ISSN 2221-5646(Print) 2523-4641(Online)
Visnyk of V.N.Karazin Kharkiv National University
Ser. “Mathematics, Applied Mathematics
and Mechanics”
2022, Vol. 96, p. 23–39
DOI: 10.26565/2221-5646-2022-96-02
УДК 532.59

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університету імені В.Н. Каразіна
Серія "Математика, прикладна
математика і механіка"
2022, Том 96, с. 23–39

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Homogeneous approximation for minimal realizations of series of iterated integrals

In the paper, realizable series of iterated integrals with scalar coefficients are considered and an algebraic approach to the homogeneous approximation problem for nonlinear control systems with output is developed. In the first section we recall the concept of the homogeneous approximation of a nonlinear control system which is linear w.r.t. the control and the concept of the series of iterated integrals. In the second section the statement of the realizability problem is given, a criterion for realizability and a method for constructing a minimal realization of the series are recalled. Also we recall some ideas of the algebraic approach to the description of the homogeneous approximation: the free graded associative algebra, which is isomorphic to the algebra of iterated integrals, the free Lie algebra, the Poincaré-Birkhoff-Witt basis, the dual basis and its construction by use of the shuffle product, the definition of the core Lie subalgebra, which defines the homogeneous approximation of a control system. In the third section we show how to find the core Lie subalgebra of the systems that is a realization of the one-dimensional series of iterated integrals without finding the system itself. The result obtained is illustrated by the example, in which we demonstrate two methods for finding the core Lie subalgebra of the realizing system. In the last section it is shown that for any graded Lie subalgebra of finite codimension there exists a one-dimensional homogeneous series such that this Lie subalgebra is the core Lie subalgebra for its minimal realization. The proof is constructive: we give a method of finding such a series; we use the dual basis to the Poincaré-Birkhoff-Witt basis of the free associative algebra, which is built by the core Lie subalgebra, and the shuffle product in this algebra. As a consequence, we get a classification of all possible homogeneous approximations of systems that are realizations of one-dimensional series of iterated integrals.

Keywords: **homogeneous approximation; series of iterated integrals; minimal realization; core Lie subalgebra.**

2010 Mathematics Subject Classification: 93B15; 93B25; 93C10.

1. Introduction

The homogeneous approximation problem has attracted great attention of experts in the control theory for several decades. We briefly recall the definition. In this paper we restrict ourselves to the class of control systems, which are linear w.r.t. the control, of the form

$$\dot{x} = \sum_{i=1}^{m} X_i(x)u_i, \tag{1}$$

where $X_1(x), \ldots, X_m(x)$ are real analytic vector fields in a neighborhood of some point x^0. Under *homogeneous system* from this class we mean a system of the polynomial form

$$\dot{x}_k = \sum_{i=1}^{m} \sum \alpha^{ik}_{q_1\cdots q_{k-1}} x_1^{q_1} \cdots x_{k-1}^{q_{k-1}} u_i, \quad \alpha^{ik}_{q_1\cdots q_{k-1}} \in \mathbb{R}, \quad k = 1, \ldots, n, \tag{2}$$

where the inner sum in the right hand side of (2) is taken over all integers $q_1, \ldots, q_{k-1} \geq 0$ such that

$$q_1 w_1 + \cdots + q_{k-1} w_{k-1} + 1 = w_k,$$

and $1 \leq w_1 \leq \cdots \leq w_n$ are some integers called *weights of the coordinates* $x_1, \ldots, x_n$. We note that a homogeneous system is *feedforward*, hence, if the controls $u_i(t)$ are known, then the components of the trajectory $x_k(t)$ can be found one by one by integrating known functions, without solving differential equations. It is convenient to deal with a coordinate-free definition. So, we say that *a system is homogeneous if it takes the form (2) after some change of variables.*

The concept of a homogeneous approximation can be introduced by different ways. Using coordinates, we can explain the definition as follows. Let us denote by $x(t;u)$ and $\widehat{x}(t;u)$ the trajectories of the systems (1) and (2) starting at x^0 and at the origin respectively and corresponding to the same control $u(t) = (u_1(t), \ldots, u_m(t))$. We denote

$$U(1) = \{u(t) = (u_1(t), \ldots, u_m(t)) : |u_i(t)| \leq 1, \ i = 1, \ldots, m, \ t \in [0,1]\}.$$

Finally, for any $u \in U(1)$, we denote by $u^{1/\theta}(t)$ the function $u^{1/\theta}(t) = u(t/\theta)$, $t \in [0, \theta]$ (i.e., $u^{1/\theta}(t)$ is obtained from $u(t)$ by "shrinking" its domain $[0,1]$ to $[0, \theta]$).

We say that a system of the form (2) is *a homogeneous approximation* of the system (1) if there exists a change of variables $y = Q(x)$ such that $Q(x^0) = 0$ and for any $u(t) \in U(1)$

$$\theta^{-w_k}\left((Q(x(\theta; u^{1/\theta})))_k - \widehat{x}_k(\theta; u^{1/\theta})\right) \to 0 \ \text{ as } \ \theta \to 0, \quad k = 1, \dots, n.$$

Informally, this means that after some change of variables trajectories of the initial system and of its approximation become equivalent at the origin for any fixed control.

Many results concerning homogeneous approximation exploited differential-geometric tools and language [3], [21], [1], [6], [2]; the results obtained within this approach were summarized in [10]. As an example of usage for a local analysis of a particular class of systems, we mention Goursat distributions [15].

Another fruitful way was initialized by M. Fliess [5]; it was based on interpreting control systems as formal series of noncommutative variables and used tools of free algebras [11], [13], [17], [18]; an overview can be found in [12]. Namely, instead of the system (1), one considers its trajectory as a *series of iterated integrals*

$$x(t; u) = x^0 + \sum \widetilde{c}_{i_1 \dots i_k} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{k-1}} u_{i_1}(\tau_1) \cdots u_{i_k}(\tau_k) d\tau_k \cdots d\tau_1,$$

where $\widetilde{c}_{i_1 \dots i_k} \in \mathbb{R}^n$ are expressed via values of the vector fields $X_i(x)$ and their derivatives at x^0. Therefore, $\widetilde{c}_{i_1 \dots i_k}$ are constant vectors. Iterated integrals are linearly independent functionals of u_i and, therefore, can be interpreted as a basis for a free associative algebra. We give more detailed explanations in the next section.

