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On integration with respect to filter

This article is devoted to the study of one generalization of the Riemann integral. Namely, in the paper, it was observed that the classical definition of the Riemann integral over a finite segment as a limit of integral sums, when the diameter of the division of the segment tends to zero, can be replaced by a limit of integral sums over a filter of sets, which can be described in a certain "good way". This idea was continued, and in the work we propose a new concept - the integral of a function over a filter on the set of all tagged partitions of a segment. Using of filters is a very good method in questions related to convergence or some of its analogues in general topological vector spaces. Namely, if the space is non-metrizable, then the concept of convergence is introduced precisely with the help of filters. Also, using filters, you can formulate the concept of completeness and its analogues. The completeness of spaces is one of the central concepts of the theory of topological vector spaces, since Banach spaces are complete. That is, using a generalization of the completeness of spaces constructed using filters, we can explore various generalizations of Banach spaces. We study standard issues related to integration. For example, does the integrability of the filter function imply its boundedness? The answer to this question is affirmative. Namely: the concept of filter boundedness of a function is introduced, and it is shown that if a function is integrable over filter, then its integral sums are bounded over the filter, and this function itself is bounded in the classical sense. Next, we showed that the filter integral satisfies the linearity property, namely, the integral over filter of the sum of two functions is the sum of the filter integrals of these functions. We introduce the concept of an exactly tagged filter, and with the help of such filters we study the filter integrability of unbounded functions on a segment. We give an example of a specific unbounded function and a specific filter under which this function is integrable. Next, we prove a theorem that describes unbounded filter-integrable functions on a segment. The last section of the article is devoted to the integration of functions relative to the filter on a subsegment of this segment.

Keywords: **integral; filter; idea; filter base.**

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1. Introduction

Let us remind main concepts which we use in this paper. Throughout this article Ω stand for a non-empty set. Non-empty family of subsets $\mathfrak{F} \subset 2^{\Omega}$ is called *filter on* Ω, if $\mathfrak{F}$ satisfies the following axioms:

1. $\emptyset \notin \mathfrak{F}$;

2. if $A,\ B \in \mathfrak{F}$ then $A \cap B \in \mathfrak{F}$;

3. if $A \in \mathfrak{F}$ and $D \supset A$ then $D \in \mathfrak{F}$.

Also very useful for us is a concept of filter base. Non-empty family of subsets $\mathfrak{B} \subset 2^{\Omega}$ is called *filter base on* Ω, if $\emptyset \notin \mathfrak{B}$ and for every $A,\ B \in \mathfrak{B}$ there exists $C \in \mathfrak{B}$ such that $C \subset A \cap B$. We say that filter base *generates filter* $\mathfrak{F}$ if and only if for each $A \in \mathfrak{F}$ there is $B \in \mathfrak{B}$ such that $B \subset A$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. For $t \in \mathbb{R}$ denote $\mathcal{O}(t)$ the family of all neighbourhoods of t. Let $\mathfrak{F}$ be a filter on $\mathbb{R}$, $y \in \mathbb{R}$. Function f is said to be *convergent to* y *over filter* $\mathfrak{F}$ (denote $y = \lim\limits_{\mathfrak{F}} f$), if for each $U \in \mathcal{O}(y)$ there exists $A \in \mathfrak{F}$ such that for each $t \in A$ the following holds true: $f(t) \in U$. We refers, for example, to [1] for more information about filter and related concepts.

The concept of filter is a very powerful tool for studying different properties of general topological vector spaces. For example, in [3] author studies convergence over ideal, generated by the modular function. Ideal is a concept dual to filter. In [2] we study completeness and its generalization using filters.

In this article we refer our attention to classical Riemann integral. Let us remind how we can construct this object. Let $[a, b] \subset \mathbb{R}$, let $f : [a, b] \to \mathbb{R}$ be a continuous function. Denote $\Pi = \{a \le \xi_1 \le \xi_2 \le ... \le \xi_n = b\}$ the partition of $[a, b]$, in other words, $\bigcup\limits_{k=1}^{n} [\xi_{k-1}, \xi_k] = [a, b]$. Consider also the set $T = \{t_1, t_2, ..., t_n\}$ such that for each $k = 1, 2, ..., n$ $t_k \in [\xi_{k-1}, \xi_k]$. Let us call the pair (Π, T) by the *tagged partition on the segment*. Denote $d(\Pi)$ *the diameter of the* Π – maximum length of $[\xi_{k-1}, \xi_k]$, where $k = 1, 2, ..., n$. Let us recall that function f is said to be *Riemann integrable* if there exist the limit $I = \lim\limits_{d(\Pi)\to 0} \sum\limits_{k=1}^{n} f(t_k) \cdot |\xi_k - \xi_{k-1}|$, and we call this limit the Riemann integral of the function f, and write $I = \int\limits_a^b f(t)dt$. We know many different properties of this integral, for example linearity, integration on subsegment of $[a, b]$ etc.

