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On some hypergeometric Sobolev orthogonal polynomials with several continuous parameters

In this paper we study the following hypergeometric polynomials:

$$\mathcal{P}_n(x) = \mathcal{P}_n(x;\alpha,\beta,\delta_1,\dots,\delta_\rho,\kappa_1,\dots,\kappa_\rho) =$$

$$= {}_{\rho+2}F_{\rho+1}(-n, n{+}\alpha{+}\beta{+}1, \delta_1{+}1,\dots,\delta_\rho{+}1; \alpha{+}1, \kappa_1{+}\delta_1{+}1,\dots,\kappa_\rho{+}\delta_\rho{+}1; x),$$

and

$$\mathcal{L}_n(x) = \mathcal{L}_n(x;\alpha,\delta_1,\dots,\delta_\rho,\kappa_1,\dots,\kappa_\rho) =$$

$$= {}_{\rho+1}F_{\rho+1}(-n, \delta_1{+}1,\dots,\delta_\rho{+}1; \alpha{+}1, \kappa_1{+}\delta_1{+}1,\dots,\kappa_\rho{+}\delta_\rho{+}1; x), \qquad n \in \mathbb{Z}_+,$$

where $\alpha,\beta,\delta_1,\dots,\delta_\rho \in (-1,+\infty)$, and $\kappa_1,\dots,\kappa_\rho \in \mathbb{Z}_+$, are some parameters. The natural number ρ of the continuous parameters $\delta_1,\dots,\delta_\rho$ can be chosen arbitrarily large. It is seen that the special case $\kappa_1 = \dots = \kappa_\rho = 0$ leads to Jacobi and Laguerre orthogonal polynomials. Of course, such polynomials and more general ones appeared in the literature earlier. Our aim here is to show that polynomials $\mathcal{P}_n(x)$ and $\mathcal{L}_n(x)$ are Sobolev orthogonal polynomials on the real line with some explicit matrices of measures.

The importance of the orthogonality property was our main reason to concentrate our attention on polynomials $\mathcal{P}_n(x)$ and $\mathcal{L}_n(x)$. Here we shall use some our tools developed earlier. In particular, it was shown recently that Sobolev orthogonal polynomials are related by a differential equation with orthogonal systems $\mathcal{A}$ of functions acting in the direct sums of usual L^2_μ spaces of square-summable (classes of the equivalence of) functions with respect to a positive measure μ. The case of a unique L^2_μ is of a special interest, since it allows to use OPRL to obtain explicit systems of Sobolev orthogonal polynomials. The main problem here is *to choose a suitable linear differential operator in order to get explicit representations for Sobolev orthogonal polynomials*. The proof of the orthogonality relations is then a verification of such a choice and it goes in another direction: we start from the already known polynomials to their properties.

We also study briefly such properties of the above polynomials: integral representations, differential equations and location of zeros. A system of such polynomials with a kind of the bispectrality property is constructed.

Keywords: **orthogonal polynomials; Sobolev orthogonality; recurrence relations.**

2010 Mathematics Subject Classification: 42C05.

1. Introduction

The theory of orthogonal polynomials on the real line (OPRL) is a classical subject of analysis having a lot of applications [29],[9],[14]. The theory of Sobolev orthogonal polynomials is less developed and recognized and it still remains to be a *terra incognita* in some aspects [21]. As this theory may be viewed as a generalization of the classical one, then one can expect that some properties and objects from the classical theory will have their mirrors and extensions in the theory of Sobolev orthogonal polynomials. For instance, the important property for OPRL is that the multiplication by x operator in the corresponding L^2_μ space is symmetric. Under some general assumptions, a weaker property of symmetry with respect to an indefinite metric holds for Sobolev orthogonal polynomials [32]. We intend to define and study some generalizations of Jacobi and Laguerre orthogonal polynomials. Namely, we shall study the following polynomials:

$$\mathcal{P}_n(x) = \mathcal{P}_n(x;\alpha,\beta,\delta_1,\dots,\delta_\rho,\kappa_1,\dots,\kappa_\rho) =$$

$$= {}_{\rho+2}F_{\rho+1}(-n, n+\alpha+\beta+1, \delta_1+1,\dots,\delta_\rho+1; \alpha+1, \kappa_1+\delta_1+1,\dots,\kappa_\rho+\delta_\rho+1; x), \tag{1}$$

and

$$\mathcal{L}_n(x) = \mathcal{L}_n(x;\alpha,\delta_1,\dots,\delta_\rho,\kappa_1,\dots,\kappa_\rho) =$$

$$= {}_{\rho+1}F_{\rho+1}(-n, \delta_1+1,\dots,\delta_\rho+1; \alpha+1, \kappa_1+\delta_1+1,\dots,\kappa_\rho+\delta_\rho+1; x), \quad n \in \mathbb{Z}_+, \tag{2}$$

where $\alpha,\beta,\delta_1,\dots,\delta_\rho \in (-1,+\infty)$, and $\kappa_1,\dots,\kappa_\rho \in \mathbb{Z}_+$, are some parameters. Observe that the number $\rho \in \mathbb{N}$ of the continuous parameters $\delta_1,\dots,\delta_\rho$ can be arbitrarily large. It is clear that the special case $\kappa_1 = \dots = \kappa_\rho = 0$ leads to the Jacobi and Laguerre orthogonal polynomials on the real line. There are also some other special cases and related systems of hypergeometric polynomials which were studied before, including Fasenmyer's polynomials, see [26]. In general, polynomials $\mathcal{P}_n(x)$ and $\mathcal{L}_n(x)$ turns out to be Sobolev orthogonal polynomials on the real line with some explicit matrix measures. This can be derived on a way proposed in papers [30] and [31].

Notice that we can consider the following more general hypergeometric polynomials:

$$\mathbf{P}_n(x) = \mathbf{P}_n(x;a,\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q) =$$
$$= {}_{p+2}F_q(-n, n+a, \alpha_1,\dots,\alpha_p; \beta_1,\dots,\beta_q; x), \tag{3}$$

and

$$\mathbf{L}_n(x) = \mathbf{L}_n(x;\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q) =$$
$$= {}_{p+1}F_q(-n,\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;x), \quad n\in\mathbb{Z}_+, \tag{4}$$

where $a\in(-1,+\infty)$; $\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q\in(0,+\infty)$, are some parameters. Here $p,q\in\mathbb{Z}_+$, and the case $p=0$ and/or $q=0$ means that α_ks and/or β_ks are absent, respectively.

Polynomials $\mathbf{P}_n(x)$, probably, appeared for the first time in a paper of Chaundy [5] (see formula (26) therein). For the case $a=1$ they appeared later in formula (21) on page 266 in [10]. Polynomials $\mathbf{L}_n(x)$ also appeared for the first time in the paper of Chaundy [5] (see formula (25) therein). Ten years later they appeared in [10] (see formula (25) on page 267).

Observe that

$$\mathcal{P}_n(x;\alpha,\beta,\delta_1,\ldots,\delta_\rho,\kappa_1,\ldots,\kappa_\rho) =$$
$$= \mathbf{P}_n(x;\alpha+\beta+1,\delta_1+1,\ldots,\delta_\rho+1;\alpha+1,\kappa_1+\delta_1+1,\ldots,\kappa_\rho+\delta_\rho+1), \quad n\in\mathbb{Z}_+, \tag{5}$$

and

$$\mathcal{L}_n(x;\alpha,\delta_1,\ldots,\delta_\rho,\kappa_1,\ldots,\kappa_\rho) =$$
$$= \mathbf{L}_n(x;\delta_1+1,\ldots,\delta_\rho+1;\alpha+1,\kappa_1+\delta_1+1,\ldots,\kappa_\rho+\delta_\rho+1), \quad n\in\mathbb{Z}_+, \tag{6}$$

where $\alpha,\beta,\delta_1,\ldots,\delta_\rho\in(-1,+\infty)$, and $\kappa_1,\ldots,\kappa_\rho\in\mathbb{Z}_+$, are arbitrary parameters; $\rho\in\mathbb{N}$.

