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On linear stabilization of a class of nonlinear systems in a critical case

In this paper, we address the stabilization problem for nonlinear systems in a critical case. Namely, we study the class of canonical nonlinear systems. Canonical nonlinear systems or chain of power integrators is an important subject of research. Studying such systems is complicated by the fact that they cannot be mapped onto linear systems. Moreover, they have the uncontrollable first approximation. Previous results on smooth stabilization of such systems were obtained under the assumption that the powers in the right-hand side are strictly decreasing. In this work, we consider a case of non-increasing powers in the right-hand side for a three-dimensional system. A popular approach for studying such systems is the backstepping method, which is a method of step-wise stabilization. This method requires a sequential investigation of lower-dimensional subsystems. Backstepping enables the study of a wide range of nonlinear triangular systems but requires technically complex and cumbersome computations. Therefore, a natural question arises about constructing stabilizing controls of a simple form. Polynomial controls can serve as an example of such controls. In the paper, we demonstrate that linear controls can be considered as stabilizing controls. We derive sufficient conditions for the coefficients of the linear control that ensure the asymptotic stability of the zero equilibrium point of the corresponding closed-loop system. The asymptotic stability is proven using the Lyapunov function method, which is found as the sum of squares. The negative definiteness of the Lyapunov function derivative in a neighborhood of the origin guarantees asymptotic stability. In contrast to the case of strictly decreasing powers, additional conditions on the control coefficients, apart from their negativity, emerge. The obtained result extends to a broader class of nonlinear systems through stabilization by nonlinear approximation. This allows the consideration of systems with higher-order terms in the right-hand side. The effectiveness of the applied approach is illustrated by several model examples. The method used in this work to investigate the case of non-increasing powers can be applied to systems of higher dimensions.

Keywords: **stabilization; nonlinear systems; Lyapunov function method; critical case; linear stabilization; linear control.**

2010 Mathematics Subject Classification: 93D15; 93D30; 93C10; 34H05.

1. Introduction

The stabilization problem for nonlinear systems in a critical case is an important problem of nonlinear control theory [1, 2, 3, 4, 5, 6, 7, 8, 9]. Significant attention has been drawn by high-order nonlinear systems that cannot be mapped to linear systems [1, 2, 3, 4, 5, 6, 7, 8, 9]. These systems exemplify a critical case. Since we are dealing with critical case, we cannot use the first (linear) approximation to find stabilizing controls for the original nonlinear system. It is natural to attempt to construct simple classes of stabilizing controls, such as linear controls. The problem of finding such stabilizing controls is called the linear stabilization problem.

In recent decades, a wide range of interest has been sparked by the systems of the following form

$$\begin{cases} \dot{x}_i = x_{i+1}^{p_i} + f_i(x_1, x_2, \dots, x_n), \quad i = 1, \dots, n-1, \\ \dot{x}_n = u^{p_n}, \end{cases} \tag{1}$$

where $p_i \geq 1$ are ratios of positive odd integers, $f_i(x_1, \dots, x_n)$ are continuous real-valued functions with $f_i(0, \dots, 0) = 0$ $(i = 1, \dots, n-1)$.

The stabilization problem for system (1) was studied in many works, see, for instance, [1, 2, 3, 4, 5, 7, 8, 9]. Works [5, 7, 8, 9] rely on the backstepping approach, which is based on recursive Lyapunov function design and leads to stabilizing controls of rather complicated structure. In [1] simple stabilizing controls of the form

$$u = a_1 x_1 + \dots + a_n x_n + a_{n+1} x_2^{p_1} + \dots + a_{2n-1} x_n^{p_{n-1}},$$

were constructed using a quadratic Lyapunov function (for $p_n = 1$). Work [3] shows that it is possible to linearly stabilize system (1).

The above-mentioned results from [3] were achieved under assumption that the powers p_i are strictly decreasing, that is, $p_1 > p_2 > \dots p_n \geq 1$. In this work we weaken the condition of powers p_i being strictly decreasing and prove that it is possible to consider non-increasing values of p_i and still be able to achieve linear stabilization.

Namely, we study the stabilizability of the system

$$\begin{cases} \dot{x}_1 = x_2^{p_1}, \\ \dot{x}_2 = x_3^{p_2}, \\ \dot{x}_3 = u^{p_3} \end{cases} \tag{2}$$

with $p_1 > 1$, $p_2 = p_3 = 1$. We find conditions on the coefficients under which a linear control stabilizes system (2). These results are generalized using nonlinear approximation.

2. Problem formulation and linear control construction

Consider the nonlinear system

$$\begin{cases} \dot{x}_1 = x_2^{p_1}, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = u, \end{cases} \tag{3}$$

where $u \in \mathbb{R}$ is a control, $p_1 > 1$ is a ratio of two positive odd integers.