If we look at the definition of Riemann integral more attentively, we realize that, in fact we can use one special filter and obtain desirable result. In next section we are going to develop this idea.

2. Integration with respect to filter

Just for simplicity we are going to consider functions, defined on $[0, 1]$. Let $f : [0, 1] \to \mathbb{R}$ be a function. As above, denote $\Pi = \{a \le \xi_1 \le \xi_2 \le ... \le$

$\xi_n = b\}$ the partition of $[0,1]$, in other words, $\bigcup\limits_{k=1}^{n}[\xi_{k-1},\xi_k] = [0,1]$. Consider also the set $T = \{t_1, t_2, ..., t_n\}$ such that for each $k = 1, 2, ..., n$ $t_k \in [\xi_{k-1}, \xi_k]$. For $k = 1, 2, ..., n$ denote $\Delta_k := |\xi_k - \xi_{k-1}|$. Denote also $\mathrm{TP}[0,1]$ the set of all tagged partition of $[0,1]$. For a tagged partition $(\Pi, T) \in \mathrm{TP}[0,1]$ denote

$$S(f, \Pi, T) = \sum_{k=1}^{n} f(t_k)\Delta_k.$$

Now we are going to introduce the central definition of this paper. It seems that the this definition is new. At least, we didn't find it in the literature.

Definition 1. Let $f : [0,1] \to \mathbb{R}$ be a function, $\mathfrak{F}$ be a filter on $\mathrm{TP}[0,1]$. We say that f is *integrable over filter* $\mathfrak{F}$ ($\mathfrak{F}$-integrable for short), if there exists $I \in \mathbb{R}$ such that $I = \lim\limits_{\mathfrak{F}} S(f, \Pi, T)$. The number I is called *the* $\mathfrak{F}$*-integral of the* f (denote $I = \int\limits_0^1 f d\mathfrak{F}$).

Remark 1. The fact that f is $\mathfrak{F}$-integrable we will write as follows:

$$f \in \mathrm{Int}(\mathfrak{F}).$$

Remark 2. Using Definition 1 we can construct the Riemann integral as follows. Let $\delta > 0$ be a real positive number. Denote

$$P_{<\delta} = \{(\Pi, T) \in \mathrm{TP}[0,1] : \ d(\Pi) < \delta\},$$

where $d(\Pi)$ stands for diameter of Π. Consider now

$$\mathfrak{B}_{<\delta} = \{P_{<\delta} : \delta > 0\}.$$

It is easy to check that $\mathfrak{B}_{<\delta}$ is a filter base. Denote $\mathfrak{F}_{<\delta}$ filter generate by $\mathfrak{B}_{<\delta}$. Let $f : [0,1] \to \mathbb{R}$ be a function. Then f is integrable by Riemann if there exists the limit $\lim\limits_{\mathfrak{F}_{<\delta}} S(f, \Pi, T)$.

Bellow we study different properties of filter integration. Let us introduce one more technical concept.

Definition 2. Let X be a non-empty set, $f : X \to \mathbb{R}$ be a function, and $\mathfrak{F}$ be a filter on X. We say that f is *bounded with respect to* $\mathfrak{F}$ ($\mathfrak{F}$-bounded for short), if there is $C > 0$ such that there exists $A \in \mathfrak{F}$ such that for every $t \in A$ $|f(t)| < C$.

The following lemma is very simple, but for readers convenient we present its proof.

Lemma 1. *Let X be a non-empty set, $f : X \to \mathbb{R}$ be a function, and $\mathfrak{F}$ be a filter on X. Suppose that there exists $I \in \mathbb{R}$, $I = \lim\limits_{\mathfrak{F}} f$. Then f is $\mathfrak{F}$-bounded.*

Proof. We know that $I = \lim\limits_{\mathfrak{F}} f$. It means that for every $\varepsilon > 0$ there exists $A \in \mathfrak{F}$ such that for all $t \in A$ $|f(t) - I| < \varepsilon$. Consider

$$|f(t)| - |I| \le |f(t) - I| < \varepsilon.$$

In other words, $|f(t)| \leq |I| + \varepsilon$. Then just put $C := |I| + \varepsilon$.

The next theorem generalizes well-know fact about Riemann integral: if function in integrable by Riemann then it's bounded.

Theorem 1. *Let $\mathfrak{F}$ be a filter on $TP[0,1]$, $f : [0,1] \to \mathbb{R}$ be a function, and $f \in Int(\mathfrak{F})$. Then $S(f, \Pi, T)$ is $\mathfrak{F}$-bounded.*

Proof. Just use Lemma 1.

Let us formulate well-known fact about Riemann integral, using filters.