The importance of the orthogonality property was our main reason to concentrate our attention on polynomials $\mathcal{P}_n(x)$ and $\mathcal{L}_n(x)$. Sobolev orthogonality for the polynomials $\mathcal{P}_n(x)$ and $\mathcal{L}_n(x)$ will be obtained in Theorem 1. Here we shall use tools developed earlier in [30] and [31]. In [31] it was shown that Sobolev orthogonal polynomials are related by a differential equation with orthogonal systems $\mathcal{A}$ of functions acting in the direct sums of usual L^2_μ spaces of square-summable (classes of the equivalence of) functions with respect to a positive measure μ. The case of a unique L^2_μ is of a special interest, since it allows to use OPRL to obtain explicit systems of Sobolev orthogonal polynomials. The main problem here is *to choose a suitable linear differential operator in order to get explicit representations for Sobolev orthogonal polynomials.* The proof of Theorem 1 is then a verification of such a choice and it goes in another direction: we start from the already known polynomials to their properties.

Differential equations for the polynomials $\mathcal{P}_n(x)$ and $\mathcal{L}_n(x)$ will be presented in Proposition 2. Obtaining linear differential operators which have orthogonal polynomials (OP) as eigenfunctions is an old and important subject. In this paper we start with hypergeometric representations of polynomials and therefore they are eigenfunctions of differential pencils quite directly. Then we move to the orthogonality. However the mainstream of this subject is to move in the opposite direction: one starts from an explicit orthogonality and then seeks for differential operators. Of course, the first classical examples of OP being eigenfunctions of a differential operator are Jacobi, Laguerre and Hermite polynomials.

H.L. Krall in [20] initiated the study of differential operators of higher orders for OPRL systems. Many years later, in 1980th, investigations of Krall were continued by Littlejohn, J. Koekoek, R. Koekoek and later by other mathematicians. In these investigations an important role was played by generalized Jacobi and Laguerre weights. This generalization includes additions of Dirac masses at endpoints of the orthogonality measure supports. For more details one can see the books [19] and [18].

The above investigations were continued by using inner products which involved derivatives (Sobolev OP), see [4],[17]. Observe that in [17] generalized Laguerre polynomials $L_n^{\alpha,M_0,M_1,...,M_N}(x)$ were related with ordinary Laguerre polynomials by a linear differential operator with real coefficients, not depending on n (the index of a polynomial). This shows that this case fits in the above new scheme from [30],[31] (cf. Condition 1 in [30]). It is shown in [17] that polynomials $L_n^{\alpha,M_0,M_1,...,M_N}(x)$ are orthogonal with respect to the following inner product:

$$< f,g >= \frac{1}{\Gamma(\alpha+1)}\int_0^\infty x^\alpha e^{-x} f(x)g(x)dx + \sum_{\nu=0}^N M_\nu f^{(\nu)}(0)g^{(\nu)}(0),$$

where $\alpha > -1$, $N \in \mathbb{N}$, and $M_\nu \geq 0$. These polynomials were called Laguerre-Sobolev orthogonal polynomials. An explicit hypergeometric representation of type $\mathbf{L}_n(x)$ from (4) was obtained, with $p = N+1$, $q = N+2$. A $(2N+3)$-term recurrence relation for the Laguerre-Sobolev OP was derived in [17] as well. The particular case $N = 1$ of Laguerre-Sobolev OP was studied in [16]. In this case, when α is a nonnegative integer it is deduced in [15] that these polynomials are eigenfunctions of a linear differential operator with polynomial coefficients. The differential operator has order $2\alpha+4$ if $M_0 > 0$, $M_1 = 0$; it has order $2\alpha+8$ if $M_0 = 0$, $M_1 > 0$; and it is of order $4\alpha+10$ if $M_0, M_1 > 0$. In the above case, but without the constraint concerning the parameter α, differential operators of infinite order, having the Laguerre-Sobolev type orthogonal polynomials as eigenfunctions, were obtained in [2].

Sobolev type Jacobi polynomials $P_n^{\alpha,\beta,M_1,M_2}(x,l_1,l_2)$ were studied by Bavinck in [3]. They are orthogonal with respect to the inner product:

$$< p,q >= \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{-1}^1 p(x)q(x)(1-x)^\alpha(1+x)^\beta dx +$$
$$+ M_1 p^{(l_1)}(-1)q^{(l_1)}(-1) + M_2 p^{(l_2)}(1)q^{(l_2)}(1),$$

where $\alpha,\beta > -1$, $l_1,l_2 \in \mathbb{N}$, $M_1,M_2 \geq 0$. $P_n^{\alpha,\beta,M_1,M_2}(x,l_1,l_2)$ are shown to be eigenfunctions of linear differential operators. Conditions which imply the finiteness of the order of operators are presented. Observe that the particular case of Gegenbauer-Sobolev OP was studied before in papers [4],[1], where similar problems were adressed. A representation as ${}_4F_3$ was given in [4].

The foregoing inner products were generalized by Durán and de la Iglesia replacing Dirac addents at the endpoints c_j by addents of the form

$$(p(c_j),p'(c_j),...,p^{(N)}(c_j))M(q(c_j),q'(c_j),...,q^{(N)}(c_j))^*,$$

where M is a positive semi-definite matrix, see [7],[8]. By using Casoratti determinants they obtained explicit representations of polynomials and showed that polynomials are eigenfunctions of a finite-order differential operators.

In [23] new representations for Jacobi Sobolev OP and Laguerre Sobolev OP were given. It was also shown that the Laguerre-Sobolev OP can be obtained from Jacobi-Sobolev OP by confluence.

Notice that some polynomial matrix perturbations of classical measures were studied in [27].

Known methods for generating functions (see, e.g., [10, Chapter XIX], [25]) can be used to obtain some additional properties of the polynomials $\mathcal{P}_n(x)$ and $\mathcal{L}_n(x)$. We shall discuss the existence of recurrence relations for these polynomials. In Theorem 2 we obtain a five-term recurrence relation for a special case of polynomials $\mathbf{L}_n(x)$, with $p = 2$, $q = 3$. The latter provides a five-term recurrence relation for $\mathcal{L}_n$ with $\rho = 2$, as a special case. In this case the polynomials $\mathcal{L}_n(x)$ ($\rho = 2$) have three important properties:

(1) the Sobolev orthogonality;

(2) these polynomials are (generalized) eigenvalues of a pencil of differential operators;

(3) these polynomials are eigenvalues of a pencil of difference operators.

Of course, each of these features is valuable and $\mathcal{L}_n$ ($\rho = 2$) possess all of them. These properties make polynomials $\mathcal{L}_n(x)$ close to classical systems of polynomials and their generalizations, see [29],[18]. Observe that properties (2) and (3) are close to the bispectral problems studied for various orthogonal systems of functions, see [6],[11],[28],[13],[8] and references therein.

Finally, some information on the location of zeros for $\mathcal{P}_n(x)$ and $\mathcal{L}_n(x)$ will be given in Proposition 3.

Notations. As usual, we denote by $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$, the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively; $\mathbb{D}_r := \{z \in \mathbb{C} : \ |z| < r\}$, $r > 0$; $\mathbb{D} := \mathbb{D}_1$. By $\mathbb{Z}_{k,l}$ we mean all integers j satisfying the following inequality: $k \leq j \leq l$; $(k, l \in \mathbb{Z})$. By $\mathbb{P}$ we denote the set of all polynomials with complex coefficients. By $\mathbb{P}_r$ we mean the set of all polynomials with real coefficients. By M^T we mean the transpose of a complex matrix M. For a complex number c we denote $(c)_0 = 1$, $(c)_k = c \cdots (c + k - 1)$, $k \in \mathbb{N}$ (*the shifted factorial or Pochhammer's symbol*). As usual, the generalized hypergeometric function is denoted by

$$
{}_mF_n(a_1, \dots, a_m; b_1, \dots, b_n; x) = {}_mF_n \left[\begin{array}{c} a_1, \dots, a_m; \\ b_1, \dots, b_n; \end{array} x \right] =
$$

$$
= \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_m)_k}{(b_1)_k \dots (b_n)_k} \frac{x^k}{k!},
$$

where a_j, b_j, x are complex numbers and b_js are not allowed to take negative integer values.

