{\min} = (f(h(\widetilde{S})))_{\min}$ equals a shuffle polynomial of d_k. Hence, as was shown in [12], $(f(S))_{\min} \in \mathcal{J}_S^{\perp}$, which proves the lemma.

Now, following the idea of the paper [2], we introduce the maximal left ideal which is orthogonal to any element $(f(S))_{\min}$. First, recall the following definition.

Definition 2. [2] *We say that a linear subspace $\mathcal{J}' \subset \mathcal{F}$ is a graded Lie generated left ideal if there exists a graded Lie subalgebra $\mathcal{L}' \subset \mathcal{L}$ such that*
$$\mathcal{J}' = \operatorname{Lin}\{a\ell : a \in \mathcal{F} + \mathbb{R},\ \ell \in \mathcal{L}'\}.$$
If this is the case, we say that $\mathcal{J}'$ is generated by $\mathcal{L}'$. We denote the set of all graded Lie generated left ideals by D.

In particular, $\mathcal{J}_S \in D$; it is generated by $\mathcal{L}_S$.

Now we introduce the following subset of graded Lie generated left ideals:
$$D_S = \{\mathcal{J} \in D : \mathcal{J} \subset (f(S))^{\perp}_{\min} \text{ for any formal function } f\}.$$
Lemma 1 implies that $\mathcal{J}_S \in D_S$, therefore, $D_S \ne \varnothing$.

Obviously, there exists the unique maximal (in the sense of inclusion) left ideal in the set D_S. We denote it by $\mathcal{J}_S^{\max}$ and denote the Lie subalgebra that generates $\mathcal{J}_S^{\max}$ by $\mathcal{L}_S^{\max}$. Let $r = \operatorname{codim}(\mathcal{L}_S^{\max})$. Since $\mathcal{L}_S \subset \mathcal{L}_S^{\max}$, we have $r \le n$.

Now we apply the construction of a dual basis described in the previous section to the Lie subalgebra $\mathcal{L}_S^{\max}$. Namely, we choose homogeneous elements $\widehat{\ell}_1, \dots, \widehat{\ell}_r \in \mathcal{L}$ such that $\mathrm{ord}(\widehat{\ell}_i) \le \mathrm{ord}(\widehat{\ell}_j)$ if $i < j$ and

$$\mathcal{L}_S^{\max} + \mathrm{Lin}\{\widehat{\ell}_1, \dots, \widehat{\ell}_r\} = \mathcal{L}.$$

Also, we choose a homogeneous basis $\{\widehat{\ell}_i\}_{i=r+1}^{\infty}$ of $\mathcal{L}_S^{\max}$. Finally, we apply the Poincaré-Birkhoff-Witt Theorem and construct a dual basis

$$\widehat{d}_{i_1 \dots i_k}^{q_1 \dots q_k} = \frac{1}{q_1! \cdots q_k!} \widehat{d}_{i_1}^{\sqcup\!\sqcup q_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \widehat{d}_{i_k}^{\sqcup\!\sqcup q_k},$$

where the notation $\widehat{d}_i = \widehat{d}_i^1$ is used. Analogously to [12] it can be shown that the set

$$\{\widehat{d}_1^{\sqcup\!\sqcup q_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \widehat{d}_r^{\sqcup\!\sqcup q_r} : q_1 + \cdots + q_r \ge 1\}$$

forms a basis of $(\mathcal{J}_S^{\max})^{\perp}$. Since $(f(S))_{\min} \subset (\mathcal{J}_S^{\max})^{\perp}$, then $(f(S))_{\min}$ is a shuffle polynomial of $\widehat{d}_1, \dots, \widehat{d}_r$ for any formal function f.