Theorem 2. *Let $f : [0,1] \to \mathbb{R}$, there exists $\lim\limits_{\mathfrak{F}_{<\delta}} S(f, \Pi, T)$. Then f is bounded, in other words, there is $C > 0$ such that for all $t \in [0,1]$ $|f(t)| \leq C$.*

The next theorem is natural generalization of the Theorem 2.

Theorem 3. *Let $f : [0,1] \to \mathbb{R}$, let $\mathfrak{F}$ be a filter on $TP[0,1]$ such that for every $A \in \mathfrak{F}$ there exists $B \in \mathfrak{F}_{<\delta}$ such that $B \subset A$ and let there exists $I \in \mathbb{R}$ such that $I = \lim\limits_{\mathfrak{F}} S(f, \Pi, T)$. Then $C > 0$ such that for each $t \in [0,1]$ we have $|f(t)| < C$.*

Proof. There exists $I \in \mathbb{R}$ such that $I = \lim\limits_{\mathfrak{F}} S(f, \Pi, T)$ if and only if for all $\varepsilon > 0$ there exists $A \in \mathfrak{F}$ such that for all $(\Pi, T) \in A$ $|S(f, \Pi, T) - I| < \varepsilon$. We know that for $A \in \mathfrak{F}$ there is $B \in \mathfrak{F}_{<\delta}$ such that $B \subset A$, then, particularly, for all $\varepsilon > 0$ there exists $A \in \mathfrak{F}$ there is $B \in \mathfrak{F}_{<\delta}$ such that $B \subset A$ such that for all $(\Pi, T) \in B$ $|S(f, \Pi, T) - I| < \varepsilon \Rightarrow$ for all $\varepsilon > 0$ there exists $B \in \mathfrak{F}_{<\delta}$ such that for all $(\Pi, T) \in B$ $|S(f, \Pi, T) - I| < \varepsilon$. So using Theorem 2, there exists $C > 0$ such that for each $t \in [0,1]$ we have $|f(t)| < C$, in other words, f is bounded.

Now we are going to demonstrate that filter integration has additive property. To demonstrate this we proof next easy two lemmas. The following Lemmas 2 and 3 are well-known, but for readers comprehension we present their proofs.

Lemma 2. *Let X be a non-empty set, $f,\ g : X \to \mathbb{R}$ be a functions, and $\mathfrak{F}$ be a filter on X. Let $x = \lim\limits_{\mathfrak{F}} f$, $y = \lim\limits_{\mathfrak{F}} g$. Then $\lim\limits_{\mathfrak{F}}(f+g) = x + y$.*

Proof. We know that $x = \lim\limits_{\mathfrak{F}} f$, so for each $U \in \mathcal{O}(x)$ there is $A \in \mathfrak{F}$ such that $f(A) \subset U$. Analogically, $y = \lim\limits_{\mathfrak{F}} f$, it means that for each $V \in \mathcal{O}(x)$ there is $B \in \mathfrak{F}$ such that $f(B) \subset V$. We have to demonstrate that for each $W \in \mathcal{O}(x+y)$ there exists $C \in \mathfrak{F}$ such that $(f+g)(C) \subset W$. Let fix $W \in \mathcal{O}(x+y)$. Then there exist $W_1 \in \mathcal{O}(x)$ and $W_2 \in \mathcal{O}(y)$ such that $W \supset W_1 + W_2$. Then there are $C_1,\ C_2 \in \mathfrak{F}$ such that $f(C_1) \subset W_1$ and $f(C_2) \subset W_2$. Denote $C := C_1 \cap C_2$. Clearly that $C \in \mathfrak{F}$. So

$$(f+g)(C) = f(C) + g(C) \subset W_1 + W_2 \subset W.$$

Lemma 3. *Let X be a non-empty set, $f : X \to \mathbb{R}$ be a function, $\mathfrak{F}$ be a filter on X, and $\alpha \in \mathbb{R}$. Let $x = \lim\limits_{\mathfrak{F}} f$. Then $\lim\limits_{\mathfrak{F}} \alpha f = \alpha x$.*

Proof. $x = \lim\limits_{\mathfrak{F}} f$, it means that for each $U \in \mathcal{O}(x)$ there is $A \in \mathfrak{F}$ such that $f(A) \subset U$. We have to demonstrate that for all $V \in \mathcal{O}(\alpha x)$ there is $B \in \mathfrak{F}$ such that $(\alpha f)(B) \subset V$. Suppose that $\alpha \neq 0$. The case $\alpha = 0$ is obvious. Remark that if $W \in \mathcal{O}(x)$ then $\alpha W \in \mathcal{O}(\alpha x)$. So just put $B := A$. Then $(\alpha f)(B) = \alpha f(B) \subset \alpha U \in \mathcal{O}(\alpha x)$.