2. Properties of some hypergeometric Sobolev orthogonal polynomials

Polynomials $\mathcal{P}_n$ and $\mathcal{L}_n$ admit some recursive integral representations. Let $\alpha, \beta > -1$. Consider the classical Jacobi and Laguerre polynomials:

$$J_n(x) = J_n(x; \alpha, \beta) := {}_2F_1(-n, n + \alpha + \beta + 1; \alpha + 1; x), \tag{7}$$

$$L_n(x) = L_n(x; \alpha) := {}_1F_1(-n; \alpha + 1; x), \qquad n \in \mathbb{Z}_+. \tag{8}$$

Proposition 1. *Let $\rho \in \mathbb{N}$, and $\alpha, \beta, \delta_1, \dots, \delta_\rho \in (-1, +\infty)$; $\kappa_1, \dots, \kappa_\rho \in \mathbb{N}$, be arbitrary parameters. If $\rho \geq 2$, then*

$$\mathcal{P}_n(z; \alpha, \beta, \delta_1, \dots, \delta_\rho, \kappa_1, \dots, \kappa_\rho) =$$
$$= \frac{\Gamma(\kappa_\rho + \delta_\rho + 1)}{\Gamma(\delta_\rho + 1)\Gamma(\kappa_\rho)} \int_0^1 t^{\delta_\rho} (1 - t)^{\kappa_\rho - 1} \mathcal{P}_n(zt; \alpha, \beta, \delta_1, \dots, \delta_{\rho-1}, \kappa_1, \dots, \kappa_{\rho-1}) dt,$$
$$z \in \mathbb{C}: \ |z| < 1, \quad n \in \mathbb{Z}_+. \tag{9}$$

If $\rho = 1$, then

$$\mathcal{P}_n(z; \alpha, \beta, \delta_1, \kappa_1) = \frac{\Gamma(\kappa_1 + \delta_1 + 1)}{\Gamma(\delta_1 + 1)\Gamma(\kappa_1)} \int_0^1 t^{\delta_1} (1 - t)^{\kappa_1 - 1} J_n(zt; \alpha, \beta) dt,$$
$$z \in \mathbb{C}: \ |z| < 1, \quad n \in \mathbb{Z}_+. \tag{10}$$

If $\rho \geq 2$, then

$$\mathcal{L}_n(z; \alpha, \delta_1, \dots, \delta_\rho, \kappa_1, \dots, \kappa_\rho) =$$
$$= \frac{\Gamma(\kappa_\rho + \delta_\rho + 1)}{\Gamma(\delta_\rho + 1)\Gamma(\kappa_\rho)} \int_0^1 t^{\delta_\rho} (1 - t)^{\kappa_\rho - 1} \mathcal{L}_n(zt; \alpha, \delta_1, \dots, \delta_{\rho-1}, \kappa_1, \dots, \kappa_{\rho-1}) dt,$$
$$z \in \mathbb{C}, \quad n \in \mathbb{Z}_+. \tag{11}$$

If $\rho = 1$, then

$$\mathcal{L}_n(z; \alpha, \delta_1, \kappa_1) = \frac{\Gamma(\kappa_1 + \delta_1 + 1)}{\Gamma(\delta_1 + 1)\Gamma(\kappa_1)} \int_0^1 t^{\delta_1} (1 - t)^{\kappa_1 - 1} L_n(zt; \alpha) dt,$$
$$z \in \mathbb{C}, \quad n \in \mathbb{Z}_+. \tag{12}$$

Proof. Use hypergeometric representations of the corresponding polynomials and Theorem 28 in [26, p. 85]. □

Fix an arbitrary $\rho \in \mathbb{N}$, and choose arbitrary parameters $\alpha, \beta, \delta_1, \dots, \delta_\rho \in (-1, +\infty)$, and $\kappa_1, \dots, \kappa_\rho \in \mathbb{N}$. Introduce the following linear differential operator $L = L(\delta, k)$ with polynomial coefficients, $\delta > -1$, $k \in \mathbb{N}$:

$$Ly(x) = \frac{1}{(\delta + 1) \dots (\delta + k)} x^{-\delta} \left(x^{k+\delta} y(x) \right)^{(k)}, \qquad y(x) \in \mathbb{P}. \tag{13}$$

Denote

$$\widehat{D} = \widehat{D}(\delta_1,\dots,\delta_\rho;\kappa_1,\dots,\kappa_\rho) = L(\delta_1,\kappa_1)L(\delta_2,\kappa_2)\dots L(\delta_\rho,\kappa_\rho) =$$

$$= \sum_{j=0}^{\kappa} c_j(x)\frac{d^j}{dx^j}, \qquad c_j(x) = c_j(x;\delta_1,\dots,\delta_\rho;\kappa_1,\dots,\kappa_\rho) \in \mathbb{P}, \tag{14}$$

where $c_\kappa(x)$ is not the null polynomial, $\kappa := \kappa_1 + \dots + \kappa_\rho$.

Now we shall show that the polynomials $\mathcal{P}_n(x)$ and $\mathcal{L}_n(x)$ are Sobolev orthogonal polynomials on the real line.

Theorem 1. *Let $\rho \in \mathbb{N}$, and $\alpha,\beta,\delta_1,\dots,\delta_\rho \in (-1,+\infty)$; $\kappa_1,\dots,\kappa_\rho \in \mathbb{N}$ be arbitrary parameters. Let $\widehat{D} = \widehat{D}(\delta_1,\dots,\delta_\rho;\kappa_1,\dots,\kappa_\rho)$ be given by (14), and*

$$M(x) := (c_0(x),\dots,c_\kappa(x))^T(c_0(x),\dots,c_\kappa(x)), \qquad x \in \mathbb{R}.$$

For polynomials $\mathcal{P}_n(x)$ and $\mathcal{L}_n(x)$, defined as in (1),(2), the following relations hold:

$$\int_0^1 (\mathcal{P}_n(x),\mathcal{P}_n'(x),\dots,\mathcal{P}_n^{(\kappa)}(x))M(x)\begin{pmatrix}\mathcal{P}_m(x)\\ \mathcal{P}_m'(x)\\ \vdots\\ \mathcal{P}_m^{(\kappa)}(x)\end{pmatrix} x^\alpha(1-x)^\beta dx =$$

$$= A_n\delta_{n,m}, \qquad A_n > 0,\ n,m\in\mathbb{Z}_+; \tag{15}$$

$$\int_0^\infty (\mathcal{L}_n(x),\mathcal{L}_n'(x),\dots,\mathcal{L}_n^{(\kappa)}(x))M(x)\begin{pmatrix}\mathcal{L}_m(x)\\ \mathcal{L}_m'(x)\\ \vdots\\ \mathcal{L}_m^{(\kappa)}(x)\end{pmatrix} x^\alpha e^{-x} dx =$$

$$= B_n\delta_{n,m}, \qquad B_n > 0,\ n,m\in\mathbb{Z}_+. \tag{16}$$

Proof. A direct calculation shows that

$$L(\delta_\rho,\kappa_\rho)\mathcal{P}_n(x;\alpha,\beta,\delta_1,\dots,\delta_\rho,\kappa_1,\dots,\kappa_\rho) =$$

$$= \begin{cases} \mathcal{P}_n(x;\alpha,\beta,\delta_1,\dots,\delta_{\rho-1},\kappa_1,\dots,\kappa_{\rho-1}), & \text{if } \rho\geq 2\\ {}_2F_1(-n,n+\alpha+\beta+1;\alpha+1;x), & \text{if } \rho = 1\end{cases};$$

and

$$L(\delta_\rho,\kappa_\rho)\mathcal{L}_n(x;\alpha,\delta_1,\dots,\delta_\rho,\kappa_1,\dots,\kappa_\rho) =$$

$$= \begin{cases} \mathcal{L}_n(x;\alpha,\delta_1,\dots,\delta_{\rho-1},\kappa_1,\dots,\kappa_{\rho-1}), & \text{if } \rho\geq 2\\ {}_1F_1(-n;\alpha+1;x), & \text{if } \rho = 1\end{cases}.$$

Therefore

$$\widehat{D}\mathcal{P}_n(x;\alpha,\beta,\delta_1,\dots,\delta_\rho,\kappa_1,\dots,\kappa_\rho) = {}_2F_1(-n,n+\alpha+\beta+1;\alpha+1;x) = J_n(x;\alpha,\beta),$$

and
$$\widehat{D}\mathcal{L}_n(x;\alpha,\delta_1,\ldots,\delta_\rho,\kappa_1,\ldots,\kappa_\rho) = {}_1F_1(-n;\alpha+1;x) = L_n(x;\alpha).$$
The latter expressions for Jacobi polynomials J_n and Laguerre polynomials L_n can be inserted into their orthogonality relations to obtain relations (15),(16). This finishes the proof. □

The hypergeometric nature of polynomials $\mathcal{P}_n$ and $\mathcal{L}_n$ provides differential equations for them.