In [7], [19] a complete classification of homogeneous approximations was obtained. It turned out that a homogeneous approximation is defined by some Lie subalgebra in the free Lie algebra with m generators called *a core Lie subalgebra*, which is defined by the system. As an important benefit of the algebraic way of finding homogeneous approximations, we mention its convenience for computer realization [20].

In the present paper we study an algebraic description of homogeneous approximations for nonlinear control systems *with output*. More specifically, we consider series of iterated integrals with scalar coefficients

$$y(t; u) = y^0 + \sum c_{i_1 \dots i_k} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{k-1}} u_{i_1}(\tau_1) \cdots u_{i_k}(\tau_k) d\tau_k \cdots d\tau_1, \qquad (3)$$

where $c_{i_1 \dots i_k} \in \mathbb{R}$. The series (3) is called *realizable* if these exists a system of the form (1) and a function $y = h(x)$ such that $y(t; u) = h(x(t; u))$ admits the representation (3); it is known that the realization of the minimal possible dimension is unique up to a change of variables [9], [4], [8].

The main results of the paper can be outlined as follows. In Section 3 we show that the core Lie subalgebra of the minimal realization can be found without finding the realization itself, i.e. directly from the series (3). In Section 4 we prove the following classification theorem: any graded Lie subalgebra of finite nonzero codimension can serve as a core Lie subalgebra of a realizing system of a (homogeneous) series of the form (3).

2. Background

2.1. Realizability problem. The realizability problem for systems with output is well known. This problem deals with a description of the output behavior for analytic nonlinear control systems. Systems are represented as differential equations of the form (1) defined in some neighborhood of a point x^0, i.e., the vector fields $X_1(x), \ldots, X_m(x)$ are defined and are analytic in a neighborhood of x^0. Let us consider also a function $y = h(x)$ that is defined in a neighborhood of x^0 and is analytic there.

We recall some basic concepts of the realizability theory. First we introduce some notation.

Below we denote by M the set of multi-indices

$$M = \{I = (i_1, \ldots, i_k) : k \geq 1, \ \ 1 \leq i_1, \ldots, i_k \leq m\}.$$

One of the most important concepts in this theory is the iterated integral, which is defined as follows

$$\eta_I(\theta, u) = \int_0^\theta \int_0^{\tau_1} \cdots \int_0^{\tau_{k-1}} u_{i_1}(\tau_1) \ldots u_{i_k}(\tau_k) d\tau_k \ldots d\tau_1.$$

It can be shown [5] that for any $\theta > 0$ iterated integrals are linearly independent as functionals on the set

$$U(\theta) = \{u(t) = (u_1(t), \ldots, u_m(t)) : |u_i(t)| \leq 1, \ i = 1, \ldots, m, \ t \in [0, \theta]\}.$$

We consider the set $\{\eta_I(\theta, u) : I \in M\}$ for an arbitrary fixed $\theta > 0$. Since the functionals $\eta_I(\theta, u)$ are linearly independent, they form a basis of some linear space. Then their linear span is a free associative algebra with the concatenation operation

$$\eta_{I_1}(\theta, u)\, \eta_{I_2}(\theta, u) = \eta_{I_1 I_2}(\theta, u);$$

we denote this algebra by $\mathcal{F}_\theta$. Note that for all $\theta > 0$ the algebras $\mathcal{F}_\theta$ are isomorphic to each other. Therefore, instead of the algebras $\mathcal{F}_\theta$, it is convenient to consider an abstract free algebra $\mathcal{F}$ isomorphic to all of them, which is generated by abstract independent elements $\eta_1, \ldots, \eta_m$. Also let us consider the free Lie algebra $\mathcal{L}$ generated by $\eta_1, \ldots, \eta_m$ with the bracket operation defined by $[a, b] = ab - ba$.

Below we use a unitary algebra $\mathcal{F}^e = \mathcal{F} + \mathbb{R}$ assuming that 1 is the unit in $\mathcal{F}^e$. In order to write elements from $\mathcal{F}$ and $\mathcal{F}^e$ in the same way, we complement M by the “empty index”,

$$M_0 = M \cup \{\varnothing\}$$

and assume that $\eta_\varnothing = 1$.

Now we can formulate the realizability problem from a formal point of view. Consider an arbitrary linear map

$$c : \mathcal{F}^e \to \mathbb{R}.$$

This map corresponds to a formal series S with scalar coefficients $c_I = c(\eta_I)$

$$S = \sum_{I \in M_0} c_I \eta_I. \tag{4}$$

Below we assume that *the map c is nontrivial*, i.e., $c(\mathcal{F}) \neq \{0\}$; then the series S has at least one nonzero term except a constant.

Definition 1. *The series (4) is called realizable if there exist vector fields $X_1(x), \ldots, X_m(x)$ and a function $h(x)$, which are analytic in some neighborhood of some point x^0, such that the functional $y(\theta; u) = h(x(\theta; u))$ where $x(\theta; u)$ is a solution of the Cauchy problem*

$$\dot{x} = \sum_{i=1}^{m} X_i(x) u_i(t), \; x(0) = x^0,$$

satisfies the equality

$$y(\theta; u) = \sum_{I \in M_0} c_I \eta_I(\theta, u).$$

In this sense, (1) is *a realizing system* for (4).

To formulate a realizability criterion, we recall the following definition.