The stabilization problem for system (3) is to find a continuous control $u(x)$ such that the equilibrium point $x = 0$ of system (3) with $u = u(x)$ is locally asymptotically stable.

Consider the linear control

$$u(x) = -k_1 x_1 - k_2 x_2 - k_3 x_3, \tag{4}$$

where $k_i \in \mathbb{R}$ are positive numbers.

Now we find conditions on the coefficients k_1, k_2, k_3 for the local asymptotic stability of the zero solution of system (3). To this end, we consider the following Lyapunov function

$$V(x) = \frac{1}{2}\left(\frac{k_2^{p_1}}{k_1}(k_1 x_1)^2 + \frac{k_3^{p_2}}{k_2}(k_1 x_1 + k_2 x_2)^2 + \frac{1}{k_3}(k_1 x_1 + k_2 x_2 + k_3 x_3)^2\right)$$

It is obvious that $V(x)$ is positive definite for $k_1 > 0$, $k_2 > 0$, $k_3 > 0$.

Applying the linear change of variables

$$e_1 = k_1 x_1, \quad e_2 = k_1 x_1 + k_2 x_2, \quad e_3 = k_1 x_1 + k_2 x_2 + k_3 x_3,$$

we get

$$V(e) = \frac{1}{2}\sum_{i=1}^{n} l_i e_i^2, \tag{5}$$

where $l_i = k_{i+1}^{p_i} k_i^{-1}$, $i = 1, 2$, $l_3 = k_3^{-1}$. The inverse change of variables is

$$x_1 = k_1^{-1} e_1, \quad x_2 = k_2^{-1}(e_2 - e_1), \quad x_3 = k_3^{-1}(e_3 - e_2).$$

Using (3), we compute $\dot{e}_1$, $\dot{e}_2$, $\dot{e}_3$ as follows:

$$\dot{e}_1 = k_1\dot{x}_1 = k_1 x_2^{p_1} = \frac{k_1}{k_2^{p_1}}(e_2 - e_1)^{p_1},$$

$$\dot{e}_2 = k_1\dot{x}_1 + k_2\dot{x}_2 = k_1 x_2^{p_1} + k_2 x_3 = \frac{k_1}{k_2^{p_1}}(e_2 - e_1)^{p_1} + \frac{k_2}{k_3}(e_3 - e_2),$$

$$\dot{e}_3 = k_1\dot{x}_1 + k_2\dot{x}_2 + k_3\dot{x}_3 = \frac{k_1}{k_2^{p_1}}(e_2 - e_1)^{p_1} + \frac{k_2}{k_3}(e_3 - e_2)$$

$$+ k_3(-k_1x_1 - k_2x_2 - k_3x_3) = \frac{k_1}{k_2^{p_1}}(e_2 - e_1)^{p_1} + \frac{k_2}{k_3}(e_3 - e_2) + k_3 u$$

Thus, applying the feedback $u = -(k_1x_1 + k_2x_2 + k_3x_3) = -e_3$, system (3) takes the form

$$\begin{cases} \dot{e}_1 = \frac{k_1}{k_2^{p_1}}(e_2 - e_1)^{p_1}, \\ \dot{e}_2 = \frac{k_1}{k_2^{p_1}}(e_2 - e_1)^{p_1} + \frac{k_2}{k_3}(e_3 - e_2), \\ \dot{e}_3 = -k_3e_3 + \frac{k_1}{k_2^{p_1}}(e_2 - e_1)^{p_1} + \frac{k_2}{k_3}(e_3 - e_2). \end{cases} \tag{6}$$

Now we calculate the derivative of $V(e)$, given by (5), along the trajectories of the closed-loop system (6)

$$\dot{V}(e) = \frac{\partial V}{\partial e_1}\frac{k_1}{k_2^{p_1}}(e_2 - e_1)^{p_1} + \frac{\partial V}{\partial e_2}\left(\frac{k_1}{k_2^{p_1}}(e_2 - e_1)^{p_1} + \frac{k_2}{k_3}(e_3 - e_2)\right)$$
$$+ \frac{\partial V}{\partial e_3}\left(-k_3e_3 + \frac{k_1}{k_2^{p_1}}(e_2 - e_1)^{p_1} + \frac{k_2}{k_3}(e_3 - e_2)\right).$$