Proposition 2. *Let $\rho \in \mathbb{N}$, and $\alpha,\beta,\delta_1,\ldots,\delta_\rho \in (-1,+\infty)$; $\kappa_1,\ldots,\kappa_\rho \in \mathbb{Z}_+$, be arbitrary parameters. Let $\theta = z\frac{d}{dz}$, and*

$$K := \theta(\theta+\alpha)\prod_{j=1}^{\rho}(\theta+\kappa_j+\delta_j), \quad L := \prod_{k=1}^{\rho}(\theta+\delta_k+1), \tag{17}$$

$$D_0 := K - z\theta(\theta+\alpha+\beta+1)L, \quad D_1 := zL, \quad D_2 := K - z\theta L. \tag{18}$$

Then $\forall n \in \mathbb{Z}_+$,

$$D_0\mathcal{P}_n(z) = -n(n+\alpha+\beta+1)D_1\mathcal{P}_n(z), \quad z \in \mathbb{D}; \tag{19}$$

$$D_2\mathcal{L}_n(z) = -nD_1\mathcal{L}_n(z), \quad z \in \mathbb{C}. \tag{20}$$

Proof. Use hypergeometric representations of the corresponding polynomials and the differential equation for ${}_pF_q$. □

We shall use a known generating function for the polynomials $\mathbf{L}_n(x)$ from [10, p. 267], formula (25). We only added the convergence fact.

Lemma 1. *Let $p,q \in \mathbb{Z}_+$: $p \le q+1$, be fixed. Let $\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q \in (0,+\infty)$, be arbitrary parameters. The following relation holds:*

$$e^t{}_pF_q\left(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;-xt\right) = \sum_{n=0}^{\infty}\mathbf{L}_n(x;\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q)\frac{t^n}{n!}, \tag{21}$$

where $t,x \in \mathbb{D}$. If $p \le q$ then relation (21) holds for all $t,x \in \mathbb{C}$.

Proof. Denote by $g(t) = g_x(t)$ the left-hand side of (21). Set

$$D := \begin{cases} \mathbb{D}, & \text{if } p = q+1, \\ \mathbb{C}, & \text{if } p \le q, \end{cases}.$$

Fix an arbitrary $x \in D$. Then $g(t) = g_x(t)$ is an analytic function of t in the domain D. Let us calculate Taylor's coefficients for its expansion at $t = 0$. By the Leibniz rule we may write:

$$g^{(n)}(0) = \sum_{k=0}^{n}\binom{n}{k}\left({}_pF_q\left(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;-xt\right)\right)_t^{(k)}\Big|_{t=0}\left(e^t\right)^{(n-k)}\Big|_{t=0} =$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left(\sum_{j=0}^{\infty} \frac{(\alpha_1)_j \dots (\alpha_p)_j}{(\beta_1)_j \dots (\beta_q)_j} \frac{(-x)^j}{j!} t^j \right)_t^{(k)} \Bigg|_{t=0} =$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{(\alpha_1)_k \dots (\alpha_p)_k}{(\beta_1)_k \dots (\beta_q)_k} (-x)^k = \sum_{k=0}^{n} (-n)_k \frac{(\alpha_1)_k \dots (\alpha_p)_k}{(\beta_1)_k \dots (\beta_q)_k} \frac{x^k}{k!} =$$

$$= \mathbf{L}_n(x; \alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q).$$

Thus, relation (21) coincides with Taylor's expansion of $g(t)$ at $t = 0$. $\square$

Let $\rho \in \mathbb{N}$, and $\alpha, \delta_1, \dots, \delta_\rho \in (-1, +\infty)$; $\kappa_1, \dots, \kappa_\rho \in \mathbb{Z}_+$, be arbitrary parameters. By Lemma 1, for all $t, x \in \mathbb{C}$ the following relation is valid:

$$e^t {}_\rho F_{\rho+1} \left[\begin{matrix} \delta_1 + 1, \dots, \delta_\rho + 1; \\ \alpha + 1, \kappa_1 + \delta_1 + 1, \dots, \kappa_\rho + \delta_\rho + 1; \end{matrix} \; -xt \right] =$$

$$= \sum_{n=0}^{\infty} \mathcal{L}_n(x; \alpha, \delta_1, \dots, \delta_\rho, \kappa_1, \dots, \kappa_\rho) \frac{t^n}{n!}. \tag{22}$$

Let us now turn to the question of the existence of some recurrence relations for polynomials $\mathbf{P}_n$ and $\mathbf{L}_n$. For big values of p and q the expressions for the coefficients of recurrence relations will be complicated and it is not clear that they will be nontrivial. Thus, the non-triviality of the recurrence relations can not be guaranteed.

We are not ready to treat effectively the case of general p and q. It looks reasonable to investigate concrete systems of polynomials $\mathbf{P}_n$ or $\mathbf{L}_n$, having some fixed values of p and q. Even in this case expressions for the coefficients can be huge and probably of few use. We shall study the case $p = 2$, $q = 3$, for the polynomials $\mathbf{L}_n$:

$$\mathbf{L}_n(x) = \mathbf{L}_n(x; \alpha_1, \alpha_2; \beta_1, \beta_2, \beta_3) = {}_3F_3(-n, \alpha_1, \alpha_2; \beta_1, \beta_2, \beta_3; x), \quad n \in \mathbb{Z}_+, \tag{23}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3 \in (0, +\infty)$. By Lemma 1 we may write:

$$e^t {}_2F_3 (\alpha_1, \alpha_2; \beta_1, \beta_2, \beta_3; -xt) = \sum_{n=0}^{\infty} \mathbf{L}_n(x) \frac{t^n}{n!}, \quad t, x \in \mathbb{C}. \tag{24}$$

Fix an arbitrary number $x \in \mathbb{C} \backslash \{0\}$. Introduce a new variable z:

$$z = -xt.$$

Relation (24) may be written in the following form:

$${}_2F_3 (\alpha_1, \alpha_2; \beta_1, \beta_2, \beta_3; z) = e^{\frac{z}{x}} \sum_{n=0}^{\infty} \mathbf{L}_n(x) \frac{(-1)^n}{x^n} \frac{z^n}{n!}, \quad z \in \mathbb{C}. \tag{25}$$

Denote the left-hand side of relation (25) by $w(z)$. It satisfies the differential equation for the hypergeometric function:

$$[\theta(\theta+\beta_1-1)(\theta+\beta_2-1)(\theta+\beta_3-1)-z(\theta+\alpha_1)(\theta+\alpha_2)]\,w(z)=0, \qquad (26)$$

where $\theta = z\frac{d}{dz}$. Set

$$b_1 := \beta_1-1, \quad b_2 := \beta_2-1, \quad b_3 := \beta_3-1, \qquad (27)$$

$$c := b_1+b_2+b_3+6, \quad \widehat{b} := 7+3(b_1+b_2+b_3)+b_1b_2+b_1b_3+b_2b_3, \qquad (28)$$

$$d := 1+b_1+b_2+b_3+b_1b_2+b_1b_3+b_2b_3+b_1b_2b_3, \quad \widehat{\alpha} = 1+\alpha_1+\alpha_2. \qquad (29)$$

