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Construction of controllability function as the time of motion

This article is devoted to the controllability function method in admissible synthesis problems for linear canonical systems. The work considers methods of constructing such control so that the controllability function is time of motion of an arbitrary point to the origin. A canonical controlled system of linear equations $\dot{x}_i = x_{i+1}, i = \overline{1, n-1}, \dot{x}_n = u$ with control constraints $|u| \leq d$ is considered. The controllability function Θ can be found as the only positive solution of the implicit equation $2a_0\Theta = (D(\Theta)FD(\Theta)x, x)$, where $D(\Theta) = diag(\Theta^{-\frac{-2n-2i+1}{2}})_{i=1}^{n}$. Matrix $F = \{f_{ij}\}_{i,j=1}^{n}$ is positive definite and $a_0 > 0$ is chosen so that the control constraints are satisfied. The controllability function is motion time if $\dot{\Theta} = -1$. From this condition, an equation is obtained, the solution of which is considered in this work. Unlike previous works on this topic, no additional restrictions are imposed on the appearance of matrix F. The task of this article is to find the parameters set of the matrix F and the column vector a, which satisfy the obtained equation and for which the controllability function is the time of movement from the point x to the origin. In this way, we get a family of controls depending on this parameters such that the trajectory of system steers the origin in finite time. In general case, difficulties may arise when finding the solution of Cauchy problem of the corresponding system. Canonical system can be reduced to Euler's equation, for which a characteristic equation can be found, and therefore a trajectory in an explicit form. Two-dimensional, three-dimensional and four-dimensional canonical systems are considered. In each case, the matrix equation is solved and sets of parameters for which the controllability functions value will be the time of movement of an arbitrary point to the origin are found. Conditions on parameters are obtained from positive definiteness of the matrix F. Some parameters and an arbitrary initial point are chosen and the solution of Cauchy problem in analytical form is found.

Keywords: **controllability; controllability function; controllability function as the time of movement.**

2010 Mathematics Subject Classification: 76A11; 76B11; 76M11.

1. Introduction

To solve the problem of admissible synthesis in 1979, Korobov V.I. the controllability function method was proposed in the article [1] and developed in the monograph [2]. In works [3, 4], the controllability function was obtained as time of motion from an arbitrary initial point to the origin. A family of controls solving synthesis problem was found. An extended control set was proposed for a two-dimensional canonical system in [5].

Let us consider the canonical system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \quad \dots \\ \dot{x}_{n-1} = x_n, \\ \dot{x}_n = u \end{cases} \tag{1}$$

with the constraint $|u| \leq d$.

To solve the control synthesis problem for the system $\dot{x} = f(x, u)$ is to construct a control $u = u(x)$ which satisfies a given constraint $|u| \leq d$. And for which the trajectory of the closed-loop system $\dot{x} = f(x, u(x))$ starts at an arbitrary point x_0 and reaches the origin in a finite time.

In the Korobov's method $\Theta(x)$ is a controllability function and the control $u(x)$ is constructed on base of $\Theta(x)$

$$u(x) = \sum_{i=1}^{n} \frac{a_i x_i}{\Theta^{n-i+1}(x)}. \tag{2}$$

Let us denote $a = (a_1, a_2, ..., a_n)^*, a_i < 0$.

The function $\Theta(x)$ needs further definition.

If the following inequality holds

$$\dot{\Theta} = \sum_{i=1}^{n} \frac{\partial \Theta(x)}{\partial x_i} f_i(x, u(x)) \leq -\beta \Theta^{1-\frac{1}{\alpha}}(x), \tag{3}$$

then the time of motion is finite.

A particular case of inequality (3) is the equation (4) for $\alpha = \beta = 1$

$$\dot{\Theta} = \sum_{i=1}^{n} \frac{\partial \Theta(x)}{\partial x_i} f_i(x, u(x)) = -1. \tag{4}$$

In this case, the controllability function is the time of motion from an arbitrary point x_0 to the origin. This problem was considered in works [3, 4]. There was highlighted a special case when the matrix F^{-1} has the form $F^{-1} = D_n C D_n$. Where C is a Hankel matrix $C = (c_{i+j})_{i,j=0}^{n-1}$ and $D_n = diag((-1)^{i-1}/(i-1)!)_{i=1}^{n}$. Our work though considers the whole class of controllability functions without constraints on matrix F^{-1}. Let us move on to the construction of a controllability function.