Theorem 4. *Let $\mathfrak{F}$ be a filter on $TP[0,1]$, $f, g : [0,1] \to \mathbb{R}$ be a functions, α, $\beta \in \mathbb{R}$, $f \in Int(\mathfrak{F})$ and $g \in Int(\mathfrak{F})$. Then $(\alpha f + \beta g) \in Int(\mathfrak{F})$*

Proof. Just use Lemmas 2 and 3.

3. Integration with respect to different filters

In the previous section we've studied arithmetic properties of integral over filter and problems deals with boundedness. This section is devoted to integration over different filters and its relations.

Remark 3. Let us note that despite the fact that this section is devoted to the integration with respect to different filters, here we describe some properties of filters deals with integration. Explicit examples of filters different from one, described in Remark 2, appear in the following sections.

For $(\Pi, T) \in \mathrm{TP}[0,1]$ and $t \in T$ we denote $\Delta(t)$ length of the element of partition of Π which covers t.

Let $(\Pi_1, T_1), (\Pi_2, T_2)$ be partitions of $[0,1]$. Consider

$$\rho((\Pi_1, T_1), (\Pi_2, T_2)) = \sum_{t \in T_1 \cap T_2} |\Delta_1(t) - \Delta_2(t)| + \sum_{T_1 \setminus T_2} \Delta_1(t) + \sum_{T_2 \setminus T_1} \Delta_2(t).$$

For easy using of concept defined above consider $\mathbb{F} : [0,1] \to l_1[0,1]$, such that $\mathbb{F}(t) = e_t$, where

$$e_t(\tau) = \begin{cases} 1, \text{if } \tau = t; \\ 0, \text{otherwise.} \end{cases}$$

It is clearly then that

$$\rho((\Pi_1, T_1), (\Pi_2, T_2)) = ||S(\mathbb{F}, \Pi_1, T_1) - S(\mathbb{F}, \Pi_2, T_2)||.$$

Now we are going to demonstrate that the mapping ρ, defined above, is a metric, or distance between two tagged partitions.

Proposition 1. *Consider $\rho : TP[0,1] \times TP[0,1] \to \mathbb{R}$, $\rho((\Pi_1, T_1), (\Pi_2, T_2)) = ||S(\mathbb{F}, \Pi_1, T_1) - S(\mathbb{F}, \Pi_2, T_2)||$. Then ρ satisfies all metric axioms.*

Proof.

1. let $(\Pi_1, T_1) = (\Pi_2, T_2)$.

   It is clear that in this case $\rho((\Pi_1, T_1), (\Pi_2, T_2)) = 0$;

2. let $\rho((\Pi_1, T_1), (\Pi_2, T_2)) = 0$.

   Then $\rho((\Pi_1, T_1), (\Pi_2, T_2)) = \sum\limits_{t \in T_1 \cap T_2} |\Delta_1(t) - \Delta_2(t)| + \sum\limits_{T_1 \setminus T_2} \Delta_1(t) + \sum\limits_{T_2 \setminus T_1} \Delta_2(t) = 0$. We have a sum of non-negative numbers equals to 0. This means that

- $\forall t \in T_1 \cap T_2 \ |\Delta_1(t) - \Delta_2(t)| = 0 \Rightarrow \forall t \in T_1 \cap T_2 \ \Delta_1(t) = \Delta_2(t)$;
- $\forall t \in T_1 \setminus T_2 \ \Delta_1(t) = 0$;
- $\forall t \in T_2 \setminus T_1 \ \Delta_2(t) = 0$;

$\Rightarrow (\Pi_1, T_1) = (\Pi_2, T_2)$.

3. consider $(\Pi_1, T_1), (\Pi_2, T_2), (\Pi_3, T_3)$. Then

$$\begin{aligned}\rho((\Pi_1, T_1), (\Pi_2, T_2)) = \\ ||S(\mathbb{F}, \Pi_1, T_1) - S(\mathbb{F}, \Pi_2, T_2) + S(\mathbb{F}, \Pi_3, T_3) - S(\mathbb{F}, \Pi_3, T_3)|| \leq \\ ||S(\mathbb{F}, \Pi_1, T_1) - S(\mathbb{F}, \Pi_3, T_3)|| + ||S(\mathbb{F}, \Pi_3, T_3) - S(\mathbb{F}, \Pi_2, T_2)|| = \\ \rho((\Pi_1, T_1), (\Pi_3, T_3)) + \rho((\Pi_3, T_3), (\Pi_2, T_2))\end{aligned}$$

Now we introduce very important concept.

Definition 3. Let $\mathfrak{F}_1, \mathfrak{F}_2$ be filters on $TP[0,1]$. We say that $\mathfrak{F}_2$ *ρ-dominates* filter $\mathfrak{F}_1$ ($\mathfrak{F}_2 \succ_\rho \mathfrak{F}_1$), if for every $\varepsilon < 0$ and for each $A_1 \in \mathfrak{F}_1$ there exists $A_2 \in \mathfrak{F}_2$ such that for all $(\Pi_2, T_2) \in A_2$ there is $(\Pi_1, T_1) \in A_1$ such that $\rho((\Pi_1, T_1), (\Pi_2, T_2)) < \varepsilon$.