Let us calculate each term separately

$$\frac{\partial V}{\partial e_1}\frac{k_1}{k_2^{p_1}}(e_2 - e_1)^{p_1} = \frac{k_2^{p_1}}{2k_1}2e_1\frac{k_1}{k_2^{p_1}}(e_2 - e_1)^{p_1} = e_1(e_2 - e_1)^{p_1},$$

$$\frac{\partial V}{\partial e_2}\left(\frac{k_1}{k_2^{p_1}}(e_2 - e_1)^{p_1} + \frac{k_2}{k_3}(e_3 - e_2)\right) = \frac{k_3}{2k_2}2e_2\left(\frac{k_1}{k_2^{p_1}}(e_2 - e_1)^{p_1}\right.$$
$$\left. + \frac{k_2}{k_3}(e_3 - e_2)\right) = e_2\left(\frac{k_1k_3}{k_2^{p_1+1}}(e_2 - e_1)^{p_1} + (e_3 - e_2)\right),$$

$$\frac{\partial V}{\partial e_3}\left(-k_3e_3 + \frac{k_1}{k_2^{p_1}}(e_2 - e_1)^{p_1} + \frac{k_2}{k_3}(e_3 - e_2)\right) = \frac{1}{2k_3}2e_3\Big(-k_3e_3$$
$$+ \frac{k_1}{k_2^{p_1}}(e_2 - e_1)^{p_1} + \frac{k_2}{k_3}(e_3 - e_2)\Big) = e_3\Big(-e_3 + \frac{k_1}{k_3k_2^{p_1}}(e_2 - e_1)^{p_1}$$
$$+ \frac{k_2}{k_3^2}(e_3 - e_2)\Big).$$

Then we have

$$\dot{V}(e) = e_1(e_2 - e_1)^{p_1} + \frac{k_1k_3}{k_2^{p_1+1}}e_2(e_2 - e_1)^{p_1} + e_2(e_3 - e_2) - e_3^2$$
$$+ \frac{k_1}{k_3k_2^{p_1}}e_3(e_2 - e_1)^{p_1} + \frac{k_2}{k_3^2}e_3(e_3 - e_2).$$

Rewrite $\dot{V}(e)$ in the form

$$\begin{aligned}\dot{V}(e) = &-e_1(e_1-e_2)^{p_1} - e_2(e_2-e_3) - e_3^2 + \frac{k_1k_3}{k_2^{p_1+1}}e_2(e_2-e_1)^{p_1} \\ &+ \frac{k_1}{k_3k_2^{p_1}}e_3(e_2-e_1)^{p_1} + \frac{k_2}{k_3^2}e_3(e_3-e_2).\end{aligned} \tag{7}$$

To estimate the derivative $\dot{V}(e)$ we use the following lemmas.

Lemma 1. *[10] For any $p \geq 1$ and any numbers $x_i \in \mathbb{R}, i = 1, \ldots, n$, the following inequality holds*

$$|x_1 + x_2 + \cdots + x_n|^p \leq n^{p-1}(|x_1|^p + |x_2|^p + \cdots + |x_n|^p).$$

Lemma 2. *[10] Suppose that $p \geq 1$ is a ratio of positive odd integers. Then the following inequality holds*

$$x(x+a)^p \geq 2^{1-p}x^{p+1} + xa^p, \forall x, a \in \mathbb{R}.$$

Lemma 3. *[6] Suppose that $m > 0$, $n > 0$ are constants. Then, given any number $\gamma > 0$, the following inequality holds*

$$|x|^m|y|^n \leq \frac{m}{m+n}\gamma|x|^{m+n} + \frac{n}{m+n}\gamma^{-\frac{m}{n}}|y|^{m+n}, \forall x, y \in \mathbb{R}.$$

First, using Lemma 1 and Lemma 2, we obtain the following inequalities:

$$\begin{aligned} -e_1(e_1-e_2)^{p_1} &\leq -2^{1-p_1}e_1^{p_1+1} + |e_2^{p_1}||e_1|, \\ -e_2(e_2-e_3) &\leq -e_2^2 + |e_2||e_3|, \\ \frac{k_1k_3}{k_2^{p_1+1}}e_2(e_2-e_1)^{p_1} &\leq \frac{k_1k_3}{k_2^{p_1+1}}2^{p_1-1}e_2^{p_1+1} + \frac{k_1k_3}{k_2^{p_1+1}}2^{p_1-1}|e_1^{p_1}||e_2|, \\ \frac{k_1}{k_3k_2^{p_1}}e_3(e_2-e_1)^{p_1} &\leq \frac{k_1}{k_3k_2^{p_1}}2^{p_1-1}|e_2^{p_1}||e_3| + \frac{k_1}{k_3k_2^{p_1}}2^{p_1-1}|e_1^{p_1}||e_3|, \\ \frac{k_2}{k_3^2}e_3(e_3-e_2) &\leq \frac{k_2}{k_3^2}e_3^2 + \frac{k_2}{k_3^2}|e_3||e_2| \end{aligned} \tag{8}$$