Definition 2 ([4], [8]). *Let $\mathfrak{B}$ denote the linear space of formal series of the form (4). Consider the map $F_c : \mathcal{L} \to \mathfrak{B}$ of the form*

$$F_c(\ell) = \sum_{I \in M_0} c(\eta_I \ell) \eta_I, \quad \ell \in \mathcal{L}. \tag{5}$$

The Lie rank of a series S is defined by the equality

$$\rho_L(c) = \dim \left\{ F_c(\ell) : \ell \in \mathcal{L} \right\}.$$

Now we are ready to recall the following criterion of realizability.

Theorem 1 ([4], [8]). *Suppose that the series $S = \sum_{I \in M_0} c(\eta_I) \eta_I$ satisfies the following growth conditions,*

$$|c_I| \leq C_1 |I|! C^{|I|} \tag{6}$$

with some $C, C_1 > 0$, where by $|I|$ we denote the length of the multi-index I. The series S is realizable if and only if $\rho_L(c) < \infty$. In this case $n = \rho_L(c)$ is the minimal dimension of a realizing system. Moreover, a minimal realization (i.e., a realization of the minimal dimension) is unique up to a change of variables.

In the language of the associative algebra $\mathcal{F}$, the realizability condition can be formulated as follows.

Theorem 2. *Consider the free associative algebra $\mathcal{F}$ and the corresponding Lie algebra $\mathcal{L}$. A formal series S satisfying the growth condition (6) is realizable if and only if there exist a natural number n and elements $\ell_1, \ldots, \ell_n \in \mathcal{L}$ satisfying the following condition: for any element $\ell \in \mathcal{L}$ there exist coefficients $\alpha_1, \ldots, \alpha_n$ such that*

$$c(a(\ell - \sum_{i=1}^{n} \alpha_i \ell_i)) = 0$$

for any element $a \in \mathcal{F}^e$.

One of the ways to construct a minimal realizing system for a given series S is as follows [8]. Since the Lie rank is n, there exist n linearly independent elements $\ell_1, \ldots, \ell_n \in \mathcal{L}$ for which the series $F_c(\ell_1), \ldots, F_c(\ell_n)$ are linearly independent. Consider the coefficients of all possible elements of the form $\eta_I \ell_j$. As the series $F_c(\ell_j)$ are linearly independent, there exist n multi-indices $I_1, \ldots, I_n \in M_0$ for which the matrix

$$\{c(\eta_{I_i} \ell_j)\}_{i,j=1}^{n} \tag{7}$$

is non-singular. We define the linear map $\widetilde{c} : \mathcal{F}^e \to \mathbb{R}^n$ by the equality

$$\widetilde{c}(\eta_I) = \begin{pmatrix} c(\eta_{I_1} \eta_I) \\ \cdots \\ c(\eta_{I_n} \eta_I) \end{pmatrix} \tag{8}$$

and consider the corresponding series

$$\widetilde{S} = \sum_{I \in M_0} \widetilde{c}(\eta_I) \eta_I \tag{9}$$

with n-dimensional coefficients. The unique system constructed by this series is a minimal realization of the series S.

2.2. Grading in the algebra $\mathcal{F}$ and homogeneous approximations of control systems. The free associative algebra $\mathcal{F}$ is *graded*, namely, it admits the following representation

$$\mathcal{F} = \sum_{k=1}^{\infty} \mathcal{F}^k, \quad \mathcal{F}^k = \mathrm{Lin}\{\eta_I : I \in M, \ |I| = k\}.$$

This grading is justified by the following observation, which concerns iterated integrals:

$$\int_0^\theta \int_0^{\tau_1} \cdots \int_0^{\tau_{k-1}} u_{i_1}(\tau_1) \cdots u_{i_k}(\tau_k) d\tau_k \cdots d\tau_1 =$$
$$= \theta^k \int_0^1 \int_0^{\tau_1} \cdots \int_0^{\tau_{k-1}} u_{i_1}(\tau_1 \theta) \cdots u_{i_k}(\tau_k \theta) d\tau_k \cdots d\tau_1.$$

Thus, $\eta_I(\theta, u^{1/\theta}) = \theta^{|I|}\eta_I(1,u)$, where $u^{1/\theta}(t) = u(t/\theta)$, $t \in [0,\theta]$. In this sense, $|I|$ denotes the order of $\eta_I(\theta, u^{1/\theta})$ as a function of θ as $\theta \to 0$.

The Lie algebra $\mathcal{L}$ inherits this grading,

$$\mathcal{L} = \sum_{k=1}^{\infty} \mathcal{L}^k, \quad \mathcal{L}^k = \mathcal{F}^k \cap \mathcal{L}.$$

Below we say that $a \in \mathcal{F}^k$ is *homogeneous* and k is its *order*; in this case we write $\mathrm{ord}(a) = k$.

Let us consider a series with vector coefficients of the form

$$\widetilde{S} = \sum_{I \in M_0} \widetilde{c}_I \eta_I, \tag{10}$$

where $\widetilde{c}_I \in \mathbb{R}^n$; it defines a linear map $\widetilde{c} : \mathcal{F}^e \to \mathbb{R}^n$ by $\widetilde{c}(\eta_I) = \widetilde{c}_I$. Assume that this map satisfies the Rashevsky-Chow condition

$$\widetilde{c}(\mathcal{L}) = \mathbb{R}^n. \tag{11}$$

Suppose also that the series $\widetilde{S}$ is realizable, that is, there exists a system of the form (1) such that its trajectory $x(\theta; u)$ is represented as $x(\theta; u) = \sum\limits_{I \in M_0} \widetilde{c}_I \eta_I(\theta, u)$. It can be shown that this system is unique, and the condition (11) means that the realizing system is locally controllable, i.e., the initial point x^0 belongs to the interior of the set of all points that are reachable from x^0 in a time $\theta > 0$.

The following definition takes into account the grading introduced above.