Assume that $z \neq 0$. We can rewrite the differential operator [...] in (26) as a sum of powers of θ, and divide the whole equality by z to obtain:

$$\Big[\frac{d}{dz}(\theta^3+(b_1+b_2+b_3)\theta^2+(b_1b_2+b_1b_3+b_2b_3)\theta+b_1b_2b_3)-$$

$$-\theta^2-(\alpha_1+\alpha_2)\theta-\alpha_1\alpha_2\Big]\,w(z)=0, \qquad z\in\mathbb{C}\backslash\{0\}. \qquad (30)$$

In terms of usual derivatives this relation can be rewritten as

$$z^3w^{(4)}+cz^2w'''+(\widehat{b}-z)zw''+(d-\widehat{\alpha}z)w'-\alpha_1\alpha_2w=0, \qquad z\in\mathbb{C}\backslash\{0\}. \quad (31)$$

Denote the left-hand side of (31) by $l(z)$. Since $w(z)$ is an entire function, then $l(z)$ is entire as well. By continuity we conclude that relation (31) holds for $z=0$. Set

$$\varphi(z)=\varphi(z;x):=\sum_{n=0}^{\infty}\mathbf{L}_n(x)\frac{(-1)^n}{x^n}\frac{z^n}{n!}, \quad z\in\mathbb{C}. \qquad (32)$$

Then

$$w(z)=e^{\frac{z}{x}}\varphi(z), \quad z\in\mathbb{C}.$$

We can calculate the derivatives of w by the Leibniz rule and substitute the resulting expressions into relation (31). If we cancel the term $e^{\frac{z}{x}}$, we shall get the following relation:

$$z^3\varphi^{(4)}+\frac{4}{x}z^3\varphi'''+\frac{6}{x^2}z^3\varphi''+\frac{4}{x^3}z^3\varphi'+\frac{1}{x^4}z^3\varphi+$$

$$+cz^2\varphi'''+c\frac{3}{x}z^2\varphi''+c\frac{3}{x^2}z^2\varphi'+c\frac{1}{x^3}z^2\varphi+$$

$$+(\widehat{b}-z)z\varphi''+(\widehat{b}-z)\frac{2}{x}z\varphi'+(\widehat{b}-z)\frac{1}{x^2}z\varphi+$$

$$+(d-\widehat{\alpha}z)\varphi'+(d-\widehat{\alpha}z)\frac{1}{x}\varphi-\alpha_1\alpha_2\varphi=0, \qquad z\in\mathbb{C}. \qquad (33)$$

Denote the left-hand side of (33) by $\widehat{l}(z)$. Observe that

$$\varphi'(z) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} \frac{\mathbf{L}_{n+1}(x)}{x^{n+1}} z^n, \quad \varphi''(z) = \sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!} \frac{\mathbf{L}_{n+2}(x)}{x^{n+2}} z^n,$$

$$\varphi'''(z) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} \frac{\mathbf{L}_{n+3}(x)}{x^{n+3}} z^n, \quad \varphi^{(4)}(z) = \sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!} \frac{\mathbf{L}_{n+4}(x)}{x^{n+4}} z^n.$$

We can substitute the latter expressions into relation (33) to get a series expansion of $\widehat{l}(z)$, which is equal to zero. Thus, every Taylor coefficient $\widehat{l}_k$ is zero, and this provides a recurrence relation for polynomials $\mathbf{L}_n$.

Theorem 2. *Let $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3 \in (0, +\infty)$. Consider polynomials*

$$\mathbf{L}_n(x) = \mathbf{L}_n(x; \alpha_1, \alpha_2; \beta_1, \beta_2, \beta_3) = {}_3F_3(-n, \alpha_1, \alpha_2; \beta_1, \beta_2, \beta_3; x), \quad n \in \mathbb{Z}_+,$$

with $\mathbf{L}_{-1}(x) = \mathbf{L}_{-2}(x) = \mathbf{L}_{-3}(x) \equiv 0$. Let $b_1, b_2, b_3, c, \widehat{b}, d, \widehat{\alpha}$ be defined as in (27)-(29). The following five-term recurrence relation holds:

$$\begin{gathered}
\left(-k(k-1)(k-2) - k(k-1)c - k\widehat{b} - d\right) \mathbf{L}_{k+1}(x) + \\
+ \left(4k(k-1)(k-2) + 3k(k-1)c + 2k\widehat{b} + d\right) \mathbf{L}_k(x) + \\
+ \left(-6k(k-1)(k-2) - 3k(k-1)c - k\widehat{b}\right) \mathbf{L}_{k-1}(x) + \\
+ \left(4k(k-1)(k-2) + k(k-1)c\right) \mathbf{L}_{k-2}(x) - k(k-1)(k-2)\mathbf{L}_{k-3}(x) = \\
= x \left[(k(k-1) + k\widehat{\alpha} + \alpha_1\alpha_2)\mathbf{L}_k(x) - \right. \\
\left. - (2k(k-1) + k\widehat{\alpha})\mathbf{L}_{k-1}(x) + k(k-1)\mathbf{L}_{k-2}(x) \right], \qquad k \in \mathbb{Z}_+. \qquad (34)
\end{gathered}$$

Proof. Calculate the Taylor coefficients $\widehat{l}_k$ of $\widehat{l}(z)$, as it was explained before the statement of the theorem. Then multiply $\widehat{l}_k$ by $(-1)^k k! x^{k+1}$ to get relation (34). $\square$

In conditions of Theorem 2 we additionally assume that

$$\beta_1, \beta_2, \beta_3 \in [1, +\infty). \qquad (35)$$

Then parameters $b_1, b_2, b_3; c, \widehat{b}, d$ are positive. This fact ensures that the coefficient by $\mathbf{L}_{k+1}(x)$ in the recurrence relation (34) is non-zero for $k \geq 3$. Since the coefficient by $\mathbf{L}_{k-3}(x)$ is also non-zero for $k \geq 3$, the recurrence relation (34) is non-trivial in this case.

Notice that by (6) we may write

$$\begin{gathered}
\mathcal{L}_n(x; \alpha, \delta_1, \delta_2, \kappa_1, \kappa_2) = \\
= \mathbf{L}_n(x; \delta_1 + 1, \delta_2 + 1; \alpha + 1, \kappa_1 + \delta_1 + 1, \kappa_2 + \delta_2 + 1), \quad n \in \mathbb{Z}_+, \qquad (36)
\end{gathered}$$

where $\alpha, \delta_1, \delta_2 \in (-1, +\infty)$, and $\kappa_1, \kappa_2 \in \mathbb{Z}_+$, are arbitrary parameters. Therefore one can write the above recurrence relation for $\mathcal{L}_n(x; \alpha, \delta_1, \delta_2, \kappa_1, \kappa_2)$.

In general, we can conjecture that polynomials $\mathbf{L}_n(x)$ from (4), with $p = q-1$, satisfy a $(q+2)$-term recurrence relation. This conjecture agrees with the classical case of Laguerre polynomials, with R. Koekoek's result mentioned in the Introduction, and with Theorem 2.

Let us now discuss the case of polynomials $\mathbf{P}_n(x)$ and their recurrence relations. We shall use a known generating function for the polynomials $\mathbf{P}_n(x)$ from [5], formula (26). As in the case of Lemma 1 we only add the convergence fact.