Now, by applying Lemma 3, we deduce

$$\begin{aligned} |e_2^{p_1}||e_1| &= \left|\frac{1}{C_1}e_2\right|^{p_1}|C_1^{p_1}e_1| \leq \frac{p_1}{(p_1+1)C_1^{p_1+1}}e_2^{p_1+1} + \frac{C_1^{p_1(p_1+1)}}{p_1+1}e_1^{p_1+1}, \\ |e_1^{p_1}||e_2| &= |C_2^{p_1}e_1|^{p_1}\left|\frac{1}{C_2^{p_1^2}}e_2\right| \leq \frac{p_1C_2^{p_1(p_1+1)}}{p_1+1}e_1^{p_1+1} + \frac{1}{(p_1+1)C_2^{p_1^2(p_1+1)}}e_2^{p_1+1}, \\ |e_2^{p_1}||e_3| &= \left|\frac{1}{C_3}e_2\right|^{p_1}|C_3^{p_1}e_3| \leq \frac{p_1}{(p_1+1)C_3^{p_1+1}}e_2^{p_1+1} + \frac{C_3^{p_1(p_1+1)}}{p_1+1}e_3^{p_1+1}, \\ |e_1^{p_1}||e_3| &= |C_4^{p_1}e_1|^{p_1}\left|\frac{1}{C_4^{p_1^2}}e_3\right| \leq \frac{p_1C_4^{p_1(p_1+1)}}{(p_1+1)}e_1^{p_1+1} + \frac{1}{(p_1+1)C_4^{p_1^2(p_1+1)}}e_3^{p_1+1}, \\ |e_2||e_3| &= \left|\frac{1}{C_5}e_2\right||C_5e_3| \leq \frac{1}{2C_5^2}e_2^2 + \frac{C_5^2}{2}e_3^2, \end{aligned} \tag{9}$$

where C_1, C_2, C_3, C_4, C_5 are sufficiently small positive numbers.

Note that (9) is true for any positive C_i, $i = 1, \dots, 5$. In order to prove asymptotic stability, we will find additional conditions on C_i to guaranty that $\dot V(e)$ is negative in some small deleted neighborhood of the origin.

Using estimates (8) and (9) sequentially, we have

$$\begin{aligned}\dot V(e) \le& -2^{1-p_1} e_1^{p_1+1} + \frac{p_1}{(p_1+1)C_1^{p_1+1}} e_2^{p_1+1} + \frac{C_1^{p_1(p_1+1)}}{p_1+1} e_1^{p_1+1} - e_2^2 \\ &+ \frac{1}{2C_5^2} e_2^2 + \frac{C_5^2}{2} e_3^2 - e_3^2 + \frac{k_1k_3}{k_2^{p_1+1}} 2^{p_1-1} e_2^{p_1+1} \\ &+ \frac{k_1k_3}{k_2^{p_1+1}} 2^{p_1-1} \frac{p_1 C_2^{p_1(p_1+1)}}{p_1+1} e_1^{p_1+1} + \frac{k_1k_3}{k_2^{p_1+1}} 2^{p_1-1} \frac{1}{(p_1+1)C_2^{p_1^2(p_1+1)}} e_2^{p_1+1} \\ &+ \frac{k_1}{k_3k_2^{p_1}} 2^{p_1-1} \frac{p_1}{p_1+1} \frac{1}{C_3^{p_1+1}} e_2^{p_1+1} + \frac{k_1}{k_3k_2^{p_1}} 2^{p_1-1} \frac{C_3^{p_1(p_1+1)}}{p_1+1} e_3^{p_1+1} \\ &+ \frac{k_1}{k_3k_2^{p_1}} 2^{p_1-1} \frac{p_1 C_4^{p_1(p_1+1)}}{p_1+1} e_1^{p_1+1} + \frac{k_1}{k_3k_2^{p_1}} 2^{p_1-1} \frac{1}{(p_1+1)C_4^{p_1^2(p_1+1)}} e_3^{p_1+1} \\ &+ \frac{k_2}{k_3^2} e_3^2 + \frac{k_2}{2k_3^2C_5^2} e_2^2 + \frac{k_2C_5^2}{2k_3^2} e_3^2.\end{aligned} \tag{10}$$

Rearranging the terms from the right-hand side of (10) we obtain the estimate for $\dot V(e)$ in the form

$$\begin{aligned}\dot V(e) \le& e_1^{p_1+1}\left(-2^{1-p_1} + \frac{C_1^{p_1(p_1+1)}}{p_1+1} + \frac{k_1k_3}{k_2^{p_1+1}} 2^{p_1-1} \frac{p_1 C_2^{p_1(p_1+1)}}{p_1+1}\right. \\ &\left.+ \frac{k_1}{k_3k_2^{p_1}} 2^{p_1-1} \frac{p_1 C_4^{p_1(p_1+1)}}{p_1+1}\right) + e_2^2\left(-1 + \frac{1}{2C_5^2} + \frac{k_2}{2k_3^2C_5^2}\right) \\ &+ e_3^2\left(-1 + \frac{C_5^2}{2} + \frac{k_2}{k_3^2} + \frac{k_2C_5^2}{2k_3^2}\right) + g(x),\end{aligned} \tag{11}$$