Definition 3 ([7],[19]). *Suppose the series (10) corresponds to the system (1). Let us define the subspaces*

$$\widetilde{\mathcal{P}}^1 = \{\ell \in \mathcal{L}^1 : \widetilde{c}(\ell) = 0\}, \quad \widetilde{\mathcal{P}}^k = \{\ell \in \mathcal{L}^k : \widetilde{c}(\ell) \in \widetilde{c}(\mathcal{L}^1 + \cdots + \mathcal{L}^{k-1})\}, \ k \geq 2,$$

and

$$\mathcal{L}_{X_1,\ldots,X_m} = \sum_{k=1}^{\infty} \widetilde{\mathcal{P}}^k.$$

Then $\mathcal{L}_{X_1,\ldots,X_m}$ is a graded Lie subalgebra; it is called a core Lie subalgebra of the system (1).

It can be shown that the core Lie subalgebra is of codimension n (in $\mathcal{L}$) and that it is invariant w.r.t. changes of variables in the system.

It turns out that the core Lie subalgebra is responsible for the homogeneous approximation of the system [7], [19]. Namely, two control systems of the form (1) have the same homogeneous approximation if and only if their core Lie subalgebras coincide. Moreover, any graded Lie subalgebra of codimension n is a core Lie subalgebra for some locally controllable system of the form (1).

In Section 3 we describe the core Lie subalgebra for a realizing system of a series of the form (4).

2.3. Basis in the algebra $\mathcal{F}$. Suppose $\{\ell_i\}_{i=1}^{\infty}$ is a (homogeneous) basis of $\mathcal{L}$. Then, due to the Poincaré-Birkhoff-Witt Theorem [16], the set

$$\{\ell_{i_1}^{q_1}\cdots\ell_{i_k}^{q_k} : k\geq 1,\ 1\leq i_1<\cdots<i_k,\ q_1,\ldots,q_k\geq 1\} \tag{12}$$

is a (homogeneous) basis of $\mathcal{F}$, where $\ell^q=\ell\cdots\ell$ (q times).

Let us introduce the *inner product* in $\mathcal{F}$ assuming the basis $\{\eta_I : I\in M\}$ is orthonormed. Also, let us introduce the *shuffle product* in $\mathcal{F}$ by the recursive formula

$$\eta_i ш \eta_j=\eta_{ij}+\eta_{ji},$$
$$\eta_{i_1I_1} ш \eta_j=\eta_j ш \eta_{i_1I_1}=\eta_{i_1}(\eta_{I_1} ш \eta_j)+\eta_{ji_1I_1},$$
$$\eta_{i_1I_1} ш \eta_{i_2I_2}=\eta_{i_1}(\eta_{I_1} ш \eta_{i_2I_2})+\eta_{i_2}(\eta_{i_1I_1} ш \eta_{I_2})$$

for any $I_1,I_2\in M$. Denote by

$$\{d_{i_1\ldots i_k}^{q_1\ldots q_k} : k\geq 1,\ 1\leq i_1<\cdots<i_k,\ q_1,\ldots,q_k\geq 1\} \tag{13}$$

a dual basis for (12) in the sense of the inner product introduced above. It can be shown [14] that

$$d_{i_1\ldots i_k}^{q_1\ldots q_k}=\frac{1}{q_1!\cdots q_k!}d_{i_1}^{ш q_1} ш \cdots ш d_{i_k}^{ш q_k},$$

where $d^{ш q}=d ш \cdots ш d$ (q times); here the notation $d_i=d_i^1$ is used for brevity. Therefore, we can rewrite the series S in the basis (13)

$$S=c(1)+\sum\frac{1}{q_1!\cdots q_k!}c(\ell_{i_1}^{q_1}\cdots\ell_{i_k}^{q_k})d_{i_1}^{ш q_1} ш \cdots ш d_{i_k}^{ш q_k},$$

where the sum is taken over all $k\geq 1$ and $1\leq i_1<\cdots<i_k$, $q_1,\ldots,q_k\geq 1$. In Section 4 we apply an analogous representation to the series $F_c(\ell)$.

3. Description of the core Lie subalgebras of realizing systems

In this section we show that the core Lie subalgebra of a realizing system (Definition 3) can be found without finding the realizing system itself.

Theorem 3. *Let S be a realizable series of the form (4) and an n-dimensional system (1) be its minimal realization. Then the core Lie subalgebra $\mathcal{L}_{X_1,\ldots,X_m}$ of this minimal realization can be found in the following way:*

$$\mathcal{L}_{X_1,\ldots,X_m}=\sum_{k=1}^{\infty}\mathcal{P}^k,$$

where

$$\mathcal{P}^1=\left\{\ell\in\mathcal{L}^1 : c(a\ell)=0 \text{ for any } a\in\mathcal{F}^e\right\},$$
$$\mathcal{P}^k=\Big\{\ell\in\mathcal{L}^k :\ \text{there exists } \ell'\in\mathcal{L}^1+\cdots+\mathcal{L}^{k-1} \text{ such that } c(a(\ell-\ell'))=0 \text{ for any } a\in\mathcal{F}^e\Big\},\quad k\geq 2. \tag{14}$$

Proof. Take an element ℓ from the subspace $\mathcal{P}^k$. It suffices to show that this element also belongs to the subspace $\widetilde{\mathcal{P}}^k$. Let $\ell \in \mathcal{P}^k$, then by formula (14) there exists an element ℓ' belonging to the sum of subspaces $\mathcal{L}^1 + \cdots + \mathcal{L}^{k-1}$ such that the equality

$$c(a(\ell - \ell')) = 0 \tag{15}$$

holds for any element a from $\mathcal{F}^e$. As an element a, we take those elements η_{I_i} for which the matrix (7) is nonsingular. Since equality (15) holds for any element a then it is true that

$$c(\eta_{I_i}(\ell - \ell')) = 0, \quad i = 1, \ldots, n. \tag{16}$$

Consider the n-dimensional mapping (8), then

$$\widetilde{c}(\ell - \ell') = \begin{pmatrix} c(\eta_{I_1}(\ell - \ell')) \\ \cdots \\ c(\eta_{I_n}(\ell - \ell')) \end{pmatrix}.$$

Since the condition (16) holds for any row, then $\widetilde{c}(\ell - \ell') = 0$. This means that the element ℓ belongs to the subspace $\widetilde{\mathcal{P}}^k$.

