ISSN 2221-5646(Print) 2523-4641(Online)
Visnyk of V.N.Karazin Kharkiv National University
Ser. "Mathematics, Applied Mathematics
and Mechanics"
2022, Vol. 96, p. 40–55
DOI: 10.26565/2221-5646-2022-96-03
УДК 517.929

Вiсник Харкiвського нацiонального
унiверситету iменi В.Н. Каразiна
Серiя "Математика, прикладна
математика i механiка"
2022, Том 96, с. 40–55

V. I. Korobov
D.Sc. in physics and mathematics, Prof.
Head Dep. of Applied Mathematics
V. N. Karazin Kharkiv National University
4 Svobody Sq., Kharkiv, Ukraine, 61022
valeriikorobov@gmail.com *http://orcid.org/0000-0001-8421-1718*

O. S. Vozniak
BS in applied mathematics
MS applied mathematics student
V. N. Karazin Kharkiv National University
Svobody Sq., 4, Kharkiv, Ukraine, 61022
o.vozniak0@gmail.com *http://orcid.org/0000-0001-9729-0742*

The explicit form of the switching surface in admissible synthesis problem

In this article we consider the problem related to positional synthesis and controllability function method and more precisely to admissible maximum principle. Unlike the more common approach the admissible maximum principle method gives discontinuous solutions to the positional synthesis problem. Let us consider the canonical system of linear equations $\dot{x}_i = x_{i+1}, i = \overline{1, n-1}, \dot{x}_n = u$ with constraints $|u| \leq d$. The problem for an arbitrary linear system $\dot{x} = Ax + bu$ can be simplified to this problem for the canonical system. A controllability function $\Theta(x)$ is given as a unique positive solution of some equation $\Phi(x, \Theta) = 0$. The control is chosen to minimize derivative of the function $\Theta(x)$ and can be written as $u(x) = -d\,\mathrm{sign}(s(x, \Theta(x)))$. The set of points $s(x, \Theta(x)) = 0$ is called the switching surface, and it determines the points where control changes its sign. Normally it contains the variable Θ which is given implicitly as the solution of equation $\Phi(x, \Theta) = 0$. Our aim in this paper is to find a representation of the switching surface that does not depend on the function $\Theta(x)$. We call this representation the explicit form. In our case the expressions $\Phi(x, \Theta)$ and $s(x, \Theta)$ are both polynomials with respect to Θ, so this problem is related to the problem of finding conditions when two polynomials have a common positive root. Earlier the solution for the 2-dimensional case was known. But during the exploration it was found out that for systems of higher dimensions there exist certain difficulties. In this article the switching surface for the three dimensional case is presented and researched. It is shown that this switching surface is a sliding surface (according to Filippov's definition). Also the other ways of constructing the switching surface using the interpolation and approximation are proposed and used for finding the trajectories of concrete points.

Keywords: **controllability; controllability function method; admissible maximum principle; switching surface.**

2010 Mathematics Subject Classification: 93C05; 93B05; 93B40.

1. Introduction

Let us consider the system of differential equations

$$\dot{x} = f(x, u), x \in \mathbb{R}^n, u \in \Omega \subset \mathbb{R}^r, \tag{1}$$

and let Q be a neighbourhood of the origin. Our aim is to construct a control $u = u(x), u \in \Omega$, such that the trajectory of the system

$$\dot{x} = f(x, u(x)), \tag{2}$$

starting at an arbitrary point $x_0 \in Q$, transfers into the origin in a some finite time $T = T(x_0)$. This problem is called the admissible positional synthesis problem.

One of the ways to solve it is the admissible maximum principle [6]. We consider constraints $|u| \leq d$ and the linear canonical system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \quad \ldots \\ \dot{x}_{n-1} = x_n, \\ \dot{x}_n = u. \end{cases} \tag{3}$$

In this case, the obtained control is discontinuous and takes only values $u = \pm\, d$, with trajectories of points sliding along the switching surface. The solution to this problem is known, but it is interesting to consider a problem of finding explicit form of the switching surface. It was earlier solved for the two-dimensional system[7], and in this work we extend it to the three-dimensional case.

The conditions for reaching the equilibrium point are important problems of mechanics and differential equations. Important results in this area were obtained by O. M. Lyapunov, and subsequently they became a part of the foundation of the mathematical theory of control.

The contributions to the development of the control theory were made by L. S. Pontryagin, V. G. Boltayanskii, R. V. Gamkrelidze, E. F. Mishchenko, R. Kalman, R. Bellman and many others. In particular, R. Bellman obtained the equation that must be satisfied by the solution of the optimal synthesis problem (finding the control that transfers an arbitrary point to the origin in the shortest time):

$$\min_{u \in \Omega} \left(\sum_{i=1}^{n} \frac{\partial T(t,x)}{\partial x_i} f_i(x, u) \right) = -1, \tag{4}$$

where $T(t,x)$ is a cost function and also a time needed to reach the origin.