where the function $g(x)$ is composed of higher order terms. The function $g(x)$ is given by

$$\begin{aligned}g(x) =& \frac{p_1}{(p_1+1)C_1^{p_1+1}} e_2^{p_1+1} + \frac{k_1k_3}{k_2^{p_1+1}} 2^{p_1-1} e_2^{p_1+1} \\ &+ \frac{k_1k_3}{k_2^{p_1+1}} 2^{p_1-1} \frac{1}{(p_1+1)C_2^{p_1^2(p_1+1)}} e_2^{p_1+1} + \frac{k_1}{k_3k_2^{p_1}} 2^{p_1-1} \frac{p_1}{p_1+1} \frac{1}{C_3^{p_1+1}} e_2^{p_1+1} \\ &+ \frac{k_1}{k_3k_2^{p_1}} 2^{p_1-1} \frac{C_3^{p_1(p_1+1)}}{p_1+1} e_3^{p_1+1} + \frac{k_1}{k_3k_2^{p_1}} 2^{p_1-1} \frac{1}{(p_1+1)C_4^{p_1^2(p_1+1)}} e_3^{p_1+1}.\end{aligned}$$

According to the Lyapunov function method, it is sufficient for $\dot V(e)$ to be negative definite to guarantee asymptotic stability. Therefore we find conditions for coefficients of $e_1^{p_1+1}$, e_2^2, e_3^2 to be negative.

We start with the coefficient of e_2^2:

$$-1+\frac{1}{2C_5^2}+\frac{k_2}{2k_3^2C_5^2}<0,$$

$$\frac{k_2}{2k_3^2C_5^2}<1-\frac{1}{2C_5^2},$$

$$\frac{k_2}{2k_3^2C_5^2}<\frac{2C_5^2-1}{2C_5^2},$$

$$\frac{k_2}{k_3^2}<2C_5^2-1,$$

$$k_2<k_3^2\big(2C_5^2-1\big). \tag{12}$$

Let us move on to the coefficient of e_3^2:

$$-1+\frac{C_5^2}{2}+\frac{k_2}{k_3^2}+\frac{k_2C_5^2}{2k_3^2}<0,$$

$$\frac{k_2}{k_3^2}+\frac{k_2C_5^2}{2k_3^2}<1-\frac{C_5^2}{2},$$

$$\frac{k_2(2+C_5^2)}{2k_3^2}<1-\frac{C_5^2}{2},$$

$$k_2(2+C_5^2)<2k_3^2\big(1-\frac{C_5^2}{2}\big),$$

$$k_2<\frac{2k_3^2-k_3^2C_5^2}{(2+C_5^2)}. \tag{13}$$

Finally, consider the coefficient of $e_1^{p_1+1}$:

$$-2^{1-p_1}+\frac{C_1^{p_1(p_1+1)}}{p_1+1}+\frac{k_1k_3}{k_2^{p_1+1}}2^{p_1-1}\frac{p_1C_2^{p_1(p_1+1)}}{p_1+1}+\frac{k_1}{k_3k_2^{p_1}}2^{p_1-1}\frac{p_1C_4^{p_1(p_1+1)}}{p_1+1}<0. \tag{14}$$

It is clear that for any k_1, k_2, k_3, there exist sufficiently small C_1, C_2, C_4 such that the coefficient of $e_1^{p_1+1}$ will be negative. Indeed, we define the function $r(C_1,C_2,C_4)$ as follows:

$$r(C_1,C_2,C_4)=\frac{C_1^{p_1(p_1+1)}}{p_1+1}+\frac{k_1k_3}{k_2^{p_1+1}}2^{p_1-1}\frac{p_1C_2^{p_1(p_1+1)}}{p_1+1}+\frac{k_1}{k_3k_2^{p_1}}2^{p_1-1}\frac{p_1C_4^{p_1(p_1+1)}}{p_1+1}$$

It is obvious that $r(C_1,C_2,C_4)$ is a continuous function and $r(0)=0$. Therefore, by choosing sufficiently small C_1,C_2,C_4 it is possible to make

$|r(C_1, C_2, C_4)|$ smaller than any given number $\varepsilon : \varepsilon \in (0, 2^{1-p_1})$. Then, for such $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|r(C_1, C_2, C_4)| \leq \varepsilon \quad \text{for all} \quad \|\hat{C}\| \leq \delta,$$

where $\hat{C} = (C_1, C_2, C_4)$. Thus,

$$2^{1-p_1} - r(C_1, C_2, C_4) > 0$$

when $\|\hat{C}\| \leq \delta$, and the inequality (14) holds. Assume that C_1, C_2, and C_4 are positive and chosen small enough to satisfy the inequality (14).