Take an element ℓ from the subspace $\widetilde{\mathcal{P}}^k$. It suffices to show that this element also belongs to the subspace $\mathcal{P}^k$. By definition, $\widetilde{c}(\ell) \in \widetilde{c}(\mathcal{L}^1 + \cdots + \mathcal{L}^{k-1})$, therefore, there exists an element $\ell' \in \mathcal{L}^1 + \cdots + \mathcal{L}^{k-1}$ such that $\widetilde{c}(\ell - \ell') = 0$. This means that $c(\eta_{I_i}(\ell - \ell')) = 0$ for $i = 1, \ldots, n$. Since the series $F_c(\ell - \ell')$ is a linear combination of the series $F_c(\ell_1), \ldots, F_c(\ell_n)$, there exist the numbers $\alpha_1, \ldots, \alpha_n$ such that for any $I \in M_0$

$$c(\eta_I(\ell - \ell')) = \sum_{j=1}^{n} \alpha_j c(\eta_I \ell_j).$$

In particular, substituting $I = I_i$, for which the matrix (7) is nonsingular, we obtain the following equality

$$\begin{pmatrix} c(\eta_{I_1}(\ell - \ell')) \\ \cdots \\ c(\eta_{I_n}(\ell - \ell')) \end{pmatrix} = \sum_{j=1}^{n} \alpha_j \begin{pmatrix} c(\eta_{I_1}\ell_j) \\ \cdots \\ c(\eta_{I_n}\ell_j) \end{pmatrix} = 0.$$

Since the matrix (7) is non-singular, the vectors $(c(\eta_{I_1}\ell_j), \ldots, c(\eta_{I_n}\ell_j))^\top$ are linearly independent. Hence, all coefficients α_j are equal to zero. This means that

$$c(a(\ell - \ell')) = 0$$

for any element $a \in \mathcal{F}^e$, therefore, $\ell \in \mathcal{P}^k$. The theorem is proved.

Example. Let a one-dimensional series

$$S = \eta_1 + \eta_{21} + \eta_{211}$$

be given. Let us show that the Lie rank of this series is equal to 2. To do this, we write down all the nonzero series of the form (5):

$$\begin{aligned} F_c(\eta_1) &= 1+\eta_2+\eta_{21}, \\ F_c([\eta_1,\eta_2]) &= -1, \\ F_c([\eta_1,[\eta_1,\eta_2]]) &= 1. \end{aligned}$$

Since two of them are linearly independent, the Lie rank of S equals 2. We can choose $\ell_1=\eta_1$, $\ell_2=[\eta_1,\eta_2]$ and $I_1=(\varnothing)$, $I_2=(2)$, then the matrix (7) is nonsingular. Then we get n-dimensional series of the form (9)

$$\widetilde{S}=\begin{pmatrix}\eta_1+\eta_{21}+\eta_{211}\\ \eta_1+\eta_{11}\end{pmatrix}.$$

Using Definition 3, let us find the core Lie subalgebra for a realization of the n-dimensional series $\widetilde{S}$. Consider the subspace

$$\widetilde{\mathcal{P}}^1=\left\{\ell\in\mathcal{L}^1:\widetilde{c}(\ell)=0\right\}.$$

We have $\mathcal{L}^1=\mathrm{Lin}\{\eta_1,\eta_2\}$. For the elements η_1,η_2 we write down their coefficients

$$\widetilde{c}(\eta_1)=\begin{pmatrix}1\\1\end{pmatrix},\quad \widetilde{c}(\eta_2)=\begin{pmatrix}0\\0\end{pmatrix}. \tag{17}$$

Then, obviously, the space $\widetilde{\mathcal{P}}^1$ is a linear span of only one element η_2

$$\widetilde{\mathcal{P}}^1=\mathrm{Lin}\left\{\eta_2\right\}.$$

For $k=2$ we get

$$\widetilde{\mathcal{P}}^2=\left\{\ell\in\mathcal{L}^2:\widetilde{c}(\ell)\in\widetilde{c}(\mathcal{L}^1)\right\}$$

and $\mathcal{L}^2=\mathrm{Lin}\{[\eta_1,\eta_2]\}$. For the element $\ell=[\eta_1,\eta_2]$ we find

$$\widetilde{c}([\eta_1,\eta_2])=\widetilde{c}(\eta_{12}-\eta_{21})=\begin{pmatrix}-1\\0\end{pmatrix}.$$

Taking into account the form of the coefficients (17), we see that $\widetilde{c}(\ell)\notin\widetilde{c}(\mathcal{L}^1)$. That is, $\widetilde{\mathcal{P}}^2=\{0\}$.

Therefore, $\dim(\widetilde{c}(\mathcal{L}^1+\mathcal{L}^2))=2$, which means that $\widetilde{\mathcal{P}}^k=\mathcal{L}^k$ for all $k\geq 3$. Thus, we have found the core Lie subalgebra for the n-dimensional series $\widetilde{S}$:

$$\mathcal{L}_{X_1,X_2}=\mathrm{Lin}\left\{\eta_2\right\}+\sum_{k=3}^{\infty}\mathcal{L}^k. \tag{18}$$

Now we show how to use Theorem 3 and find this core Lie subalgebra using only the one-dimensional series S. We write down the non-zero coefficients of this series:

$$c(\eta_1)=1,\ c(\eta_{21})=1,\ c(\eta_{211})=1.$$

Consider the subspaces (14). For $k = 1$ we have

$$\mathcal{P}^1 = \left\{ \ell \in \mathcal{L}^1 : c(a\ell) = 0 \text{ for any } a \in \mathcal{F}^e \right\}.$$