Definition 3. *For a given set $A \subset \mathcal{F}$, we define a shuffle span of the set A as*

$$A^{sh} = \mathrm{Lin}\{a_1^{\sqcup\!\sqcup i_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup a_k^{\sqcup\!\sqcup i_k} : k \ge 1, a_1, \dots, a_k \in A,\ i_1, \dots, i_k \ge 0\}.$$

Let us consider the subspace

$$N_S = \{(f(S))_{\min} : f \text{ is a formal function}\}. \tag{13}$$

As is shown above, any element of N_S is a shuffle polynomial of $\widehat{d}_1, \dots, \widehat{d}_r$, that is,

$$N_S \subset \{\widehat{d}_1, \dots, \widehat{d}_r\}^{sh}.$$

Thus, any element of N_S is a shuffle polynomial of r elements, where $r \le n = \rho_L(c)$. However, elements $\widehat{d}_i$ may not belong to the set N_S. We show how one can find a "shuffle basis" of the set N_S, that is, elements of N_S that generate the set N_S by using shuffles. Below we say that several elements are *polynomially independent* if any of them does not equal a shuffle polynomial of the others.

Lemma 2. *There exist $q \le p$ homogeneous polynomially independent elements $\widehat{a}_1, \dots, \widehat{a}_q \in N_S$ such that*

$$N_S = \{\widehat{a}_1, \dots, \widehat{a}_q\}^{sh}. \tag{14}$$

Proof. We describe the algorithm for finding such elements $\widehat{a}_i$. It is a generalization of the algorithm [3], [14] for finding a homogeneous approximation of a series of Lie rank n satisfying the Rashevsky-Chow condition (9).

Step 1. Assume that the components of S are nonzero. Find the minimal order of all components,

$$\alpha_1 = \min\{\mathrm{ord}((S_i)_{\min}) : i = 1, \dots, p\}.$$

Find a linear nonsingular mapping F such that the elements $((F(S))_i)_{\min} \in \mathcal{F}^{\alpha_1}$ for $i = 1, \dots, n_1$ are linearly independent and $(F(S))_i$ contain only elements of order greater than α_1, $i = n_1 + 1, \dots, p$. Denote $S^1 = F(S)$.

Step $k \geq 2$. If $n_{k-1} = p$, then stop. If not, suppose that after the $(k-1)$-th step we obtain the series S^{k-1} for which the elements $(S_1^{k-1})_{\min}, \dots, (S_{n_{k-1}}^{k-1})_{\min}$ of order no greater than α_{k-1} are polynomially independent and S_i^{k-1} equal zero or contain only elements of order greater than α_{k-1} for $i = n_{k-1}+1, \dots, p$. Here $\alpha_1 < \cdots < \alpha_{k-1}$ and $n_1 < \cdots < n_{k-1}$. On the current step we find the mapping that does not change the components $S_1^{k-1}, \dots, S_{n_{k-1}}^{k-1}$.

Consider components S_i^{k-1}, $i = n_{k-1}+1, \dots, p$. If all of them are zero, then stop. Otherwise, find the minimal order of all nonzero components,

$$\alpha_k = \min\{\mathrm{ord}((S_i^{k-1})_{\min}) : i = n_{k-1}+1, \dots, p,\ S_i^{k-1} \neq 0\} > \alpha_{k-1}.$$

Without loss of generality assume that $(S_i^{k-1})_{\min} \in \mathcal{F}^{\alpha_k}$, $i = n_{k-1}+1, \dots, n'_k$, and S_i^{k-1} contain only elements of order greater than α_k or $S_i^{k-1} = 0$ for $i > n'_k$. (This can be achieved by swapping components of the series.)

Now consider the components S_i^{k-1} successively, for $i = n_{k-1}+1, \dots, n'_k$.

Case 1. If $(S_i^{k-1})_{\min}$ belongs to the shuffle span of $(S_1^{k-1})_{\min}, \dots, (S_{i-1}^{k-1})_{\min}$, then there exists a polynomial $F_i(x) = x_i + p_i(x_1, \dots, x_{i-1})$ such that $F_i(S^{k-1})$ equals zero or contains only elements of order greater than α_k. Then replace the i-th component of the series by $F_i(S^{k-1})$ leaving the other components unchanged and pass to the next i.

Case 2. If $(S_i^{k-1})_{\min}$ is polynomially independent of $(S_1^{k-1})_{\min}, \dots, (S_{i-1}^{k-1})_{\min}$, then pass to the next i.