So, from the conditions on the coefficients k_1, k_2 and k_3, given by (12) and (13), we obtain the following constraints:

$$\begin{cases} k_2 < k_3^2(2C_5^2 - 1), \\ k_2 < \frac{2k_3^2 - k_3^2 C_5^2}{(2+C_5^2)}, \\ k_1, k_2, k_3 > 0. \end{cases} \tag{15}$$

Using inequality (12), we deduce

$$C_5^2 > \frac{k_3^2 + k_2}{2k_3^2}.$$

From (13) we obtain

$$C_5^2 < \frac{2k_3^2 - 2k_2}{k_2 + k_3^2}.$$

Combining the last two equations, we derive the constraint for C_5^2:

$$\frac{k_3^2 + k_2}{2k_3^2} < C_5^2 < \frac{2k_3^2 - 2k_2}{k_2 + k_3^2}. \tag{16}$$

To ensure the existence of $C_5 > 0$, it is necessary for the following inequality to hold

$$\frac{k_3^2 + k_2}{2k_3^2} < \frac{2k_3^2 - 2k_2}{k_2 + k_3^2} \tag{17}$$

from which follows:

$$\frac{k_3^2 + k_2}{2k_3^2} - \frac{2k_3^2 - 2k_2}{k_2 + k_3^2} < 0,$$

$$\frac{-3k_3^4 + 6k_3^2 k_2 + k_2^2}{2k_3^2(k_2 + k_3^2)} < 0.$$

It is clear that $k_3^2(k_2 + k_3^2) > 0$, which yields

$$-3k_3^4 + 6k_3^2 k_2 + k_2^2 < 0.$$

First we find the roots of the equations

$$-3k_3^4 + 6k_3^2 k_2 + k_2^2 = 0.$$

We put $z = k_3^2$, then

$$-3z^2 + 6zk_2 + k_2^2 = 0,$$

and

$$z = \frac{k_2(3 \pm 2\sqrt{3})}{3}.$$

Recall that $z = k_3^2$ is positive number, then $z = \frac{k_2(3+2\sqrt{3})}{3}$. Therefore, we conclude that inequality (17) holds for

$$k_3^2 > \frac{k_2(3 + 2\sqrt{3})}{3}. \tag{18}$$

Thus, condition (16) is non-contradictory and determines C_5 so that system (15) is consistent.

Now suppose that C_5 is chosen to satisfy condition (16), C_3 is any positive number. Recall that C_1, C_2, C_4 satisfy (14). This implies that by choosing k_1, k_2, and k_3 satisfying condition (18), we render $\dot{V}(e)$ negative definite in a neighborhood of the origin. Indeed, the function $g(x)$ is composed of higher order terms, since $p_1 > 1$. So, if the coefficients of e_1, e_2, and e_3 are negative, then in some sufficiently small neighborhood of the origin $U(0) \in \mathbb{R}^n$ we have

$$\dot{V}(e) < 0 \quad \text{for all} \quad e \in U(0) \setminus \{0\}\,.$$

This, by the Lyapunov function method, means that the zero equilibrium point $e = 0$ of the system (6) is asymptotically stable. Therefore, since the change of variables $x_1 = k_1^{-1} e_1$, $x_2 = k_2^{-1}(e_2 - e_1)$, $x_3 = k_3^{-1}(e_3 - e_2)$ is continuous, $x = 0$ is a locally asymptotically stable equilibrium point of system (3) with $u = u(x)$ given by (4). So, we have proved the following theorem.

Theorem 1. *Let $k_1 > 0$. Suppose that $k_2 > 0$ and k_3 satisfy the inequality*

$$k_3^2 > \frac{k_2(3 + 2\sqrt{3})}{3} = (2.154700538\ldots)k_2. \tag{19}$$

Then the linear control $u = -k_1x_1 - k_2x_2 - k_3x_3$ solves the stabilization problem for system (3).

Condition (19) distinguishes our case from the case of strictly decreasing powers, in which there is no additional requirements for k_2 and k_3 except that they should be positive.

Example 1. Consider the stabilization problem for the nonlinear system:

$$\begin{cases} \dot{x}_1 = x_2^5, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = u. \end{cases} \tag{20}$$

In this case $p_1 = 5$, $p_2 = p_3 = 1$.