First, as an element ℓ we take η_1. In particular, for $a = 1$ we get $c(a\eta_1) = c(\eta_1) = 1$, hence, $\eta_1 \notin \mathcal{P}^1$. Now we choose $\ell = \eta_2$, then $c(a\eta_2) = 0$ for all $a \in \mathcal{F}$. This means that $\mathcal{P}^1 = \mathrm{Lin}\,\{\eta_2\}$. Now consider the subspace

$$\mathcal{P}^2 = \left\{ \ell \in \mathcal{L}^2 : \text{there exists } \ell' \in \mathcal{L}^1 \text{ such that } c(a(\ell - \ell')) = 0 \text{ for any } a \in \mathcal{F}^e \right\}.$$

As an element ℓ, we take the bracket $[\eta_1, \eta_2] = \eta_{12} - \eta_{21}$, and $\ell' \in \mathcal{L}^1$ is a linear combination $\alpha\eta_1 + \beta\eta_2$, where α, β are numbers. In the definition (14) for $k = 2$, we first take $a = 1$. Then

$$c(a(\ell - \ell')) = c(\eta_{12} - \eta_{21} - \alpha\eta_1 - \beta\eta_2) = -1 - \alpha = 0,$$

which means that $\alpha = -1$. Now we choose $a = \eta_2$, which gives

$$c(a(\ell - \ell')) = c(\eta_{212} - \eta_{221} - \alpha\eta_{21} - \beta\eta_{22}) = -\alpha = 0,$$

hence, $\alpha = 0$. We have got a contradiction, therefore, $[\eta_1, \eta_2] \notin \mathcal{P}^2$. This means that $\mathcal{P}^2 = \{0\}$. Finally, we consider the subspace

$$\mathcal{P}^3 = \left\{ \ell \in \mathcal{L}^3 : \text{there exists } \ell' \in \mathcal{L}^1 + \mathcal{L}^2 \text{ such that } c(a(\ell - \ell')) = 0 \text{ for any } a \in \mathcal{F}^e \right\}$$

and take into account that $\mathcal{L}^3 = \mathrm{Lin}\{[\eta_1, [\eta_1, \eta_2]], [\eta_2, [\eta_1, \eta_2]]\}$. First we take $\ell = [\eta_1, [\eta_1, \eta_2]] = \eta_{112} - 2\eta_{121} + \eta_{211}$ and $\ell' = \alpha\eta_1 + \beta(\eta_{12} - \eta_{21})$. Then for $a = 1$ we get

$$c(a(\ell - \ell')) = c(\eta_{112} - 2\eta_{121} + \eta_{211} - \alpha\eta_1 - \beta\eta_{12} + \beta\eta_{21}) = 1 - \alpha + \beta = 0$$

while for $a = \eta_2$ we get

$$c(a(\ell - \ell')) = c(\eta_{2112} - 2\eta_{2121} + \eta_{2211} - \alpha\eta_{21} - \beta\eta_{212} + \beta\eta_{221}) = -\alpha = 0.$$

This gives $\alpha = 0$ and $\beta = -1$, that is, $\ell' = -\eta_{12} + \eta_{21}$. One easily checks that $c(a\ell) = c(a\ell')$ for any $a \in \mathcal{F}^e$, hence, $[\eta_1, [\eta_1, \eta_2]] \in \mathcal{P}^3$. Using similar reasoning for the element $\ell = [\eta_2, [\eta_1, \eta_2]]$, we see that $c(a\ell) = 0$ for any $a \in \mathcal{F}^e$. This means that $\mathcal{P}^3 = \mathcal{L}^3$. Since $c(\mathcal{L}^k) = 0$ for $k \geq 4$, we get $\mathcal{P}^k = \mathcal{L}^k$. Thus, we have obtained the core Lie subalgebra (18) using only the initial one-dimensional series S.

One can check that a realization of the series S in a neighborhood of the point $x^0 = 0$ can be chosen in the following form

$$\begin{aligned} &\dot{x}_1 = u_1 + x_2 u_2, \\ &\dot{x}_2 = \sqrt{1 + 2x_2}\, u_1, \\ &y = x_1, \end{aligned}$$

that is, $X_1(x) = (1, \sqrt{1+2x_2})^\top$, $X_2(x) = (x_2, 0)^\top$, $h(x) = x_1$. As a homogeneous approximation for this system, we can choose a homogeneous system with the same core Lie subalgebra

$$\dot{x}_1 = u_1,$$
$$\dot{x}_2 = x_1 u_2.$$

We observe that manipulating with the series for finding the core Lie subalgebra is more convenient than with vector fields directly.

4. Description of all possible homogeneous approximations of realizing systems

In this section we show that any graded Lie subalgebra of finite nonzero codimension is the core Lie subalgebra of a realizing system of some series (3). We introduce such a series using the dual basis (13); the corresponding linear map is defined by formula (19) below. The following lemma describes one property of this map.

Lemma 1. *Suppose $\{\ell_i\}_{i=1}^{\infty}$ is a homogeneous basis of the Lie algebra $\mathcal{L}$. Let a linear map $c : \mathcal{F} \to \mathbb{R}$ be defined on the elements of the corresponding Poincaré-Birkhoff-Witt basis (12) as follows: for any $k \geq 1$ and any $1 \leq i_1 \leq \cdots \leq i_k$*

$$c(\ell_{i_1} \cdots \ell_{i_k}) = \begin{cases} 1 & \text{if } \ k = n \ \text{ and } \ (i_1, \ldots, i_n) = (1, \ldots, n), \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Consider any k-tuple $(j_1, \ldots, j_k)$ of natural numbers, where $1 \leq k \leq n$. Then

$$c(\ell_{j_1} \cdots \ell_{j_k}) = 0 \quad \text{if } 1 \leq k \leq n-1, \tag{20}$$

$$c(\ell_{j_1} \cdots \ell_{j_n}) = \begin{cases} 1 & \text{if } (j_1, \ldots, j_n) \text{ is a permutation of } \{1, \ldots, n\}, \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

Proof. Let us denote by $\mathrm{inv}(j_1, \ldots, j_k)$ the number of inversions in the tuple $(j_1, \ldots, j_k)$, i.e., the number of pairs (s', s'') such that $s' < s''$ and $j_{s'} > j_{s''}$. If $\mathrm{inv}(j_1, \ldots, j_k) = r$, then sorting the tuple in non-decreasing order requires r adjacent transpositions. Below we use the notation

$$N_{k,r} = \{(j_1, \ldots, j_k) : \mathrm{inv}(j_1, \ldots, j_k) = r\}, \quad k \geq 1, \ r \geq 0.$$

For any k the maximal possible number of inversions is $\frac{1}{2}k(k-1)$ (this number of inversions is achieved when the numbers in the tuple strictly decrease). Therefore, if $r > \frac{1}{2}k(k-1)$, then $N_{k,r} = \varnothing$. Hence, the set of all tuples of natural numbers can be represented as a union of the sets $N_{k,r}$ where $k \geq 1$, $0 \leq r \leq \frac{1}{2}k(k-1)$. We are interested in k such that $1 \leq k \leq n$.