Lemma 2. *Let $p, q \in \mathbb{Z}_+$: $p \leq q-1$, and c: $0 < c < \frac{1}{2}$, be fixed. Let $a; \alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q \in (0, +\infty)$, be arbitrary parameters. The following relation holds:*

$$(1-t)^{-a} {}_{p+2}F_q \left(\frac{a}{2}, \frac{a+1}{2}, \alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; -\frac{4xt}{(1-t)^2} \right) =$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n}{n!} \mathbf{P}_n(x; a, \alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q) t^n, \tag{37}$$

where

$$t, x \in \mathbb{C}: \ |t| < c, \ |x| < \frac{1}{4c} - \frac{1}{2}. \tag{38}$$

Proof. Notice that condition (38) provides that

$$\left| \frac{4xt}{(1-t)^2} \right| < 1. \tag{39}$$

In fact, we may write:

$$\left| \frac{4xt}{(1-t)^2} \right| = \frac{4|x||t|}{|1-t|^2} < \frac{4\left(\frac{1}{4c} - \frac{1}{2}\right)c}{(1-c)^2} = \frac{1-2c}{(1-c)^2} \leq \frac{1-2c+c^2}{(1-c)^2} = 1.$$

Therefore the left-hand side of (37) is well-defined for all t, x satisfying condition (38). Denote by R_1 the right-hand side of (37). At this point we do not know if the series in R_1 converges. Consider the following two iterated series which differ by the order of summation:

$$R_2 := \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (a)_n \frac{t^n}{n!} (-n)_k (n+a)_k u_k \frac{x^k}{k!}, \tag{40}$$

$$R_3 := \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} (a)_n \frac{t^n}{n!} (-n)_k (n+a)_k u_k \frac{x^k}{k!}, \tag{41}$$

where for brevity we denoted

$$u_j := \frac{(\alpha_1)_j \dots (\alpha_p)_j}{(\beta_1)_j \dots (\beta_q)_j}, \qquad j \in \mathbb{Z}_+, \tag{42}$$

and t, x are satisfying condition (38). We are going to prove that the series R_3 converges absolutely. Denote

$$\widehat{R}_3 := \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \left| (a)_n \frac{t^n}{n!} (-n)_k (n+a)_k u_k \frac{x^k}{k!} \right| =$$

$$= \sum_{k=0}^{\infty} u_k \frac{|x|^k}{k!} \sum_{n=k}^{\infty} (a)_n (n+a)_k |(-n)_k| \frac{|t|^n}{n!} =$$

$$= \sum_{k=0}^{\infty} u_k \frac{|x|^k}{k!} \sum_{n=k}^{\infty} (a)_n (n+a)_k \frac{|t|^n}{(n-k)!}, \tag{43}$$

where we have removed the null terms. Denote the inner sum in the last row of (43) by S_k. By the ratio test it converges for all $t \in \mathbb{D}_c$. Changing the summation index $j = n-k$ we get

$$S_k = \sum_{j=0}^{\infty} (a)_{j+k} (j+k+a)_k \frac{|t|^{j+k}}{j!} = (a)_{2k} |t|^k \sum_{j=0}^{\infty} (a+2k)_j \frac{|t|^j}{j!} =$$

$$= (a)_{2k} |t|^k (1-|t|)^{-a-2k}, \qquad t \in \mathbb{D}_c.$$

Then

$$\widehat{R}_3 = (1-|t|)^{-a} \sum_{k=0}^{\infty} (a)_{2k} \frac{u_k}{k!} \left(\frac{|xt|}{(1-|t|)^2} \right)^k =$$

$$= (1-|t|)^{-a} \sum_{k=0}^{\infty} \left(\frac{a}{2}\right)_k \left(\frac{a+1}{2}\right)_k \frac{u_k}{k!} \left(\frac{4|xt|}{(1-|t|)^2} \right)^k =$$

$$= (1-|t|)^{-a} {}_{p+2}F_q \left(\frac{a}{2}, \frac{a+1}{2}, \alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; \frac{4|x||t|}{(1-|t|)^2} \right), \tag{44}$$

where we have used the following relation (see Lemma 5 in [26, p. 22]):

$$(a)_{2k} = 4^k \left(\frac{a}{2}\right)_k \left(\frac{a+1}{2}\right)_k.$$

By virtue of (39) with parameters $|x|, |t|$ instead of x, t, we obtain that $\frac{4|x||t|}{(1-|t|)^2} < 1$, and this proves the last line of (44). Thus, the series R_3 converges absolutely. Let

$$R_3 = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_{k,n}, \quad a_{k,n} = u_{k,n} + i v_{k,n},\ u_{k,n}, v_{k,n} \in \mathbb{R}.$$

By Theorem 2 in [12, p. 34] we conclude that

$$\sum_{j=0}^{\infty} |a_j| < \infty,$$

where the series is composed of elements $a_{k,j}$, placed in an arbitrary order. Let $a_j = u_j + i v_j$, $u_j, v_j \in \mathbb{R}$. By the comparison test it follows that

$$\sum_{j=0}^{\infty} |u_j| < \infty,\ \sum_{j=0}^{\infty} |v_j| < \infty.$$

By Theorem 1 in [12, p. 32] we obtain that

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} u_{k,n} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} u_{k,n} = \sum_{j=0}^{\infty} u_j; \tag{45}$$

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} iv_{k,n} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} iv_{k,n} = \sum_{j=0}^{\infty} iv_j. \tag{46}$$

Summing relations (45) and (46) we get

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} a_{k,n} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{k,n}. \tag{47}$$

Therefore $R_3 = R_2$. It remains to check that R_3 coincides with the left-hand side of (37). We may write:

$$R_3 = \sum_{k=0}^{\infty} u_k \frac{x^k}{k!} \sum_{n=k}^{\infty} (a)_n (-n)_k (n+a)_k \frac{t^n}{n!}.$$

Denote

$$T_k := \sum_{n=k}^{\infty} (a)_n (-n)_k (n+a)_k \frac{t^n}{n!}.$$

The series T_k converges absolutely by the ratio test. Proceeding in a similar manner as for S_k, we change the summation index $j = n - k$:

$$T_k = \sum_{j=0}^{\infty} (a)_{j+k} (j+k+a)_k (-1)^k \frac{t^{j+k}}{j!} = (a)_{2k} (-t)^k \sum_{j=0}^{\infty} (a+2k)_j \frac{t^j}{j!} =$$

$$= (a)_{2k} (-t)^k (1-t)^{-a-2k}, \qquad t \in \mathbb{D}_c.$$

Therefore

$$R_3 = (1-t)^{-a} \sum_{k=0}^{\infty} (a)_{2k} \frac{u_k}{k!} \left(\frac{-xt}{(1-t)^2} \right)^k =$$

$$= (1-t)^{-a} \sum_{k=0}^{\infty} \left(\frac{a}{2}\right)_k \left(\frac{a+1}{2}\right)_k \frac{u_k}{k!} \left(-\frac{4xt}{(1-t)^2} \right)^k =$$

$$= (1-t)^{-a} {}_{p+2}F_q \left(\frac{a}{2}, \frac{a+1}{2}, \alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; \frac{-4xt}{(1-t)^2} \right), \tag{48}$$

where we have used relation (39). Since $R_3 = R_2 = R_1$, the proof is complete. □

As an immediate consequence of Lemmas 1 and 2 we have the following result.