Let us consider the canonical system with the constraint $|u| \leq d$. We will choose a control according to the formula (2).

And the controllability function is defined as the only positive solution of the equation

$$2a_0\Theta = (D(\Theta)FD(\Theta)x, x),$$

where $x \neq 0$ and $\Theta(0) = 0$, if $x = 0$. Here $D(\Theta) = diag\left(\Theta^{-\frac{-2n-2i+1}{2}}\right)_{i=1}^{n}$.Matrix $F = \{f_{ij}\}_{i,j=1}^{n}$ is a positive definite matrix and $a_0 > 0$ is such a number that the constraints on a control are satisfied. The value of a_0 found [2] $2a_0 = \frac{1}{(F^{-1}a,a)}$.

Let us denote $y(\Theta, x) = D(\Theta)x$. Then the control function satisfies the equation

$$2a_0\Theta(x) = (Fy(\Theta(x), x), y(\Theta(x), x)). \tag{5}$$

Derivative $\dot{\Theta}(x)$ of the controllability function $\Theta(x)$ has the following form

$$\dot{\Theta}(x) = \frac{((F(A_0 + b_0a^*) + (A_0 + b_0a^*)^*F)y(\Theta(x), x), y(\Theta(x), x))}{((F - HF - FH)y(\Theta(x), x), y(\Theta(x), x)))}, \tag{6}$$

where $b_0 = \begin{pmatrix} 0 \\ 0 \\ ... \\ 0 \\ 1 \end{pmatrix}$, $A_0 = \begin{pmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \\ 0 & 0 & 0 & ... & 0 \end{pmatrix}$ and $H = diag(-\frac{-2n-2i+1}{2})_{i=1}^{n}$.

We equate the derivative of $\Theta(x)$ (6) to -1 and get

$$F\left(A_0 + b_0a^* + \frac{1}{2}I - H\right) + \left(A_0 + b_0a^* + \frac{1}{2}I - H\right)^* F = 0. \tag{7}$$

Denote $A = \left(A_0 + b_0a^* + \frac{1}{2}I - H\right)$. It has the following form

$$\begin{pmatrix} n & 1 & 0 & ... & 0 & 0 \\ 0 & n-1 & 1 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 2 & 1 \\ a_1 & a_2 & a_3 & ... & a_{n-1} & 1 + a_n \end{pmatrix}. \tag{8}$$

We obtain

$$FA + A^*F = 0. \tag{9}$$

Our task is to find parameters of the matrix F and the column vector a so that the equation (9) is fulfilled. Then the controllability function $\Theta(x)$ is time of motion from an arbitrary point x_0 to the origin.

Let us move on to the construction of the controllability function. From the lemma [2, p. 79] we get $a_n = -\frac{n(n+1)}{2}$. Consider $det(A-\lambda E)$. Matrix A is similar to the skew-symmetric matrix $F^{\frac{1}{2}}AF^{-\frac{1}{2}}$, therefore real parts of eigenvalues are equal to zero. All coefficients of λ^{n-k} are zero where k is odd and $k \le n$. In this way we obtain equations for parameters a_i. We substitute these parameters into the matrix A and solve the equation (9). It should be noted, that the matrix FA is skew-symmetric, therefore all main-diagonal elements are zeros. We obtain the matrix F, which is positive definite as was said earlier. We use Sylvester's criterion and find conditions for parameters of the matrix F and a_i. In this way we describe the whole class of controllability functions $\Theta(x)$ and controls $u(x)$, which transfer some initial point to the origin of coordinates. Moreover, the controllability function is the time of motion.

Next, we find a trajectory of the canonical system (1), which reduces to an Euler equation $(\Theta_0 - t)^n x_1^{(n)} - (\Theta_0 - t)^{n-1} a_n x_1^{(n-1)} - ... - a_1 x_1 = 0$.