Let us choose arbitrary $k_1 > 0$. Choose k_2, k_3 by the condition (19). For example, we put $k_1 = 5$, $k_2 = 2$, $k_3 = 10$. Then, by Theorem 1, the linear stabilizing control (4) has the form $u = u(x)$, where

$$u(x) = -5x_1 - 2x_2 - 10x_3.$$

Let us substitute the control $u(x)$ into system (20). By Theorem 1 the closed-loop system has asymptotically stable equilibrium point. We will illustrate the behavior of the closed-loop system trajectory, for example, for initial conditions

$$x_1(0) = 1, \quad x_2(0) = 1, \quad x_3(0) = 1.$$

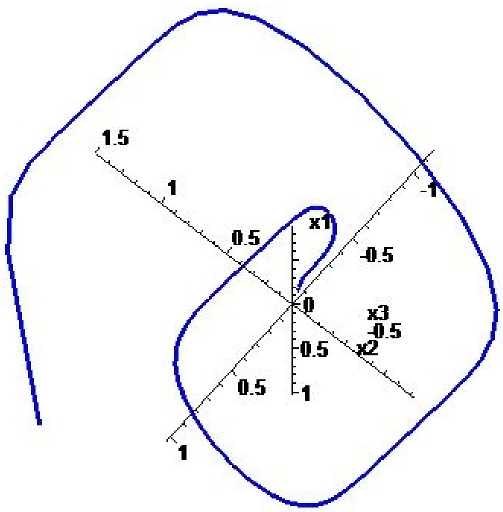

Fig. 1. The trajectory of system (20) with $u = u(x)$.

3. Stabilization by nonlinear approximation

The results obtained in Section 2 can be generalized by considering the following nonlinear system:

$$\begin{cases} \dot{x}_1 = x_2^{p_1} + \varphi_1(x_1, x_2, x_3), \\ \dot{x}_2 = x_3 + \varphi_2(x_1, x_2, x_3), \\ \dot{x}_3 = u, \end{cases} \tag{21}$$

where $\varphi_i(x_1, x_2, x_3)$ are continuous functions, $i = 1, 2, 3$.

To stabilize system (21), we use the same control $u = u(x)$ as in the case of system (3):

$$u(x) = -k_1 x_1 - k_2 x_2 - k_3 x_3.$$

So, suppose k_i satisfy Theorem 1, therefore; $u(x)$ stabilizes system (3). Assume that the functions $\varphi_i(x_1, x_2, x_3)$ satisfy the following inequalities:

$$|\varphi_1(x_1, x_2, x_3)| \leq \rho_1(x_1, x_2, x_3)\big(|x_2|^{p_1+\delta_1} + |x_3|^{p_1+\delta_1}\big),$$

$$|\varphi_2(x_1, x_2, x_3)| \leq \rho_2(x_1, x_2, x_3)\big(|x_3|^{1+\delta_2}\big)$$

in a neighborhood of the origin, where $\rho_i(x_1, x_2, x_3) \geq 0$ are some continuous functions $(i = 1, 2)$, $\delta_1 > 0$ and $\delta_2 > 0$ are some real numbers.

The control $u = u(x)$ stabilizes system (21), since the functions $\varphi_i(x_1, x_2, x_3)$ has higher order then $x_{i+1}^{p_i}$, $i = 1, 2$ $(p_1 > 1,\ p_2 = 1)$. Indeed, we can use the same change of variables and Lyapunov function as for system (3). Note that higher-order terms generated by the functions $\varphi_i(x_1, \dots, x_n)$ should be attributed to the function $g(x)$. These terms will not affect the sign of the derivative of the Lyapunov function $\dot{V}$ in a sufficiently small neighborhood of zero. Therefore, the control $u(x)$ stabilizes not only system (3) but also system (21). Thus, such an approach is similar to the stabilization by first-order approximation. It should be noted that system (3) is used as a nonlinear approximation of system (21).

We will illustrate this approach with the following example.

Example 2. We find a stabilizing control for the following nonlinear system

$$\begin{cases} \dot{x}_1 = x_2^5 + x_2^6 \sin(x_1 + x_2), \\ \dot{x}_2 = x_3 + x_3^2 \cos(x_1), \\ \dot{x}_3 = u. \end{cases} \tag{22}$$

We use system (20) as a nonlinear approximation of system (22). Therefore, system (22) can be stabilized by the same control as system (20).

So, consider the control $u = u(x)$ of the form

$$u(x) = -5x_1 - 2x_2 - 10x_3.$$

We recall that $k_1 = 5, k_2 = 2, k_3 = 10, p_1 = 5$, then condition (19) is satisfied. Put $\rho_1(x_1, x_2, x_3) = 1, \rho_2(x_1, x_2, x_3) = 1, \delta_1 = 1, \delta_2 = 1$. Then, it is clear that for the functions $\varphi_1(x_1, x_2, x_3) = x_2^6 \sin(x_1 + x_2)$ and $\varphi_2(x_1, x_2, x_3) = x_3^2 \cos(x_1)$ the following estimates hold:

$$|\varphi_1(x_1, x_2, x_3)| \leq \rho_1(x_1, x_2, x_3)\left(|x_2|^{p_1+\delta_1} + |x_3|^{p_1+\delta_1}\right) = x_2^6 + x_3^6,$$

$$|\varphi_2(x_1, x_2, x_3)| \leq \rho_2(x_1, x_2, x_3)|x_3|^{1+\delta_2} = x_3^2$$

in the entire space $\mathbb{R}^3$.