Corollary 1. *Let $\rho \in \mathbb{N}$, and $\delta_1, \dots, \delta_\rho \in (-1, +\infty)$, $\kappa_1, \dots, \kappa_\rho \in \mathbb{Z}_+$, be arbitrary parameters. If $\alpha > -1$ then*

$$\mathcal{L}_n(x; \alpha, \delta_1, \dots, \delta_\rho, \kappa_1, \dots, \kappa_\rho) =$$

$$= \frac{n!}{2\pi i} \oint_{|\zeta|=1} \zeta^{-n-1} e^{\zeta} {}_\rho F_{\rho+1} \left[\begin{matrix} \delta_1 + 1, \dots, \delta_\rho + 1; \\ \alpha + 1, \kappa_1 + \delta_1 + 1, \dots, \kappa_\rho + \delta_\rho + 1; \end{matrix} \; -x\zeta \right] d\zeta,$$

$$x \in \mathbb{C}, \quad n \in \mathbb{Z}_+. \tag{49}$$

If $\alpha, \beta \in (-1, +\infty): \ \alpha + \beta > -1$, then

$$\mathcal{P}_n(x; \alpha, \beta, \delta_1, \dots, \delta_\rho, \kappa_1, \dots, \kappa_\rho) =$$

$$= \frac{1}{2\pi i} \frac{n!}{(\alpha+\beta+1)_n} \oint_{|\zeta|=\frac{1}{4}} \zeta^{-n-1} (1-\zeta)^{-\alpha-\beta-1} *$$

$$*_{\rho+2}F_{\rho+1}\left[\begin{array}{c} \frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta+2}{2}, \delta_1+1,\dots,\delta_\rho+1; \\ \alpha+1, \kappa_1+\delta_1+1,\dots,\kappa_\rho+\delta_\rho+1; \end{array} \quad -\frac{4x\zeta}{(1-\zeta)^2}\right] d\zeta,$$

$$x\in\mathbb{C}:\ |x|<\frac{1}{4},\quad n\in\mathbb{Z}_+. \tag{50}$$

Proof. The proof follows by calculating the corresponding Taylor coefficients in the above Lemmas (with $c=\frac{1}{3}$). $\Box$

In formula (37) on the left we see ${}_{p+2}F_q$ with an argument $-\frac{4xt}{(1-t)^2}$. If we proceed as for the case of $\mathbf{L}_n$ we shall get huge expressions because of this composition of functions. We also have $(1-t)^{-a}$ instead of e^{-x} which also has effect on the complexification.

Observe that ${}_3F_2$ polynomials of type $\mathbf{P}_n$ were already studied in [30]. A recurrence relation for them was obtained by Fasenmyer's method. This recurrence relation was very large and, probably, of restricted use. It should be noticed that Fasenmyer's method seems to be more preferable in the case of polynomials $\mathbf{P}_n$.

Let us turn to the question about the location of zeros of polynomials $\mathbf{P}_n$ and $\mathbf{L}_n$. As usual, it is useful to use the Eneström–Kakeya Theorem ([22, p. 136]).

Proposition 3. *Let $p,q\in\mathbb{Z}_+$: $p\geq q+1$, and*

$$a\in(-1,+\infty);\ \alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q\in(0,+\infty),$$

are some parameters. If

$$\alpha_j\geq\beta_j,\ j\in\mathbb{Z}_{1,q};\quad \alpha_k\geq 1,\ k\in\mathbb{Z}_{q+1,p}, \tag{51}$$

then all zeros of polynomials $\mathbf{P}_n(x)=\mathbf{P}_n(x;a,\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q)$ and all zeros of polynomials $\mathbf{L}_n(x)=\mathbf{L}_n(x;\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q)$ lie in the unit disc $\mathbb{D}$.

Proof. Fix an arbitrary $n\in\mathbb{N}$. Since

$$\mathbf{P}_n(x;a,\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q) = {}_{p+2}F_q(-n,n+a,\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;x)=$$

$$=\sum_{k=0}^n(-n)_k(n+a)_k\frac{(\alpha_1)_k\dots(\alpha_p)_k}{(\beta_1)_k\dots(\beta_q)_k}\frac{x^k}{k!}=$$

$$=\sum_{k=0}^n\frac{n!}{(n-k)!}(n+a)_k\frac{(\alpha_1)_k\dots(\alpha_p)_k}{(\beta_1)_k\dots(\beta_q)_k}\frac{(-x)^k}{k!}=\sum_{k=0}^n d_kz^k=:p(z),$$

where

$$d_k:=\frac{n!}{(n-k)!}(n+a)_k\frac{(\alpha_1)_k\dots(\alpha_p)_k}{(\beta_1)_k\dots(\beta_q)_k}\frac{1}{k!}>0,\quad z:=-x.$$

Thus, the polynomial $p(z)$ has degree n and positive coefficients. The reversed polynomial:

$$p^*(z):=z^np(1/z)$$

has degree n and positive coefficients as well. Observe that

$$d_k/d_{k+1}=\frac{1}{(n-k)}\frac{1}{(n+a+k)}\frac{(\beta_1+k)\dots(\beta_q+k)}{(\alpha_1+k)\dots(\alpha_p+k)}(k+1)\leq 1,\quad k\in\mathbb{Z}_{0,n-1},$$

where we used condition (51). We can apply the Eneström–Kakeya Theorem ([22, p. 136]) for the polynomial $p^*(z)$ to obtain that all its zeros lie in the domain $D_e:=\{z\in\mathbb{C}:\ |z|>1\}$. Therefore the zeros of $\mathbf{P}_n$ lie in $\mathbb{D}$.

We may proceed for polynomials $\mathbf{L}_n$ in a similar way:

$$\mathbf{L}_n(x;\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q) = {}_{p+1}F_q(-n,\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;x) =$$

$$= \sum_{k=0}^{n}(-n)_k \frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\frac{x^k}{k!} =$$

$$= \sum_{k=0}^{n} \frac{n!}{(n-k)!}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\frac{(-x)^k}{k!} = \sum_{k=0}^{n}\widehat{d}_k z^k =: \widehat{p}(z),$$

where

$$\widehat{d}_k := \frac{n!}{(n-k)!}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\frac{1}{k!} > 0, \quad z := -x.$$

Since

$$\widehat{d}_k/\widehat{d}_{k+1} \le 1, \qquad k \in \mathbb{Z}_{0,n-1},$$

by the Eneström–Kakeya Theorem we conclude that the reversed polynomial $\widehat{p}^*$ has its zeros in D_e. Thus, the zeros of $\mathbf{L}_n$ lie in $\mathbb{D}$ as well. □

Let us make an illustration on the last result. Consider the following three systems of polynomials:

$$f_n(x) = {}_3F_1(-n,\pi,5;3;x), \quad g_n(x) = {}_4F_1(-n,n+1,\pi,5;3;x),$$

and

$$h_n(x) = {}_3F_3(-n,\pi+1,2\pi+1;1,\pi+8,2\pi+201;x), \qquad n \in \mathbb{Z}_+.$$

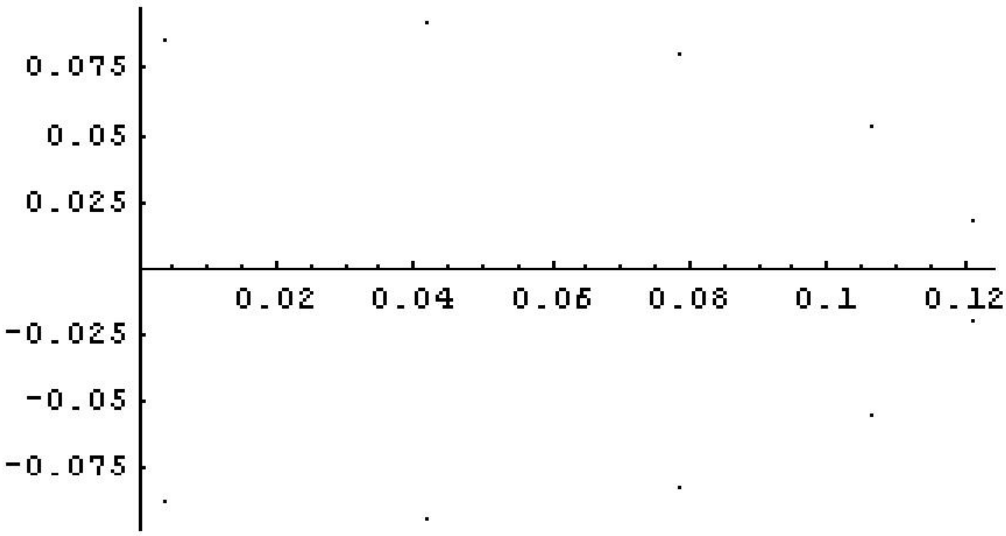

Figure 1. Zeros of $f_{10}(x)$.