Looking for a solution in the form $x_1(t) = (\Theta_0 - t)^{\lambda}$ we get the characteristic equation. After solving we get an analytical solution.

2. Construction of the controllability function in the two-dimensional case

Consider a solution of the synthesis problem. We find the controllability function $\Theta(x)$. On base of $\Theta(x)$, we construct the control $u(x)$, which transfers an arbitrary given point to the origin.

System has the following form

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = u. \end{cases} \tag{10}$$

Theorem 1. *Let*

$$a_1 < -\frac{9}{2}, \quad f_{22} > 0. \tag{11}$$

The controllability function $\Theta = \Theta(x)$ *is defined as the only positive root of the equation*

$$-\frac{4+a_1}{a_1(3+a1)}\Theta^4 = -a_1 x_1^2 + 4x_1 x_2 \Theta + x_2^2 \Theta^2, \tag{12}$$

at $x = 0$, *we put* $\Theta(0) = 0$.

Then control

$$u(x) = \frac{a_1 x_1}{\Theta^2(x)} - \frac{3x_2}{\Theta(x)}. \tag{13}$$

transfers an arbitrary initial point $x_0 \in \mathbb{R}^2$ *to the origin in time* $\Theta(x_0)$.

Indeed, in this case

$$F = \begin{pmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 1 \\ a_1 & a_2+1 \end{pmatrix}$$

Recall that the real parts of the eigenvalues are equal to zero, hence the coefficients near odd powers of λ are equal to zero. We have

$$det(A - \lambda E) = 2 - a_1 + 2a_2 - (a_2+3)\lambda + \lambda^2,$$

then $a_2 = -3$.

Equation (9) has the form

$$\begin{cases} 2f_{11} + a_1 f_{12} = 0, \\ f_{12} + (a_2+1) f_{22} = 0, \\ f_{11} + (a_2+3) f_{12} + a_1 f_{22} = 0. \end{cases}$$

It follows that

$$F = \begin{pmatrix} -a_1 f_{22} & 2f_{22} \\ 2f_{22} & f_{22} \end{pmatrix}, \; A = \begin{pmatrix} 2 & 1 \\ a_1 & -2 \end{pmatrix}, \; F^1 = \begin{pmatrix} -4a_1 f_{22} & 6f_{22} \\ 6f_{22} & 2f_{22} \end{pmatrix}. \tag{14}$$

We use Sylvester criterion and get (11).

Therefore, form (5) where $y(\Theta(x), x) = (x_1\Theta^{-3/2}, x_2\Theta^{-1/2})$ and $2a_0 = \frac{1}{(F^{-1}a,a)}$ we get (12). The solution is any control (13), where $a_1 < -\frac{9}{2}$. It should be noted that we choose only parameters a_1 and f_{22}. The other ones we calculate from (14) according to the formulas: $f_{11} = -a_1 f_{22}, f_{12} = 2f_{22}$ and inequalities (11) must be fulfilled.

For example, let us choose $a_1 = -6, f_{22} = 1$, then $f_{11} = 6, f_{12} = 2$, conditions (11) fulfilled. We have $a_0 = \frac{1}{18}$. We obtain the equation relating Θ

$$\frac{1}{9}\Theta^4 = 6x_1^2 + 4x_1x_2\Theta + x_2^2\Theta^2$$

The system has the form (10), where

$$u = -\frac{6x_1(t)}{\Theta^2(x_1,x_2)} - \frac{3x_2(t)}{\Theta(x_1,x_2)}.$$

So, a control is found that satisfies the constraints and translates any given initial point to the origin in a finite time. Let $\{1,1\}$ be the initial point. Let's find the trajectory of the system. Equation (12) takes the form

$$\frac{1}{9}\Theta^4 = 6 + 4\Theta + \Theta^2$$

It has a unique positive solution $\Theta_0 \approx 4.4512$.