Proposition 2. *Let $\mathfrak{F}_2 \supset \mathfrak{F}_1$. Then $\mathfrak{F}_2$ ρ-dominates $\mathfrak{F}_1$.*

Proof. As $\mathfrak{F}_2 \supset \mathfrak{F}_1$ we obtain that if $A \in \mathfrak{F}_1$ then $A \in \mathfrak{F}_2$. Consider an arbitrary $\varepsilon > 0$. Then for every $A_1 \in \mathfrak{F}_1$ there is $A_2 \in \mathfrak{F}_2$, $A_2 := A_1$ such that for each $(\Pi_2, T_2) \in A_2$ there exists $(\Pi_1, T_1) \in A_1$, $(\Pi_1, T_1) := (\Pi_2, T_2)$ such that $\rho\left((\Pi_1, T_1), (\Pi_2, T_2)\right) = \rho\left((\Pi_2, T_2), (\Pi_2, T_2)\right) = 0 < \varepsilon$.

Previous proposition shows us that ρ-dominance generates some relation of order on TP$[0,1]$ and is more general concept that relation of inclusion.

It is clear that if $\mathfrak{F}_1 \subset \mathfrak{F}_2$ and $f \in \mathrm{Int}(\mathfrak{F}_1)$ then $f \in \mathrm{Int}(\mathfrak{F}_2)$ – just use the definition of function limit over filter. So we can formulate next easy proposition.

Proposition 3. *Let $f : [0,1] \to \mathbb{R}$ be a function, $\mathfrak{F}_1$, $\mathfrak{F}_2$ be filters on $TP[0,1]$ such that $\mathfrak{F}_1 \subset \mathfrak{F}_2$ and $f \in Int(\mathfrak{F}_1)$. Then $f \in Int(\mathfrak{F}_2)$.*

Theorem 5. *Let $\mathfrak{F}_1, \mathfrak{F}_2$ be filters on $[0,1]$. Let $f : [0,1] \to \mathbb{R}$ be a bounded function. Let $I = \lim\limits_{\mathfrak{F}_1} S(f, \Pi, T)$ and $\mathfrak{F}_2 \succ_\rho \mathfrak{F}_1$. Then $I = \lim\limits_{\mathfrak{F}_2} S(f, \Pi, T)$.*

Proof. Denote $C := \sup\limits_{t \in [0,1]} |f(t)|$.

We have to proof that for every $\varepsilon > 0$ there exists $B \in \mathfrak{F}_2$ such that for each $(\Pi_B, T_B) \in B$ we have $|S(f, \Pi_B, T_B) - I| < \varepsilon$.

We know that for every $\varepsilon > 0$ there exists $A \in \mathfrak{F}_1$ such that for each $(\Pi_1, T_1) \in A$ we have $|S(f, \Pi_1, T_1) - I| < \varepsilon$.

Now for an arbitrary $\varepsilon > 0$ and $A \in \mathfrak{F}_1$ found above one can find $A_2 \in \mathfrak{F}_2$ such that for all $(\Pi_2, T_2) \in A_2$ there is $(\Pi_1, T_1) \in A_1$ such that $\rho((\Pi_1, T_1), (\Pi_2, T_2)) < \varepsilon$.

Then put $B := A_2$. Then for all $(\Pi_B, T_B) \in B$ we have $(\Pi_1, T_1) \in A_1$ such that

$$|S(f,\Pi_B,T_B) - I| =$$
$$|S(f,\Pi_B,T_B) - S(f,\Pi_1,T_1) + S(f,\Pi_1,T_1) - I| \leq$$
$$|S(f,\Pi_B,T_B) - S(f,\Pi_1,T_1)| + |S(f,\Pi_1,T_1) - I| =$$
$$\sum_{t\in T_B\cap T_1} |f(t)|\cdot|\Delta_B(t) - \Delta_1(t)| + \sum_{t\in T_B\setminus T_1} |f(t)|\cdot\Delta_B(t) +$$
$$\sum_{t\in T_1\setminus T_B} |f(t)|\cdot\Delta_1(t) + \varepsilon \leq C\cdot\rho((\Pi_B,T_B),(\Pi_1,T)) + \varepsilon \leq$$
$$C\varepsilon + \varepsilon \leq \varepsilon(1+C).$$

4. Exactly tagged filters

In this part of our paper we consider problems deals filter integration of unbounded functions.

Definition 4. Let $\mathfrak{B}$ be a filter base on $TP[0,1]$. We say that $\mathfrak{B}$ is *exactly tagged* if there exist $A \subset [0,1]$ – a strictly decreasing sequence of numbers such that for each $B \in \mathfrak{B}$ and for every $(\Pi, T) \in B$ we have that $T \cap A = \emptyset$.