We use induction on the set of pairs (k, r) such that $k \geq 1$, $0 \leq r \leq \frac{1}{2}k(k-1)$ ordered lexicographically. Namely, we assume

$$(k', r') < (k'', r'') \ \text{ if } \ k' < k'' \text{ or } k' = k'' \text{ and } r' < r''.$$

If $k = 1$, then the required equalities (20), (21) follow from (19).

If $2 \le k \le n$ and $(j_1, \dots, j_k) \in N_{k,0}$, then $j_1 \le \dots \le j_k$. Therefore, $\ell_{j_1} \cdots \ell_{j_k}$ belongs to the Poincaré-Birkhoff-Witt basis. Hence, equalities (20), (21) follow from (19).

Let us consider any pair (k, r) such that $2 \le k \le n$ and $1 \le r \le \frac{1}{2}k(k-1)$ and suppose that the equalities (20), (21) hold for any element $\ell_{q_1} \cdots \ell_{q_{k'}}$ where $(q_1, \dots, q_{k'}) \in N_{k',r'}$ and $(k', r') < (k, r)$. This means that $c(\ell_{q_1} \cdots \ell_{q_{k'}}) = 0$ except the case when $(k', r') = (n, r')$ and $\{q_1, \dots, q_{k'}\} = \{1, \dots, n\}$; in this case $c(\ell_{q_1} \cdots \ell_{q_{k'}}) = 1$.

Consider any $(j_1, \dots, j_k) \in N_{k,r}$. Since $r \ge 1$, there exists $1 \le s \le k-1$ such that $j_s > j_{s+1}$. Since

$$\ell_{j_s}\ell_{j_{s+1}} = [\ell_{j_s}, \ell_{j_{s+1}}] + \ell_{j_{s+1}}\ell_{j_s},$$

we can express

$$\ell_{j_1} \cdots \ell_{j_k} = a_1 + a_2,$$

where

$$a_1 = \ell_{j_1} \cdots \ell_{j_{s-1}}[\ell_{j_s}, \ell_{j_{s+1}}]\ell_{j_{s+2}} \cdots \ell_{j_k},$$
$$a_2 = \ell_{j_1} \cdots \ell_{j_{s-1}}\ell_{j_{s+1}}\ell_{j_s}\ell_{j_{s+2}} \cdots \ell_{j_k}.$$

First we consider a_1. Since the element $[\ell_{j_s}, \ell_{j_{s+1}}]$ belongs to the Lie algebra $\mathcal{L}$, it equals a linear combination of basis elements, $[\ell_{j_s}, \ell_{j_{s+1}}] = \sum \alpha_p \ell_p$, where $\alpha_p \in \mathbb{R}$. Then

$$a_1 = \sum \alpha_p \ell_{j_1} \cdots \ell_{j_{s-1}}\ell_p\ell_{j_{s+2}} \cdots \ell_{j_k},$$

where $(j_1, \dots, j_{s-1}, j_p, j_{s+2}, \dots, j_k) \in N_{k-1,r'}$ for some r'. Since $(k-1, r') < (k, r)$, we get $c(a_1) = 0$ by the induction supposition (we take into account that $k \le n$).

Therefore, $c(\ell_{j_1} \cdots \ell_{j_k}) = c(a_2)$. Obviously, $a_2 \in N_{k,r-1}$ and $(k, r-1) < (k, r)$. Hence, the equalities (20), (21) hold for the element $\ell_{j_1} \cdots \ell_{j_k}$ since, due to the induction supposition, they hold for a_2. This completes the proof of Lemma 1.

The following theorem is the main result of this section.

Theorem 4. *Let $\mathcal{L}'$ be a graded Lie subalgebra of codimension $n \ge 1$. Then there exists a one-dimensional homogeneous series of Lie rank n such that $\mathcal{L}'$ is a core Lie subalgebra of its (minimal) realization.*

Proof. Since $\mathcal{L}'$ is a graded Lie subalgebra of codimension n, we can choose homogeneous elements $\ell_1, \dots, \ell_n \in \mathcal{L}$ such that $\mathcal{L}' + \mathrm{Lin}\{\ell_1, \dots, \ell_n\} = \mathcal{L}$. Without loss of generality we assume $\mathrm{ord}(\ell_i) \le \mathrm{ord}(\ell_j)$ if $i < j$. Then choose a homogeneous basis $\{\ell_i\}_{i=n+1}^{\infty}$ of $\mathcal{L}'$ and consider the corresponding Poincaré-Birkhoff-Witt basis (12) and its dual basis (13). Introduce the series

$$S = d_1 ш \cdots ш d_n. \tag{22}$$

We note that this series corresponds to a linear map $c : \mathcal{F}^e \to \mathbb{R}$ defined by (19) and such that $c(1) = 0$.

We show that the series (22) is of Lie rank n. In fact, its Lie rank is not greater than n since the series has an n-dimensional realization, namely, the n-dimensional system corresponding to the series

$$\widetilde{S} = \begin{pmatrix} d_1 \\ \dots \\ d_n \end{pmatrix}$$

with the output $y = h(x) = x_1 \cdots x_n$. Such a system can be explicitly found as is described in [19]. It satisfies the Rashevsky-Chow condition (11) since $\widetilde{c}(\ell_i) = e_i$, $i = 1, \dots, n$. Obviously, $\widetilde{c}(\mathcal{L}') = 0$, hence, the core Lie subalgebra of this system equals $\mathcal{L}'$. Now we show that this realization is minimal.