As was mentioned earlier, the system (10) reduces to Euler equation

$$(\Theta_0 - t)^2 \ddot{x}_1 + 3(\Theta_0 - t)\dot{x}_1 + 6x_1 = 0.$$

Looking for a solution in the form $x_1(t) = (\Theta_0 - t)^\lambda$ we get a characteristic equation

$$\lambda^2 - 4\lambda + 6 = 0.$$

We find roots $\lambda_{1,2} = 2 \pm i\sqrt{2}$ and obtain

$$x_1(t) = (\theta_0 - t)^2 \left(c_1 \cos(\sqrt{2}\ln(\theta_0 - t)) + c_2 \sin(\sqrt{2}\ln(\theta_0 - t)) \right).$$

From the initial conditions $x_1(0) = 1, x_2(0) = 1$ we find $c_1 = 0.17, c_2 = 0.16$. Let us denote $\tau(t) = \sqrt{2}\ln(\Theta_0 - t)$. Finally, we have the solution in analytical form

$$x_1(t) = (\Theta_0 - t)^2(0.17\cos(\tau(t)) + 0.16\sin(\tau(t))),$$
$$x_2(t) = -2(\Theta_0 - t)(0.29\cos(\tau(t)) + 0.04\sin(\tau(t))).$$

The trajectory is shown in Fig. 1 and time of motion $\Theta_0 \approx 4.4512$.

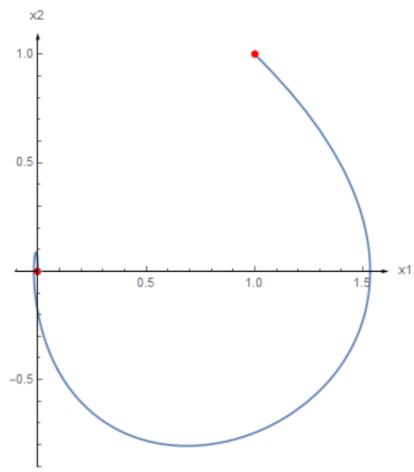

Fig. 1. The trajectory $(x_1(t), x_2(t))$ of the point $\{1, 1, 1\}$ which reaches the origin in time $\Theta_0 \approx 4.4512$.

3. Construction of the controllability function in the three-dimensional case

Similarly to the previous case, we will consider a solution of the synthesis problem.

The system has the following form

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = u. \end{cases} \tag{15}$$

Theorem 2. *Let*

$$a_1 < -\frac{75}{2}, f_{23} > 0,$$

$$\frac{15 f_{23}}{8} < f_{13} < -\frac{a_1 f_{23}}{20}. \tag{16}$$

The controllability function $\Theta = \Theta(x)$ at $x \neq 0$ is defined as the only positive root of the equation (5), at $x = 0$ we put $\Theta(0) = 0$.

Then view control

$$u = \frac{a_1 x_1(t)}{\Theta^3(x)} + \frac{(a_1 - 30) x_2(t)}{3\Theta^2(x)} - \frac{6 x_3(t)}{\Theta(x)}. \tag{17}$$

translates an arbitrary point $x_0 \in \mathbb{R}^3$ to the origin in time $\Theta(x_0)$.

Here

$$F = \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{pmatrix}, \quad A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ a_1 & a_2 & 1 + a_3 \end{pmatrix}.$$

Recall that the real parts of the eigenvalues are equal to zero, therefore the coefficients for even powers of λ are equal to zero. Then $a_3 = -6, a_2 = \frac{1}{3} a_1 - 10$.

We solve an equation (9) and get

$$F = \begin{pmatrix} -\frac{1}{3} a_1 f_{13} & 2 f_{13} - \frac{1}{5} a_1 f_{23} & f_{13} \\ 2 f_{13} - \frac{1}{5} a_1 f_{23} & -f_{13} + (5 - \frac{1}{15} a_1) f_{23} & f_{23} \\ f_{13} & f_{23} & \frac{1}{5} f_{23} \end{pmatrix}, \tag{18}$$

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ a_1 & \frac{1}{3} a_1 - 10 & -5 \end{pmatrix}, \tag{19}$$

$$F^1 = \begin{pmatrix} -2 a_1 f_{13} & 10 f_{13} - a_1 f_{23} & 4 f_{13} \\ 10 f_{13} - a_1 f_{23} & -4 f_{13} + 4(5 - \frac{1}{15} a_1) f_{23} & 3 f_{23} \\ 4 f_{13} & 3 f_{23} & \frac{2}{5} f_{23} \end{pmatrix}.$$

We use Sylvester's criterion for F and F^1 and obtain (16).