If for all i only Case 1 occurs, we obtain a mapping F such that $(F(S^{k-1}))_i$ for all $i = n_{k-1}+1, \dots, n'_k$ equals zero or contains only elements of order greater than α_k. Then repeat the k-th step with the series $F(S^{k-1})$.

If not, then we obtain the series $S^k = F(S^{k-1})$ such that $S_i^k = S_i^{k-1}$ for $i = 1, \dots, n_{k-1}$, and $(S_1^k)_{\min}, \dots, (S_{n_k}^k)_{\min}$ have the order no greater than α_k and are polynomially independent, $n_k \geq n_{k-1}+1$, and S_i^k equal zero or contain only elements of order greater than α_k for $i = n_k+1, \dots, p$. In this case, pass to the $(k+1)$-th step.

We emphasize that the case when the algorithm needs an infinite number of steps is not excluded. In this case, after an infinite number of steps one or several components of the series become zero.

As a result, we obtain the series

$$F(S) = \begin{pmatrix} \widehat{a}_1 + R_1 \\ \cdots \\ \widehat{a}_q + R_q \\ 0 \\ \cdots \\ 0 \end{pmatrix}, \tag{15}$$

where $\widehat{a}_1,\dots,\widehat{a}_q$ are polynomially independent, that is, any of them does not equal a shuffle polynomial of the others, and R_i contain elements of order greater than $\mathrm{ord}(\widehat{a}_i)$. We notice that the mapping F constructed by this algorithm is invertible. Hence, for any formal function f we obtain that $(f(S))_{\min} = (f(F^{-1}(F(S))))_{\min}$ is a shuffle polynomial of $\widehat{a}_1,\dots,\widehat{a}_q$. Moreover, the elements $\widehat{a}_i$ and any shuffle polynomial of them can be obtained as $(f(S))_{\min}$ by some formal function f, which proves the lemma.

As follows from the proof, elements $\widehat{a}_i$ are defined uniquely up to shuffle polynomials. Moreover, all elements $\widehat{a}_i$ belong to N_S and are polynomially independent. Taking into account the equality (14), we say that the set $\{\widehat{a}_1,\dots,\widehat{a}_q\}$ given by the algorithm is *a shuffle basis* of the set N_S.

Remark. We notice that the number q can be less than, equal, or greater than r. For example, for the one-dimensional series $S=\eta_{21}$, we have $S_{\min}=S$, that is, $q=p=1$. In this case, $r=n=2$, and the dual basis can be chosen as $d_1=\widehat{d_1}=\eta_1$, $d_2=\widehat{d_2}=\eta_{21}$. However, for the series

$$S=\begin{pmatrix}\eta_1\\ \eta_{12}+\eta_{21}\\ \eta_{122}+\eta_{212}+\eta_{221}\end{pmatrix}=\begin{pmatrix}\eta_1\\ \eta_1 ш \eta_2\\ \eta_1 ш \eta_2 ш \eta_2\end{pmatrix} \tag{16}$$

with $q=p=3$, we obviously get $r=n=2$.

Example. For the following series, the algorithm described above requires infinite number of steps:

$$S=\begin{pmatrix}\eta_1\\ \eta_1+\frac{1}{2!}\eta_1^{ш 2}+\cdots+\frac{1}{k!}\eta_1^{ш k}+\cdots\end{pmatrix}$$

Actually, the map $F(x)=(x_1,x_2-e^{x_1})^\top$ reduces S to the form $F(S)=(\eta_1,0)^\top$.

Finally, we notice that elements $\widehat{a}_i$, which are polynomially independent, can satisfy shuffle-polynomial equalities. For example, for the series (16), we can choose $\widehat{a}_i=S_i$ and we have $\widehat{a}_1 ш \widehat{a}_3=\widehat{a}_2 ш \widehat{a}_2$.

Taking into account Lemma 2, we propose the following definition of a homogeneous approximation of a series of the form (7).

Definition 4. *We say that the series*

$$\widehat{S}=\begin{pmatrix}\widehat{a}_1\\ \cdots\\ \widehat{a}_q\end{pmatrix},$$

where $\widehat{a}_i$ are homogeneous polynomially independent elements, is a homogeneous approximation of the series (7) *if there exists an invertible formal mapping F such that the series $F(S)$ has the form* (15).