Based on the results of the work, it can be concluded that the zero equilibrium point of system (22) under the linear control law $u = u(x)$ is asymptotically stable. Specifically, as shown above, since the control $u = u(x)$ stabilizes the system of the nonlinear approximation (20), it also stabilizes the original nonlinear system (22) with higher-order terms in the right-hand side.

To demonstrate the behavior of solutions of the closed-loop system (22) under the chosen linear control $u(x)$, we construct the trajectory, for example, using the following initial conditions:

$$x_1(0) = 0.8, x_2(0) = 0.7, x_3(0) = 1.$$

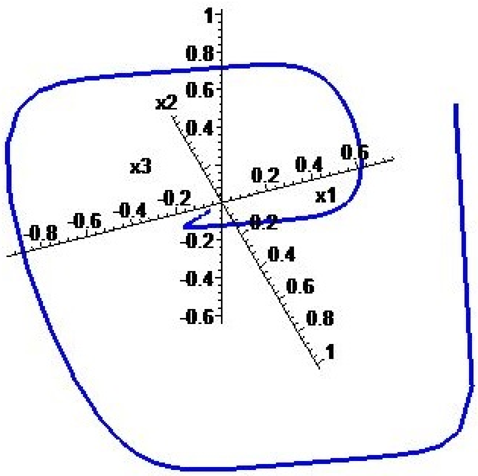

Fig. 2. The trajectory of system (22) with $u = u(x)$.

Conclusion

This work presents a constructive method for stabilizing a class of high-order nonlinear systems in a critical case. Namely, the class of three-dimensional canonical nonlinear systems is considered. Compared to previous results, the condition of decreasing powers was relaxed to a condition of non-increasing powers. It has been shown that for such systems, a linear control can be chosen to ensure that the equilibrium point $x = 0$ is locally asymptotically stable.

Furthermore, an additional condition on the coefficients k_1, k_2, and k_3 was found, compared to the case of strictly decreasing powers, to achieve local asymptotic stability of the zero equilibrium point. Moreover, the class of systems was extended by using stabilization through nonlinear approximation.

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Про лінійну стабілізацію одного класу нелінійних систем у критичному випадку

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В статті розглядається задача стабілізації нелінійних систем у критичному випадку. А саме, вивчається клас канонічних нелінійних систем. Клас канонічних нелінійних систем або ланцюг степеневих інтеграторів є важливим об'єктом дослідження. Вивчення таких систем ускладнюється тим фактом, що їх не можна відобразити на лінійні системи. Крім того, вони є некерованими за першим наближенням. Відомі результати щодо гладкої стабілізації таких систем було отримано при умові строгого спадання степенів правої частини. У цій роботі розглянуто один з випадків нестрогого спадання степенів у правій частині для тривимірної системи. Популярним підходом до дослідження таких систем є метод покрокової побудови стабілізуючих керувань - backstepping. Він потребує послідовного дослідження підсистем меншої розмірності. Цей метод дає можливість досліджувати широкі класи нелінійних трикутних систем, але потребує технічно складних, громіздких обчислень. Тому виникає природне питання про побудову стабілізуючих керувань простого вигляду. Прикладом таких керувань можуть служити поліноміальні керування. У статті показано, що можна розглядати лінійні керування в якості стабілізуючих. Отримано умови на коефіцієнти лінійного керування, які є достатніми для асимптотичної стійкості нульової точки спокою відповідної замкнутої системи. Для доведення асимптотичної стійкості використано метод функції Ляпунова, яку вдається знайти як суму квадратів. Від'ємна визначеність похідної функції Ляпунова в околі нуля гарантує асимптотичну стійкість. На відміну від випадку строгого спадання степенів, виникають додаткові умови на коефіцієнти керування окрім їх від'ємності. Отриманий результат розширюється на більш широкий клас нелінійних систем за рахунок стабілізації по нелінійному наближенню. Це дає змогу розглядати системи з доданками більш високого порядку у правій частині. Ефективність застосованого підходу проілюстровано на кількох модельних прикладах. Використаний в роботі метод дослідження випадку нестрогого спадання степенів може бути застосовано для систем більш високої розмірності.

Ключові слова: **стабілізація; нелінійні системи; метод функції Ляпунова; критичний випадок; лінійна стабілізація; лінійні керування.**

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