Therefore, from the equation (5), where $y(\Theta(x), x) = (x_1 \Theta^{-\frac{5}{2}}, x_2 \Theta^{-\frac{3}{2}}, x_3 \Theta^{-\frac{1}{2}})$ we get (12). And control (17) is a solution of the synthesis problem. Note that we choose only the parameters a_1, f_{13} and f_{23}, and we calculate the others from (18),(19) according to the formulas: $a_2 = \frac{1}{3} a_1 - 10, f_{11} = -\frac{1}{3} a_1 f_{13}, f_{12} = 2 f_{13} - \frac{1}{5} a_1 f_{23}, f_{22} = -f_{13} + (5 - \frac{1}{15} a_1) f_{23}, f_{33} = \frac{1}{5} f_{23}$. Inequalities (16) must be satisfied.

For example, let's choose $a_1 = -57, f_{13} = 2, f_{23} = \frac{19}{20}$, then $a_2 = -29, f_{11} = 38, f_{12} = \frac{1483}{100}, f_{22} = \frac{159}{25}, f_{33} = \frac{19}{100}$, conditions (16) fulfilled. We have $a_0 = \frac{667}{259000}$. Equation (5) for chosen parameters

$$\frac{667}{1295} \Theta^6 = 3800 x_1^2 + 2966 x_1 x_2 \Theta + (636 x_2^2 + 400 x_1 x_2) \Theta^2 + 190 x_2 x_3 \Theta^3 + 19 x_3^2 \Theta^4.$$

And control solving synthesis problem

$$u = -\frac{57x_1(t)}{(\Theta_0 - t)^3} - \frac{29x_2(t)}{(\Theta_0 - t)^2} - \frac{6x_3(t)}{\Theta_0 - t}.$$

Let $\{1, 1, 1\}$ be an initial point. We get

$$\frac{667}{1295}\Theta^6 = 3800 + 2966\Theta + 1036\Theta^2 + 190\Theta^3 + 19\Theta^4$$

We have a unique positive solution $\Theta_0 \approx 10.0131$.

The system (15) reduces to an Euler equation

$$(\Theta_0 - t)^3\,\dddot{x}_1 + 6(\Theta_0 - t)^2\ddot{x}_1 + (10 - \frac{1}{3}a_1)(\Theta_0 - t)\dot{x}_1 - a_1 x_1 = 0.$$

As earlier, we find the characteristic equation

$$-\lambda^3 + 9\lambda^2 - 37\lambda + 57 = 0.$$

We get roots $\lambda_1 = 3, \lambda_{2,3} = 3 \pm i\sqrt{10}$ and obtain an expression for x_1.

From the initial conditions we find $c_1 = 0.017, c_2 = -0.005, c_3 = -0.016$. Let us denote $\tau(t) = \sqrt{10}\ln(\Theta_0 - t)$. Finally, we have an analytical solution

$$\begin{aligned} x_1(t) &= (\Theta_0 - t)^3(0.017 - 0.005\cos\tau(t) - 0.016\sin\tau(t)),\\ x_2(t) &= -3(\Theta_0 - t)^2(0.017 - 0.021\cos\tau(t) - 0.01\sin\tau(t)),\\ x_3(t) &= 6(\Theta_0 - t)(0.017 - 0.038\cos\tau(t) + 0.024\sin\tau(t)).\end{aligned}$$

The trajectory is shown in Fig. 2 and time of motion $\Theta_0 \approx 10.0131$.

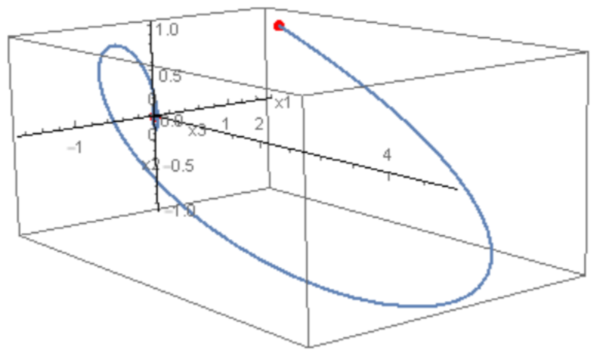

Fig. 2. The trajectory$(x_1(t), x_2(t), x_3(t))$ of the point $\{1, 1, 1\}$ which reaches the origin in time $\Theta_0 \approx 10.0131$.