Remark. If a series is such that $p=\rho_L(c)=n$ and satisfies the Rashevsky-Chow condition (9), then this definition coincides with the usual definition of homogeneous approximation [12], [3]; in this case $q = p = n$ and $\widehat{a}_i$ can be chosen as $\widehat{a}_i = d_i$, $i = 1, \dots, n$. On the other hand, if $p = 1$, then this definition coincides with the definition of homogeneous approximation proposed in [2]; in this case $\widehat{a}_1$ can be chosen as $\widehat{a}_1 = \widehat{S} = S_{\min}$.

Definition 5. *We say that two series are algebraically equivalent if they have the same homogeneous approximation.*

Lemma 2 implies the following result.

Theorem 1. *Two series S^1 and S^2 are algebraically equivalent if and only if $N_{S^1} = N_{S^2}$, where the sets N_{S^i} are defined for series S^i as in* (13)*, i.e.,*

$$N_{S^i} = \{(f(S^i))_{\min} : f \text{ is a formal function}\}, \quad i = 1, 2.$$

As elements $\widehat{a}_i$ of a homogeneous approximation, any shuffle basis of the set N_{S^i} can be chosen.

We emphasize that two algebraically equivalent series can have unequal dimensions.

Definition 5 generalizes the definition of A-equivalence for series with $p = \rho_L(c) = n$ satisfying the Rashevsky-Chow condition [5]. However, this definition is not so natural for general series (7). For example, the series (16) and the series

$$S' = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

are very similar since they can be reconstructed from the same two-dimensional system though they are not algebraically equivalent: obviously, $N_S \neq N_{S'}$. This is because in the general case the set N_S is not completely defined by the maximal left ideal. In order to formulate this property, we propose the following definition.

Definition 6. *We say that two series S^1 and S^2 are weakly algebraically equivalent if their maximal left ideals coincide, i.e., $\mathcal{J}_{S^1}^{\max} = \mathcal{J}_{S^2}^{\max}$.*

Obviously, if $N_{S^1} = N_{S^2}$, then $\mathcal{J}_{S^1}^{\max} = \mathcal{J}_{S^2}^{\max}$. Therefore, we get the following corollary.

Corollary 1. *If two series S^1 and S^2 are algebraically equivalent, then they are weakly algebraically equivalent.*

Example. Let us consider two one-dimensional series

$$S^1 = \eta_1 \quad \text{and} \quad S^2 = \eta_{11}.$$

Recall that $\eta_{11} = \frac{1}{2}\eta_1 ш \eta_1$, therefore, both series have the same maximal left ideal; their one-dimensional realization is $\dot{x}_1 = u_1$. Hence, S^1 and S^2 are weakly algebraically equivalent. However, the sets N_{S^1} and N_{S^2} do not coincide since $\eta_1 \in N_{S^1}$ but $\eta_1 \notin N_{S^2}$. Thus, S^1 and S^2 are not algebraically equivalent.

Example. Consider the series

$$S = \begin{pmatrix} \eta_2 + \eta_{21} \\ \eta_{22} + \eta_{221} \end{pmatrix}.$$

Applying the algorithm described in the proof of Lemma 2, we use the mapping $F(x) = (x_1, -x_2 + \frac{1}{2}x_1^2)^\top$. Since

$$-\eta_{22} - \eta_{221} + \tfrac{1}{2}(\eta_2 + \eta_{21}) ш (\eta_2 + \eta_{21}) = -\eta_{221} + \eta_2 ш \eta_{21} + R = \eta_{221} + \eta_{212} + R,$$

where $\mathrm{ord}(R) = 4$, we obtain

$$F(S) = \begin{pmatrix} \eta_2 + \eta_{21} \\ \eta_{221} + \eta_{212} + R \end{pmatrix}.$$

Hence, as a homogeneous approximation of the series S we can take $(F(S))_{\min}$, i.e.,

$$\widehat{S} = \begin{pmatrix} \widehat{a}_1 \\ \widehat{a}_2 \end{pmatrix} = \begin{pmatrix} \eta_2 \\ \eta_{221} + \eta_{212} \end{pmatrix}.$$