Definition 5. We say that filter $\mathfrak{F}$ on $TP[0,1]$ is *exactly tagged* if there exists exactly tagged base $\mathfrak{B}$ of $\mathfrak{F}$.

Theorem 6. *If filter $\mathfrak{F}$ on $TP[0,1]$ is exactly tagged then there exists unbounded function $f : [0,1] \to \mathbb{R}$ such that $f \in Int(\mathfrak{F})$.*

Proof. Denote $\mathbb{N}^{-1} = \left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$ and consider next filter base $\mathfrak{B} = (B_n)_{n\in\mathbb{N}}$ on $TP[0,1]$:

$B_1 = \big\{(\Pi,T) : T\cap\mathbb{N}^{-1} = \emptyset \text{ and } d(\Pi) < 1\big\}$;

$B_2 = \left\{(\Pi,T) : T\cap\mathbb{N}^{-1} = \emptyset \text{ and } d(\Pi) < \frac{1}{2}\right\}$;

$B_3 = \left\{(\Pi,T) : T\cap\mathbb{N}^{-1} = \emptyset \text{ and } d(\Pi) < \frac{1}{3}\right\}$;

...

$B_m = \left\{(\Pi,T) : T\cap\mathbb{N}^{-1} = \emptyset \text{ and } d(\Pi) < \frac{1}{m}\right\}$.

Consider now

$$f(t) = \begin{cases} n, \text{ if } t = \frac{1}{n},\ n\in\mathbb{N} \\ 0, \text{ otherwise} \end{cases} .$$

Then for each $n \in \mathbb{N}$ and for every $(\Pi, T) \in B_n$ we have that $S(f, \Pi, T) = 0$, so $\lim\limits_{\mathfrak{B}} S(f, \Pi, T) = 0$.

For a tagged partition (Π, T) of $[0, 1]$ and $\tau \in [0, 1]$ denote $\ell(\Pi, T, \tau)$ the number which is equal to the length of the segment $\Delta \in \Pi$, for which $\tau \in \Delta$, if

$\tau \in T$. If $\tau \notin T$, we put $\ell(\Pi, T, \tau) = 0$. In this notation

$$S(f, \Pi, T) = \sum_{t \in [0,1]} f(t)\ell(\Pi, T, t).$$

Theorem 7. *For a filter $\mathfrak{F}$ on $TP[0,1]$ the following assertions are equivalent:*

1. *There exists an unbounded function $f : [0,1] \to [0, +\infty)$ such that $S(f, \Pi, T)$ is $\mathfrak{F}$-bounded;*

2. *There exists a countable subset $\{t_n\}_{n\in\mathbb{N}} \subset [0,1]$ such that there is $A \in \mathfrak{F}$ such that for every $(\Pi, T) \in A$*

$$\sum_{n\in\mathbb{N}} n \cdot \ell(\Pi, T, t_n) < 1.$$

Proof. **(1)⇒(2)**: Let f be a non-negative, unbounded function on $[0,1]$ such that there is $C > 0$ and $B \in \mathfrak{F}$ such that for each $(\Pi, T) \in B$ we have $\sum\limits_{t\in[0,1]} f(t) \cdot \ell(\Pi, T, t) < C$. As f is unbounded, there exists $(\alpha_n) \subset [0,1]$ such that for every $n \in \mathbb{N}$ $f(\alpha_n) \geq Cn$. Then there exists $(\alpha_n) \subset [0,1]$, $C > 0$, there is $A \in \mathfrak{F}$, $A := B$ such that for all $(\Pi, T) \in A$ we obtain:

$$\sum_{t\in[0,1]} n \cdot \ell(\Pi, T, \alpha_n) \leq \sum_{n\in\mathbb{N}} \frac{f(\alpha_n)}{C} \cdot \ell(\Pi, T, \alpha_n) \leq$$
$$\frac{1}{C} \sum_{t\in[0,1]} f(t) \cdot \ell(\Pi, T, t) < \frac{1}{C} \cdot C = 1.$$

(2)⇒(1): Let there exists a countable subset $\{t_n\}_{n\in\mathbb{N}} \subset [0,1]$ and $C > 0$ such that there is $A \in \mathfrak{F}$ such that for every $(\Pi, T) \in A$ $\sum\limits_{n\in\mathbb{N}} n \cdot \ell(\Pi, T, t_n) < C$. Consider function

$$f(t) = \begin{cases} n, \text{ if } t = \alpha_n, \ n \in \mathbb{N} \\ 0, \text{ if } t \neq \alpha_n \end{cases}.$$

Obviously, $f(t)$ is unbounded. Then there is $C > 0$ and there is $B \in \mathfrak{F}$, $B := A$ such that for every $(\Pi, T) \in A$

$$\sum_{t\in[0,1]} f(t) \cdot \ell(\Pi, T, t) \leq \sum_{n\in\mathbb{N}} f(\alpha_n) \cdot \ell(\Pi, T, \alpha_n) \leq$$
$$\sum_{n\in\mathbb{N}} n \cdot \ell(\Pi, T, \alpha_n) < C$$

.