Note that in order to use the methods described [3, 4] in three-dimensional space, matrix F^{-1} must have a representation $F^{-1} = D_3CD_3$. For this, the matrix $D_3^{-1}F^{-1}D_3^{-1}$ must be Hankel. This holds only if $f_{13} = 2f_{23}$. In our example, we selected such parameters for which it was not fulfilled.

4. Construction of the controllability function in four-dimensional case

Let us consider a solution of the synthesis problem. We have a system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = u. \end{cases} \tag{20}$$

Theorem 3. *Let*

$$a_1 < -\tfrac{3675}{8}, \quad \tfrac{1}{16}a_1 - 39 < a_3 < -23 - 2\sqrt{-a_1},$$
$$\tfrac{23+a3+\sqrt{4a_1+(23+a_3)^2}}{8a_1} f_{14} < f_{44} < \tfrac{23+a3-\sqrt{4a_1+(23+a_3)^2}}{8a_1} f_{14}, \;\; f_{14} > 0,$$

$$(30+a_3)(3a_1 - 49(30+a_3))f_{14}^2 + 6a_1(-2a_1 + 49(30+a_3))f_{14}f_{44} - 441a_1^2 f_{44}^2 > 0,$$

$$\tfrac{1}{2}(-a_1(155+6a_3) + 2(30+a_3)(1770+49a_3))f_{14}^3 + 2(6a_1^2 + 98(30+a_3)^2(33+a_3) -$$
$$-a_1(8175 + a_3(415+6a_3)))f_{14}^2 f_{44} + a_1(-98(30+a_3)(636+17a_3) +$$
$$+3a_1(945+34a_3))f_{14}f_{44}^2 - 72a_1^2(3a_1 - 49(39+a_3))f_{44}^3 > 0,$$

$$\tfrac{1}{4}(f_{14}^2 + 4(21+a_3)f_{14}f_{44} - 18a_1 f_{44}^2)((5125 - 8a_1 + 130a_3)f_{14}^2 + 4(-a_1(107+6a_3) +$$
$$+49(1425 + a_3(115+2a_3)))f_{14}f_{44} + 2a_1(48a_1 - 49(633+16a_3))f_{44}^2) > 0.$$

(21)

The controllability function $\Theta = \Theta(x)$ at $x \neq 0$ is defined as the only positive root of the equation (5), at $x = 0$ we put $\Theta(0) = 0$.

Control

$$u = \frac{a_1 x_1(t)}{\Theta^4(x)} + \frac{7(30+a_3)x_2(t)}{\Theta^3(x)} + \frac{a_3 x_3(t)}{\Theta^2(x)} - \frac{10x_3(t)}{\Theta(x)}. \tag{22}$$

translates an arbitrary point $x_0 \in \mathbb{R}^3$ to the origin in time $\Theta(x_0)$.

In this case

$$F = \begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{12} & f_{22} & f_{23} & f_{24} \\ f_{13} & f_{23} & f_{33} & f_{34} \\ f_{14} & f_{24} & f_{34} & f_{44} \end{pmatrix}, \quad A = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ a_1 & a_2 & a_3 & 1+a_4 \end{pmatrix}.$$

As earlier we obtain $a_4 = -10, a_2 = 7(30 + a_3)$.