In many cases, finding a control that is a solution to this equation is quite difficult. This is one of the reasons why V. I. Korobov introduced the problem of admissible positional synthesis. Admissibility means that the chosen control does not necessarily provide the given or the shortest time, but ensures its finiteness.

The solution of the admissible positional synthesis problem, called the controllability function method, was proposed by V. I. Korobov in [4] and later developed in many other works. This method is based on the construction of the control $u(x)$, such that for the system (2) there exists a function $\Theta(x)$ which is an analogue of the Lyapunov function in the stability theory, but also satisfies a condition which ensures finiteness of the time. More precisely the following theorem holds.

Theorem 1 ([4]). *Suppose that in the system (1) at any set of points $K_1(\rho_1,\rho_2) = \{(x,u) : 0 < \rho_1 \leq ||x|| \leq \rho_2,\ u \in \Omega\}$ the vector function $f(x,u)$ satisfies the Lipschitz continuity condition:*

$$\|f(x',u') - f(x'',u'')\| \leq L_1(\rho_1,\rho_2)(\|x'' - x'\| + \|u'' - u'\|),$$

for any $(x',u'), (x'',u'') \in K_1(\rho_1,\rho_2)$.

And suppose that there exists a function $\Theta(x)$, such that the following conditions hold:

1. *$\Theta(x) \geq 0$ if $x \neq 0$ and $\Theta(0) = 0$;*
2. *$\Theta(x)$ is continuous everywhere and continuously differentiable at any point except, perhaps, the point $x = 0$;*
3. *there exists a number $c > 0$ such that the set $Q = \{x : \Theta(x) \leq c\}$ is bounded and there exists $R > 0$ such that $Q \subset \{x : \|x\| < R\}$;*
4. *there exists a function $u(x) : Q \to \Omega$, that satisfies the inequality*

$$\dot{\Theta} = \sum_{i=1}^{n} \frac{\partial \Theta(x)}{\partial x_i} f_i(x,u(x)) \leq -\beta \Theta^{1-\frac{1}{\alpha}}(x)$$

for some $\alpha > 0$, $\beta > 0$. And $u(x)$ is Lipschitz continuous at any point of the set $K(\rho_1,\rho_2) = \{x \in Q : 0 < \rho_1 \leq \|x\| \leq \rho_2\}$, that is

$$\|u(x'') - u(x')\| \leq L_2(\rho_1,\rho_2)\|x'' - x'\|,$$

for any $x', x'' \in K(\rho_1,\rho_2)$.

Then the trajectory $x(t)$ of the system $\dot{x} = f(x,u(x))$, which starts at an arbitrary point $x \in Q$, ends at the point $x_1 = 0$ at a certain finite moment of time (which depends on x_0) $T(x_0) \leq (\alpha/\beta)\Theta^{\frac{1}{\alpha}}(x_0)$. Moreover if $\alpha = \infty$, then $x(t) \to 0$ as $t \to \infty$.

The function $\Theta(x)$ is called the controllability function. The conditions 1-3 of this theorem coincide with the conditions of Lyapunov theorem on asymptotic stability, and the condition 4 ensures the finiteness of the time for an arbitrary point to reach the origin. In the case where $\alpha = \infty$ the function $\Theta(x)$ is a Lyapunov function for the obtained system.

Also in the case when $\alpha = \beta = 1$, and instead of inequality, equality is fulfilled, i.e.

$$\sum_{i=1}^{n} \frac{\partial \Theta(x)}{\partial x_i} f_i(x, u(x)) = -1, \tag{5}$$

the controllability function is also a motion time from an arbitrary point to the origin. If, in addition, the Bellman equation is satisfied:

$$\min_{u \in \Omega} \left(\sum_{i=1}^{n} \frac{\partial \Theta(x)}{\partial x_i} f_i(x, u) \right) = \left(\sum_{i=1}^{n} \frac{\partial \Theta(x)}{\partial x_i} f_i(x, u(x)) \right) = -1, \tag{6}$$

the function $\Theta(x)$ is also an optimal time.

The function $\Theta(x)$ is naturally constructed implicitly as a solution of some equation $\Phi(x, \Theta) = 0$. It makes it different from the Lyapunov function which is constructed in explicit form. On the other hand, in the linear optimal control problem, the motion time is also found implicitly[5].

Let us consider the canonical system:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \quad \cdots \\ \dot{x}_{n-1} = x_n, \\ \dot{x}_n = u, \end{cases} \tag{7}$$

with the constraint on control $|u| \leq d$. It is a linear system $\dot{x} = A_0 x + b_0 u$, where

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad b_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \ldots \\ 1 \end{pmatrix}. \tag{8}$$

An admissible position synthesis problem for an arbitrary linear system $\dot{x} = Ax + bu$ can be simplified to this problem for the canonical system[4].