Therefore, $N_S = \{\eta_2, \eta_{221} + \eta_{212}\}^{sh}$. Hence, $\mathcal{L}_S^{\max}$ cannot contain η_2 and $[\eta_2, [\eta_2, \eta_1]]$ since these elements are not orthogonal to the elements $\widehat{a}_1$, $\widehat{a}_2$ respectively and cannot contain η_1 since $\eta_{22}\eta_1$ is not orthogonal to $\widehat{a}_2$. Actually, $\mathcal{L}_S^{\max} = \mathrm{Lin}\{[\eta_1, \eta_2], [\eta_1, [\eta_1, \eta_2]]\} + \sum_{k=4}^{\infty} \mathcal{L}^k$ and therefore the minimal realization of $\widehat{S}$ can be chosen as

$$\dot{x}_1 = u_1$$
$$\dot{x}_2 = u_2$$
$$\dot{x}_3 = x_1 x_2 u_2$$

with the dual basis elements $d_1 = \eta_1$, $d_2 = \eta_2$, $d_3 = \eta_{221} + \eta_{212}$. Obviously, $N_S \subset \{d_1, d_2, d_3\}^{sh}$. Let us consider the series

$$S' = \begin{pmatrix} \eta_1 \\ \eta_{221} + \eta_{212} \end{pmatrix}.$$

Obviously, it is weakly algebraically equivalent but not algebraically equivalent to S.

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Однорідні апроксимації нелінійних керованих систем з виходом і слабка алгебраїчна еквівалентність

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У роботі ми розглядаємо нелінійні керовані системи, які є лінійними за керуванням, з виходом; векторні поля, що визначають систему, і вихід вважаються дійсно аналітичними. Слідуючи алгебраїчному підходу, ми розглядаємо ряди S ітерованих інтегралів, що відповідають таким системам. Ітеровані інтеграли утворюють вільну асоціативну алгебру, і всі наші конструкції використовують її властивості. Спочатку ми розглядаємо множину всіх (формальних) функцій таких рядів $f(S)$ і визначаємо множину N_S членів мінімального порядку для всіх таких функцій. Ми вводимо означення максимального градуйованого Лі-породженого лівого ідеалу $\mathcal{J}_S^{\max}$, який є ортогональним до множини N_S. Ми описуємо зв'язки між цим максимальним лівим ідеалом і лівим ідеалом $\mathcal{J}_S$, що породжений кореневою підалгеброю Лі системи, яка реалізує ряд. А саме, ми показуємо, що $\mathcal{J}_S \subset \mathcal{J}_S^{\max}$. Зокрема, з цього випливає, що градуйована підалгебра Лі, яка породжує лівий ідеал $\mathcal{J}_S^{\max}$, має скінченну ковимірність. Також ми даємо алгоритм, який приводить ряд S до трикутної форми, і пропонуємо означення однорідної апроксимації ряду S. А саме, однорідною апроксимацією є однорідний ряд, компоненти якого – доданки мінімального порядку в кожній компоненті цієї трикутної форми. Ми доводимо, що N_S збігається з множиною тасуючих поліномів компонентів однорідної апроксимації. На відміну від випадку, коли вихід є тотожним, однорідна апроксимація не визначається повністю ідеалом $\mathcal{J}_S^{\max}$. Для того, щоб описати цю властивість, ми вводимо два різних означення еквівалентності рядів: алгебраїчну еквівалентність (коли два ряди мають одну й ту саму однорідну апроксимацію) і слабку алгебраїчну еквівалентність (коли два ряди мають один і той самий максимальний лівий ідеал і, отже, мають одну й ту саму мінімальну реалізуючу систему). Ми доводимо, що якщо два ряди є алгебраїчно еквівалентними, то вони є слабко алгебраїчно еквівалентними. Приклади показують, що обернене твердження не є правильним.

Ключові слова: **Однорідна апроксимація; нелінійна керована система; ряд ітерованих інтегралів; коренева підалгебра Лі; максимальний лівий ідеал.**

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