We solve the equation (9) and get$F = \{f_{ij}\}_{i,j=1}^4$

$$\begin{aligned}
&f_{11} = -\tfrac{a_1}{4} f_{14},\\
&f_{12} = -(30 + a_3)f_{14} - 3a_1 f_{44},\\
&f_{13} = 5f_{14} - a_1 f_{44},\\
&f_{22} = -\tfrac{1}{4}(30 + a_3)f_{14} + (a_1 - 49(30 + a_3))f_{44},\\
&f_{23} = \tfrac{1}{2} f_{14} - 7(12 + a_3)f_{44},\\
&f_{24} = \tfrac{1}{4} f_{14} + 21f_{44},\\
&f_{33} = -\tfrac{1}{4} f_{14} - (a_3 - 42)f_{44},\\
&f_{34} = 9f_{44},
\end{aligned} \tag{23}$$

$$A = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ a_1 & 7(30 + a_3) & a_3 & -9 \end{pmatrix}.$$

From Sylvester criterion we get (21).

From (5), where $y(\Theta(x), x) = (x_1\Theta^{-\frac{7}{2}}, x_2\Theta^{-\frac{5}{2}}, x_3\Theta^{-\frac{3}{2}}, x_4\Theta^{-\frac{1}{2}})$ we get an equation relative to $\Theta(x)$. Note that we choose only parameters a_1, a_3, f_{14} and f_{44} so that inequalities (21) are fulfilled, and we calculate the others according to the formulas $a_2 = 7(30 + a_3), a_4 = -10$ and (23).

For example, let's choose parameters $a_1 = -550, a_3 = -73$ and $f_{14} = 75, f_{44} = 1$, conditions (21) are fulfilled. Then we have $a_2 = -301$, $f_{11} = \frac{20625}{2}, f_{12} = 4875, f_{13} = 925, f_{22} = \frac{9453}{4}, f_{23} = \frac{929}{2}, f_{24} = \frac{159}{4}, f_{33} = \frac{385}{4}, f_{34} = 9$ and $a_0 = \frac{23}{6536}$. Equation is relative to Θ

$$\begin{aligned}\tfrac{23}{3268}\Theta^8 = {}& \tfrac{20625}{2}x_1^2 + 9750x_1x_2\Theta + (\tfrac{9453}{4}x_2^2 + 1850x_1x_3)\Theta^2 + (929x_2x_3 + \\ &+150x_1x_4)\Theta^3 + (\tfrac{385}{4}x_3^2 + \tfrac{159}{2}x_2x_4)\Theta^4 + 18x_3x_4\Theta^5 + x_4^2\Theta^6.\end{aligned} \tag{24}$$

We obtain control

$$u = -\frac{550x_1(t)}{\Theta^4(x)} - \frac{301x_2(t)}{\Theta^3(x)} - \frac{73x_3(t)}{\Theta^2(x)} - \frac{10x_3(t)}{\Theta(x)}.$$

Let $\{1, 1, 1, 1\}$ be the initial point. From (24) we get

$$\tfrac{23}{1634}\Theta^8 = \tfrac{20625}{2} + 9750\Theta + \tfrac{16853}{4}\Theta^2 + 1079\Theta^3 + \tfrac{703}{4}\Theta^4 + 18\Theta^5 + \Theta^6.$$

Here the solution is $\Theta_0 \approx 19.2179$.

We have Euler equation

$$(\Theta_0 - t)^4 \ddot{\ddot{x}}_1 - a_4(\Theta_0 - t)^3 \dddot{x}_1 - a_3(\Theta_0 - t)^2\ddot{x}_1 - a_2(\Theta_0 - t)\dot{x}_1 - a_1x_1 = 0,$$

We find characteristic equation and get the roots $\lambda_{1,2} = 4 \pm 1.31129i, \lambda_{3,4} = 4 \pm 0.964628i$.

We use the initial conditions to find constants $c_1 = 0.0083, c_2 = 0.0015, c_3 = 0.00005, c_4 = 0.0011$. Let us denote $\tau(t) = \ln(\Theta_0 - t)$, $\beta_1 = \sqrt{9 - 5\sqrt{3}}$, $\beta_2 = \sqrt{9 + 5\sqrt{3}}$. Finally, we have a solution in analytical form