Let us describe the algorithm of constructing the control using the admissible maximum principle described in [7]. We determine the controllability function $\Theta(x)$ at an arbitrary point x as a positive root of the equation

$$\Phi(x, \Theta) = 2a_0\Theta - (D(\Theta)FD(\Theta)x, x) = 0, \tag{9}$$

(it can be proved that this root is unique at every point [7]), where F is a positive definite matrix,

$$D(\Theta) = \operatorname{diag}\left(\Theta^{-\frac{m+n-2i+1}{2\alpha}}\right)_{i=1}^{n}, \tag{10}$$

and numbers $m \in \mathbb{N}, \alpha \geq 1$ are chosen so that the matrix

$$F^{\alpha} = \left(\left(1 + \frac{m+n-i-j+1}{\alpha}\right) f_{ij}\right)_{i,j=1}^{n}$$

is positive definite. In particular, we will consider $m = n$, $\alpha = 1$. The number a_0 is chosen to satisfy the constraint on control.

The derivative $\dot{\Theta}$ of the function $\Theta(x)$ can be written in the following form:

$$\dot{\Theta} = \frac{\Theta((FA_0 + A_0^* F)y(x,\Theta), y(x,\Theta)) + 2u\Theta(D(\Theta)FD(\Theta)x, b_0)}{(F^{\alpha} y(x,\Theta), y(x,\Theta))}, \tag{11}$$

where and $y(x,\Theta) = D(\Theta)x$. Let us denote

$$s(x, \Theta(x)) = (D(\Theta(x))FD(\Theta(x))x, b_0), \tag{12}$$

that is,

$$\begin{gathered} s(x_1, x_2, \ldots, x_n, \Theta(x_1, x_2, \ldots, x_n)) = \\ = f_{n1}x_1 + f_{n2}\Theta(x_1, x_2, \ldots x_n)x_2 + ... + f_{nn}\Theta^{n-1}(x_1, x_2, \ldots, x_n)x_n. \end{gathered} \tag{13}$$

We choose the control as $u(x) = -d\,\operatorname{sign}(s(x, \Theta(x)))$ and call the set of points satisfying the equation

$$s(x, \Theta(x)) = 0 \tag{14}$$

the switching surface S.

This control gives the minimum value of the derivative $\dot{\Theta}$ of the function $\Theta(x)$ that can be obtained under given constraints. We note that this control is not continuous. It takes only boundary values and has discontinuity at points of the surface (14).

After substitution of the control to the system (7) we obtain:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \quad \ldots \\ \dot{x}_{n-1} = x_n, \\ \dot{x}_n = -d\,\operatorname{sign} s(x_1, x_2, ..., x_n, \Theta(x_1, x_2, ..., x_n)). \end{cases} \tag{15}$$

Algorithm of finding the concrete trajectory from the point x_0 to the point $x_1 = 0$ in the case when the switching surface is given by the equation (14) is the following. At the point x_0 we find a unique positive solution Θ_0 of the equation

(9) and add the equation (11) to the system (15). After that we find the trajectory $(x_1(t), x_2(t), \dots x_n(t))$ as the solution of the Cauchy problem:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \quad \dots \\ \dot{x}_{n-1} = x_n, \\ \dot{x}_n = -d \operatorname{sign} s(x_1, x_2, ..., x_n, \Theta), \\ \dot{\Theta} = \frac{2\Theta(F(\Theta)x, A_0 x) - 2d\Theta|(D(\Theta)FD(\Theta)x, b_0)|}{(F^1 y(\Theta,x), y(\Theta,x))}, \end{cases} \tag{16}$$

$$x_1(0) = x_{10}, x_2(0) = x_{20}, \dots, x_n(0) = x_{n0}, \Theta(0) = \Theta_0. \tag{17}$$

2. The explicit form of the switching surface

The formula $s(x, \Theta(x)) = 0$ gives the implicit form of the switching surface, that is, it contains the function $\Theta(x)$ as an implicit solution of the equation (9). We are considering the problem of finding the switching surface in the explicit form. Hence, we need to exclude the variable Θ from the equation for the surface.