5. Integration over filter on a subsegment

Our next goal is as follows: if function f is integrable on $[0,1]$ over filter $\mathfrak{F}$ on TP$[0,1]$ then for an arbitrary $[\alpha,\beta] \subset [0,1]$ function f is is integrable on $[\alpha,\beta]$ over filter $\mathfrak{F}$.

To achieve this purpose we need to construct some restriction of filter $\mathfrak{F}$ on subsegment $[\alpha,\beta] \subset [0,1]$. Now we present how we can construct such restriction.

Consider an arbitrary $[\alpha,\beta] \subset [0,1]$. We consider only T such that $T \cap (\alpha,\beta) \neq \emptyset$. Consider an arbitrary $(\Pi, T) \in TP[0,1]$.

We have four cases:

1. $\min\{T \cap (\alpha,\beta)\} > \min\{\Pi \cap (\alpha,\beta)\}$
   $\max\{T \cap (\alpha,\beta)\} < \max\{\Pi \cap (\alpha,\beta)\}$;

2. $\min\{T \cap (\alpha,\beta)\} > \max\{\Pi \cap (0,\alpha)\}$
   $\max\{T \cap (\alpha,\beta)\} < \max\{\Pi \cap (\alpha,\beta)\}$;

3. $\min\{T \cap (\alpha,\beta)\} > \min\{\Pi \cap (\alpha,\beta)\}$
   $\max\{T \cap (\alpha,\beta)\} < \min\{\Pi \cap (\beta,1)\}$;

4. $\min\{T \cap (\alpha,\beta)\} > \max\{\Pi \cap (0,\alpha)$
   $\max\{T \cap (\alpha,\beta)\} < \min\{\Pi \cap (\beta,1)\}$.

We have to construct a restriction of (Π, T) on $[\alpha,\beta]$. In each of four described cases we have such $(\Pi_k, T_k) \in TP[\alpha,\beta]$, $k = 1,2,3,4$:

1. $\Pi_1 = \Big(\Pi \setminus \big((\Pi \cap [0,\alpha)) \cup (\Pi \cap (\beta,1)) \cup \min\{\Pi \cap (\alpha,\beta)\} \cup \max\{\Pi \cap (\alpha,\beta)\}\big)\Big) \cup$
   $\{\alpha,\beta\}$
   $T_1 = T \setminus \big((T \cap [0,\alpha)) \cup (T \cap (\beta,1])\big)$;

2. $\Pi_2 = \Big(\Pi \setminus \big((\Pi \cap [0,\alpha)) \cup \max\{\Pi \cap (\alpha,\beta)\} \cup (\Pi \cap (\beta,1))\big)\Big) \cup \{\alpha,\beta\}$
   $T_2 = T_1$;

3. $\Pi_3 = \Big(\Pi \setminus \big((\Pi \cap [0,\alpha)) \cup \min\{\Pi \cap (\alpha)\} \cup (\Pi \cap [\beta,1))\big)\Big) \cup \{\alpha,\beta\}$
   $T_3 = T_1$;

4. $\Pi_4 = \Big(\Pi \setminus \big((\Pi \cap [0,\alpha)) \cup (\Pi \cap [\beta,1))\big)\Big) \cup \{\alpha,\beta\}$
   $T_4 = T_1$.

Now if we have an arbitrary filter $\mathfrak{F}$ on $TP[0,1]$ we can construct filter $\mathfrak{F}_{[\alpha,\beta]}$ on $TP[\alpha,\beta]$, induced with $\mathfrak{F}$ in such way: consider an arbitrary $A \in \mathfrak{F}$ and for each $(\Pi, T) \in A$ we have to execute an algorithm, described above. For each $A \in \mathfrak{F}$ denote A_α^β the restriction of A on $[\alpha,\beta]$, described above.

Definition 6. Let $\mathfrak{F}$ be a filter on $TP[0,1]$, $[\alpha,\beta] \subset [0,1]$. We call the filter $\mathfrak{F}$ *$[\alpha,\beta]$-complemented* if for each $A \in \mathfrak{F}$, for every (Π_1, T_1), $(\Pi_2, T_2) \in A_\alpha^\beta$ there exists $(\Pi^*, T^*) \in TP[0,\alpha]$ and $(\Pi^{**}, T^{**}) \in TP[\beta, 1]$ such that

$$(\Pi^*, T^*) \cup (\Pi_1, T_1) \cup (\Pi^{**}, T^{**}) \in A,$$

$$(\Pi^*, T^*) \cup (\Pi_2, T_2) \cup (\Pi^{**}, T^{**}) \in A.$$

Here we present promised result about filter integration on subsegment.