$$
\begin{aligned}
x_1(t) = {} & (\Theta_0 - t)^4(0.0083 \cos \beta_1\tau(t) + 0.0015 \sin \beta_1\tau(t)+ \\
& 0.00005 \cos \beta_2\tau(t) + 0.0011 \sin \beta_2\tau(t)) \\
x_2(t) = {} & -4(\Theta_0 - t)^3(0.0085 \cos \beta_1\tau(t) - 0.0003 \sin \beta_1\tau(t)+ \\
& +0.0011 \cos \beta_2\tau(t) - 0.0012 \sin \beta_2\tau(t)) \\
x_3(t) = {} & 12(\Theta_0 - t)^2(0.0086 \cos \beta_1\tau(t) - 0.0014 \sin \beta_1\tau(t)+ \\
& +0.0028 \cos \beta_2\tau(t) - 0.0004 \sin \beta_2\tau(t)) \\
x_4(t) = {} & -24(\Theta_0 - t)(0.0082 \cos \beta_1\tau(t) - 0.0039 \sin \beta_1\tau(t)+ \\
& +0.0020 \cos \beta_2\tau(t) - 0.0064 \sin \beta_2\tau(t)).
\end{aligned}
$$

The trajectory is shown in Fig. 3 and time of motion $\Theta_0 \approx 19.2179$.

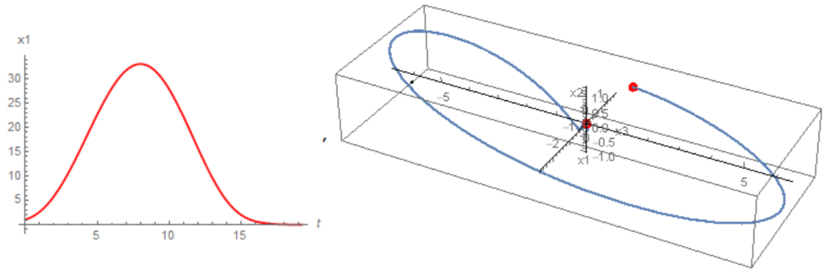

Fig. 3. The trajectory$(x_1(t), x_2(t), x_3(t), x_4(t))$ of the point $\{1, 1, 1, 1\}$ which reaches the origin in time $\Theta_0 \approx 19.2179$

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Побудова функції керованості як часу руху

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Дана стаття присвячена методу функції керованості в задачах допустимого синтезу для лінійних канонічних систем. В роботі розглянуто спосіб побудови такого керування, щоб функція керованості була часом руху довільної точки в початок координат. Розглядається канонічна керована система лінійних рівнянь $\dot{x}_i = x_{i+1}, i = \overline{1, n-1}, \dot{x}_n = u$ з обмеженнями на управління $|u| \leq d$. Функція керованості Θ знаходиться як єдиний додатній розв'язок неявного рівняння $2a_0\Theta = (D(\Theta)FD(\Theta)x, x)$, де $D(\Theta) = diag(\Theta^{-\frac{-2n-2i+1}{2}})_{i=1}^{n}$. Матриця $F = \{f_{ij}\}_{i,j=1}^{n}$ додатно визначена, а $a_0 > 0$ обирається так, щоб виконувались обмеження на керування. Функція керованості є часом руху, якщо $\dot{\Theta} = -1$. З цієї умови отримано рівняння, розв'язання якого розглядається у даній роботі. На відміну від попередніх робіт з цієї теми, на вигляд матриці F не накладено додаткові обмеження. В цій статті знайдено множину параметрів матриці F та вектор-стовпця a, які задовільняють отриманому рівнянню та для яких функція керованості час руху із точки x у початок координат. Таким чином описується весь клас функцій керованості, які є часом руху. У загальному випадку при знаходження розв'язку задачі Коші відповідної системи можуть виникати труднощі. Система, яка розглядалась у даній роботі зводиться до рівняння Ейлера, для якого можна знайти характеристичне рівняння, а отже і траєкторію у явному вигляді. Розглянуто двовимірну, тривимірну та чотиривимірну канонічні системи. У кожному випадку розв'язано матричне рівняння та знайдено множини параметрів, при яких значення функції керованості буде часом руху довільної точки в початок координат. Також обрано деякий довільний набір параметрів, які задовільняють умовам додатної визначеності матриці F та побудовано траєкторії з обраних початкових точок в початок координат.

Ключові слова: **керованість; функція керованості; функція керованості як час руху.**

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