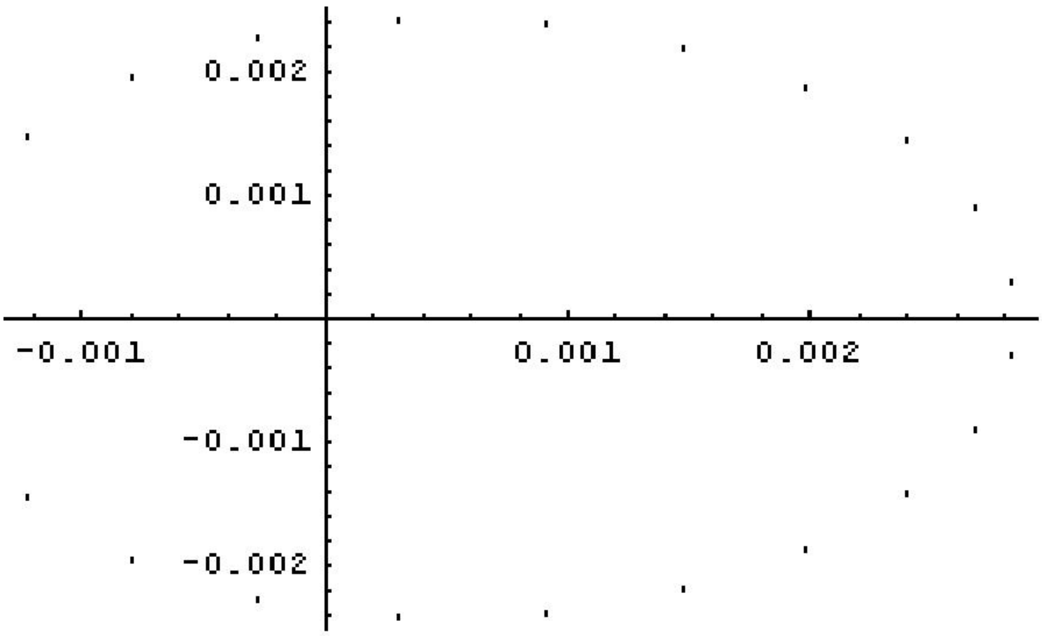

Figure 2. Zeros of $g_{20}(x)$.

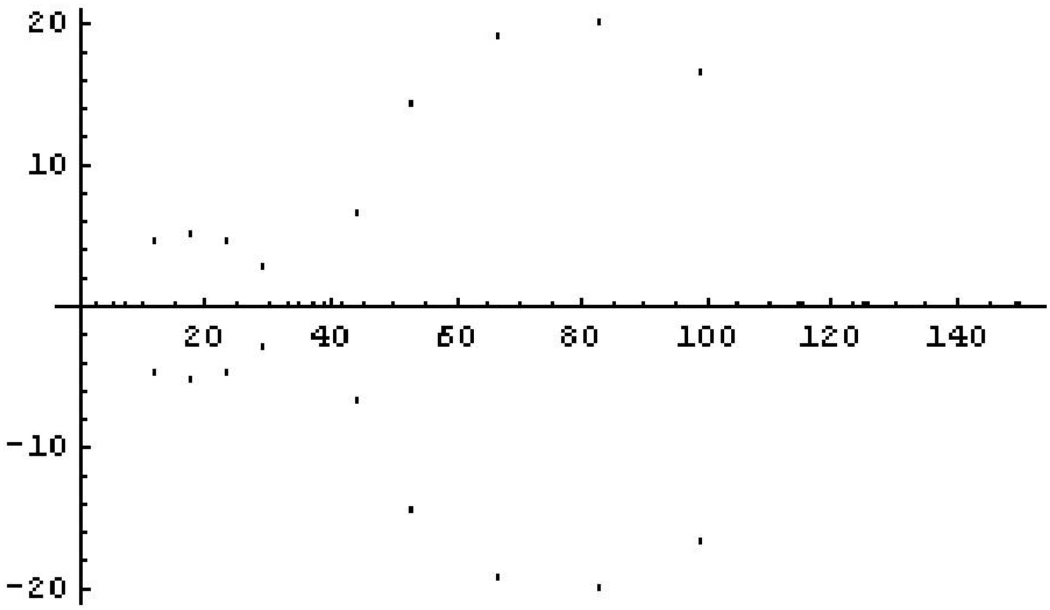

Figure 3. Zeros of $h_{30}(x)$.

Polynomials f_n and g_n fit into the conditions of Proposition 3, while polynomials h_n do not satisfy these conditions. Numerical calculations were performed by using *Mathematica*, while by final formatting we used *Paint*.

In Figures 1 and 2 we see that all zeros of $f_{10}(x)$ and $g_{20}(x)$ are close to the origin and they lie symmetrically (which is not surprising since polynomials have real coefficients). It seems that all zeros are located on certain algebraic curves.

Figure 3 shows that zeros of $g_{20}(x)$ can lie outside the unit disc. They are located on an interesting curve as well. Of course, the nature of the above mentioned curves is not yet clear. However this encourages some further investigations on the location of zeros of hypergeometric polynomials $\mathbf{P}_n$ and $\mathbf{L}_n$.

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Про деякі гіпергеометричні соболевські ортогональні многочлени з кількома неперервними параметрами

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В цій статті ми вивчаємо наступні гіпергеометричні многочлени:

$$\mathcal{P}_n(x) = \mathcal{P}_n(x;\alpha,\beta,\delta_1,\dots,\delta_\rho,\kappa_1,\dots,\kappa_\rho) =$$

$$= {}_{\rho+2}F_{\rho+1}(-n, n+\alpha+\beta+1, \delta_1+1,\dots,\delta_\rho+1; \alpha+1, \kappa_1+\delta_1+1,\dots,\kappa_\rho+\delta_\rho+1; x),$$

та

$$\mathcal{L}_n(x) = \mathcal{L}_n(x;\alpha,\delta_1,\dots,\delta_\rho,\kappa_1,\dots,\kappa_\rho) =$$

$$= {}_{\rho+1}F_{\rho+1}(-n, \delta_1+1,\dots,\delta_\rho+1; \alpha+1, \kappa_1+\delta_1+1,\dots,\kappa_\rho+\delta_\rho+1; x), \qquad n\in\mathbb{Z}_+,$$

де $\alpha,\beta,\delta_1,\dots,\delta_\rho\in(-1,+\infty)$, та $\kappa_1,\dots,\kappa_\rho\in\mathbb{Z}_+$, є деякими параметрами. Натуральне число ρ неперервних параметрів $\delta_1,\dots,\delta_\rho$ може бути обраним довільно великим. Ясно, що спеціальний випадок $\kappa_1=\dots=\kappa_\rho=0$ призводить до многочленів Якобі

та Лагерра. Звичайно, подібні та більш загальні поліноми виникали в літературі раніше. Наша мета тут полягає в тому, щоб показати, що поліноми $\mathcal{P}_n(x)$ та $\mathcal{L}_n(x)$ є соболевськими ортогональними многочленами на дійсній осі з деякими явними матричними мірами.

Важливість ортогональності була нашою головною причиною зосередити нашу увагу на поліномах $\mathcal{P}_n(x)$ та $\mathcal{L}_n(x)$. Тут ми використовуємо деякі наші інструменти, отримані раніше. Зокрема, нещодавно було показано, що соболевські ортогональні многочлени пов'язані через диференціальне рівняння з ортогональними системами $\mathcal{A}$ функцій, що діють у прямих сумах звичайних L^2_μ просторів квадратично сумованих (класів еквівалентності) функцій відносно позитивної міри μ. Випадок одного L^2_μ має додаткову цікавість, оскільки він дозволяє використовувати OPRL для отримання явних систем соболевських ортогональних многочленів. Основна проблема тут полягає в *виборі підходящого лінійного диференціального оператора з метою отримання явних представлень соболевських ортогональних многочленів*. Після цього доказ співвідношень ортогональності є перевіркою такого вибору і проводиться в іншому напрямку: ми починаємо з вже відомих многочленів та йдемо до їх властивостей.

Ми також коротко вивчаємо такі властивості вищенаведених поліномів: інтегральні представлення, диференціальні рівняння та розташування нулів. Побудовано систему таких поліномів з біспектральністю певного виду.

Ключові слова: **ортогональні поліноми; соболевська ортогональність; рекурентні співвідношення.**

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