To this end, we show that the Lie rank of the series (22) is not less than n. By definition, the Lie rank equals the dimension of the set of series of the form (5). It is convenient to re-expand the series w.r.t. the dual basis (13). Thus, the Lie rank equals the dimension of the set of series of the form

$$F_c(\ell) = c(\ell) + \sum_{j_1 < \cdots < j_k} \frac{1}{q_1! \dots q_k!} c(\ell_{j_1}^{q_1} \cdots \ell_{j_k}^{q_k} \ell) d_{j_1}^{\sqcup\!\sqcup q_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup d_{j_k}^{\sqcup\!\sqcup q_k}.$$

Now we show that the series $F_c(\ell_1), \dots, F_c(\ell_n)$ are linearly independent. For $n = 1$, there is nothing to prove. Suppose $n \geq 2$. Let us introduce the notation

$$\overline{d_1} = d_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup d_n, \quad \overline{d_n} = d_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup d_{n-1},$$
$$\overline{d_r} = d_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup d_{r-1} \sqcup\!\sqcup d_{r+1} \cdots \sqcup\!\sqcup d_n, \quad r = 2, \dots, n-1.$$

In other words, $\overline{d_r}$ is the shuffle product of all elements $d_1, \dots, d_n$ except d_r. Analogously, define

$$\overline{\ell_1} = \ell_2 \cdots \ell_n, \quad \overline{\ell_n} = \ell_1 \cdots \ell_{n-1},$$
$$\overline{\ell_r} = \ell_1 \cdots \ell_{r-1} \ell_{r+1} \cdots \ell_n, \quad r = 2, \dots, n-1.$$

Then the coefficient of $\overline{d_r}$ in the series $F_c(\ell_i)$ equals $c(\overline{\ell_r}\ell_i)$. Due to Lemma 1,

$$c(\overline{\ell_r}\ell_i) = \begin{cases} 1 & \text{if } i = r, \\ 0 & \text{otherwise.} \end{cases}$$

This means that the matrix $n \times n$ formed by the coefficients of elements $\overline{d_1}, \dots, \overline{d_n}$ in the series $F_c(\ell_1), \dots, F_c(\ell_n)$ is identity. Hence, series $F_c(\ell_1), \dots, F_c(\ell_n)$ are linearly independent, and therefore, the Lie rank of the series (22) is not less than n.

Thus, the series (22) is of Lie rank n, therefore, its minimal realization is of dimension n. As was mentioned before, this series has a realization with the core Lie subalgebra $\mathcal{L}'$. Since the minimal realization is unique up to a change of variables, the mentioned realization is minimal. The theorem is proved.

Theorem 4 has the following classification corollary close to [19].

Corollary 1. *Any graded Lie subalgebra of a finite (nonzero) codimension is a core Lie subalgebra of the minimal realization of some one-dimensional (nontrivial) series, and the dimension of this realization equals the codimension of the Lie subalgebra.*

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Article history: Received: 24 August 2022; Final form: 29 August 2022
Accepted: 24 December 2022.

How to cite this article:

D. M. Andreieva, S. Yu. Ignatovich, Homogeneous approximation for minimal realizations of series of iterated integrals, Visnyk of V. N. Karazin Kharkiv National University. Ser. Mathematics, Applied Mathematics and Mechanics, Vol. 96, 2022, p. 23–39. DOI: 10.26565/2221-5646-2022-96-02

Однорідна апроксимація мінімальних реалізацій рядів ітерованих інтегралів

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У статті розглядаються реалізовні ряди ітерованих інтегралів зі скалярними коефіцієнтами і розвивається алгебраїчний підхід до задачі однорідної апроксимації

нелінійних керованих систем з виходом. У першому розділі ми нагадуємо поняття однорідної апроксимації нелінійної керованої системи, лінійної за керуванням, та поняття ряду ітерованих інтегралів. У другому розділі наведено постановку задачі реалізовності, нагадано критерій реалізовності ряду ітерованих інтегралів та спосіб побудови мінімальної реалізації ряду. Також ми нагадуємо деякі ідеї алгебраїчного підходу до опису однорідної апроксимації: вільна градуйована асоціативна алгебра, що ізоморфна алгебрі ітерованих інтегралів, вільна алгебра Лі, базис Пуанкаре-Біркгофа-Вітта, біортогональний базис і його побудова за допомогою тасуючого добутку, означення кореневої підалгебри Лі, яка визначає однорідну апроксимацію керованої системи. У третьому розділі ми показуємо, як можна знайти кореневу підалгебру Лі системи, яка є реалізацією одновимірного ряду ітерованих інтегралів, не знаходячи самої системи. Отриманий результат проілюстровано прикладом, в якому продемонстровано два способи знаходження кореневої підалгебри Лі реалізуючої системи. В останньому розділі показано, що для будь-якої градуйованої підалгебри Лі скінченної ковимірності існує такий одновимірний однорідний ряд, що ця підалгебра Лі є кореневою підалгеброю Лі його мінімальної реалізації. Доведення є конструктивним: ми наводимо спосіб побудови такого ряду, в якому використовується біортогональний базис до базису Пуанкаре-Біркгофа-Вітта вільної асоціативної алгебри, побудований за кореневою підалгеброю Лі, і тасуючий добуток в цій алгебрі. Як наслідок, отримуємо класифікацію всіх можливих однорідних апроксимацій систем, які є реалізаціями одновимірних рядів ітерованих інтегралів.
Ключові слова: **однорідна апроксимація; ряд ітерованих інтегралів; мінімальна реалізація; коренева підалгебра Лі.**

Історія статті: отримана: 24 серпня 2022; останній варіант: 29 серпня 2022
прийнята: 24 грудня 2022.