Theorem 8. *Let $f : [0,1] \to \mathbb{R}$, $\mathfrak{F}$ be a filter on $TP[0,1]$ such that for each $[\alpha,\beta] \subset [0,1]$ $\mathfrak{F}$ is $[\alpha,\beta]$-complemented. Let f be integrable of $[0,1]$ with respect to $\mathfrak{F}$. Then for every $[\alpha,\beta] \subset [0,1]$ f is integrable on $[\alpha,\beta]$ with respect to $\mathfrak{F}$*

Proof. We know that for an arbitrary $\varepsilon > 0$ there exists $A \in \mathfrak{F}$ such that for all $(\Pi_1, T_1), (\Pi_2, T_2) \in A$ we have: $|S(f, \Pi_1, T_1) - S(f, \Pi_2, T_2)| < \varepsilon$.

Let fix $\varepsilon > 0$ and consider an arbitrary $[\alpha,\beta] \subset [0,1]$. For $A \in \mathfrak{F}$ consider an arbitrary $(\Pi^1, T^1), (\Pi^2, T^2) \in A_\alpha^\beta$. As $\mathfrak{F}$ is $[\alpha,\beta]$-complemented we can find $(\Pi^*, T^*) \in A_0^\alpha$ and $(\Pi^{**}, T^{**}) \in A_\beta^1$ such that $(\Pi_{11}, T_{11}) := (\Pi^*, T^*) \cup (\Pi^1, T^1) \cup (\Pi^{**}, T^{**}) \in A$ and $(\Pi_{22}, T_{22}) := (\Pi^*, T^*) \cup (\Pi^2, T^2) \cup (\Pi^{**}, T^{**}) \in A$. Then $\varepsilon > |S(f, \Pi_{11}, T_{11}) - S(f, \Pi_{22}, T_{22})| = |S(f, \Pi^1, T^1) - S(f, \Pi^2, T^2)|$.

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Про iнтегрування вiдносно фiльтра

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Дану статтю присвячено дослiдженню одного узагальнення iнтеграла Рiмана. А саме, в роботi помiчено, що класичне означення iнтеграла Рiмана по скiнченному вiдрiзку як границi iнтегральних сум, коли дiаметр розбиття вiдрiзка прямує до нуля, може бути замiнено на границю iнтегральних сум по фiльтру множин, якi можна описати певним "хорошим чином". Цю iдею продовжено, i в роботi запропоновано нове поняття – iнтеграла функцiї по фiльтру на множинi всiх вiдмiчених розбиттiв вiдрiзка. Використання фiльтрiв є дуже хорошим методом в питаннях, пов'язаних зi збiжнiстю або деякими її аналогами в загальних топологiчних векторних просторах. А саме, якщо простiр не є метризовним, то поняття збiжностi вводиться саме за допомогою фiльтрiв. Також, використовуючи фiльтри, можна формулювати поняття повноти та її аналогiв. Повнота просторiв є одним iз центральних понять теорiї топологiчних векторних просторiв, оскiльки банаховi простори є повними. Тобто, використовуючи узагальнення повноти просторiв, побудованих з використанням фiльтрiв, ми можемо дослiджувати рiзнi узагальнення банахових просторiв. Далi в статтi дослiджуються стандартнi питання, пов'язанi з iнтегруванням. Наприклад, чи витiкає з iнтегровностi функцiї по фiльтру її обмеженiсть? На це питання дано ствердну вiдповiдь. Докладнiше: введено поняття обмеженостi функцiї за фiльтром, i показано, що якщо функцiя є iнтегровною за фiльтром, то її iнтегральнi суми є обмеженими за фiльтром, а сама ця функцiя є обмеженою в класичному розумiннi. Далi ми показали, що iнтеграл за фiльтром задовольняє властивiсть лiнiйностi, а саме iнтеграл за фiльтром вiд суми двох функцiй є сумою iнтегралiв за фiльтром цих доданкiв. Ми вводимо поняття точно вiдмiченого фiльтра, i за допомогою таких фiльтрiв вивчаємо iнтегровнiсть за фiльтром необмежених на вiдрiзку функцiй. Ми наводимо приклад конкретної необмеженої функцiї та конкретного фiльтра, за яким дана функцiя є iнтегровною Далi ми доводимо теорему, яка описує необмеженi, iнтегровнi за фiльтром, функцiї на вiдрiзку. Останнiй роздiл статтi присвячено iнтегрегрування функцiй вiдносно фiльтра по пiдвiдрiзку даного вiдрiзка.
Ключовi слова: **iнтеграл; фiльтр; iдеал; база фiльтра.**

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