For this let us write the equation (9) and the formula for the switching surface in the following form:

$$\Phi(x, \Theta) = 2a_0\Theta^{2n} - \sum_{i,j=1}^{n} f_{ij}\Theta^{i+j-2}x_i x_j = 0, \tag{18}$$

$$s(x, \Theta) = f_{n1}x_1 + f_{n2}\Theta x_2 + ... + f_{nn}\Theta^{n-1}x_n = 0. \tag{19}$$

One way to remove a common factor from two equations is to use the resultant. Let $x \in S$, $x \neq 0$ be a fixed point, then $\Phi(x, \Theta)$, $s(x, \Theta)$ are the polynomials of variable Θ. If $\Phi(x, \Theta)$ and $s(x, \Theta)$ have a common root, then their resultant $R(\Phi, s)$ is equal to zero. Hence, the set of all points where they have a common root can be given by the equation:

$$R(\Phi, s) = 0. \tag{20}$$

But the surface given by equation (20) is larger than the switching surface, because it also contains points where $\Phi(x, \Theta)$, $s(x, \Theta)$ have common negative root, or this root equals zero. Instead, the switching surface contains only those points where a common root $\Theta > 0$. Therefore, we have certain difficulties related to the fact that we need to find a way to separate the points where $\Theta(x) > 0$ from the entire set. Hence, further we will use the resultant only for obtaining this wider set.

As an example let us consider the process of finding switching surface for the case $n = 2$ described in [7].

Let us determine Θ with the equation

$$\Phi(\Theta, x) = \frac{2}{9}\Theta^4 - \Theta^2 x_2^2 - 2\Theta x_1 x_2 - 3x_1^2 = 0 \tag{21}$$

(the algorithm for finding such equations is described in [7]). Then the switching surface has the equation:

$$s(\Theta, x) = x_1 + \Theta x_2 = 0. \tag{22}$$

Using the formula (20) we obtain the surface given by resultant:

$$R(\Phi, s) = \begin{vmatrix} \frac{2}{9} & 0 & -x_2^2 & -2x_1x_2 & -3x_1^2 \\ x_2 & x_1 & 0 & 0 & 0 \\ 0 & x_2 & x_1 & 0 & 0 \\ 0 & 0 & x_2 & x_1 & 0 \\ 0 & 0 & 0 & x_2 & x_1 \end{vmatrix} = \frac{1}{9}x_1^4 - x_1^2 x_2^4 = 0. \tag{23}$$

To separate points where the common root of equations (21) and (22) is positive we use the fact that the equation (22) has only one root $\Theta = -\frac{x_1}{x_2}$, which is positive only when $x_1 x_2 < 0$. The part of surface (23) that satisfies this condition can be written in the form:

$$x_1 = -3x_2|x_2|. \tag{24}$$

This formula gives the equation of the switching surface. But for systems of higher dimensions, overcoming such difficulties can be more complicated. Now we give the explicit form of the switching surface in the case $n = 3$.

Let us determine the controllability function by the equation:

$$\Phi(x, \Theta) = \frac{9}{1625}\Theta^6 - 38x_1^2 - 30\frac{4}{5}x_1x_2\Theta - 4x_1x_3\Theta^2 - 6\frac{4}{5}x_2\Theta^2 - 2x_2x_3\Theta^3 - \frac{1}{5}x_3^2\Theta^4 = 0. \tag{25}$$

Then the switching surface has the form:

$$s(x, \Theta) = 10x_1 + 5\Theta x_2 + \Theta^2 x_3 = 0, \tag{26}$$

and equation defined by the resultant is as follows:

$$\begin{aligned} R(\Phi, s) = x_1^2\big(&160x_1^4 - 1625x_2^6 + 5200x_1x_2^4x_3 - 4940x_1^2x_2^2x_3^2 + 1040x_1^3x_3^2 + \\ &+ 845x_2^4x_3^4 - 2366x_1x_2^2x_3^5 + 1690x_1^2x_3^6\big) = 0. \end{aligned} \tag{27}$$

We are searching for the points where there exists a common root $\Theta > 0$. Let us show that the factor x_1^2 can be discarded. Indeed, if $x_1 = 0$ then

$$\Phi(x, \Theta) = \frac{9}{1625}\Theta^6 - 6\frac{4}{5}x_2\Theta^2 - 2x_2x_3\Theta^3 - \frac{1}{5}x_3^2\Theta^4 = 0, \tag{28}$$

$$s(x, \Theta) = 5\Theta x_2 + \Theta^2 x_3 = 0. \tag{29}$$

These polynomials always have a common root $\Theta = 0$. The second root $\Theta = \frac{-5x_2}{x_3}$ of equation (29) is also a root for (28) if:

$$\frac{1125x_2^6}{13x_3^6} - \frac{45x_2^4}{x_3^2} = 0. \tag{30}$$

That is,

$$x_2^2 = \frac{13}{25}x_3^4. \tag{31}$$

But the points $\left\{x_1 = 0, x_2^2 = \frac{13}{25}x_3^4\right\}$ are also solutions for the equation

$$\begin{gathered}160x_1^4 - 1625x_2^6 + 5200x_1x_2^4x_3 - 4940x_1^2x_2^2x_3^2 + 1040x_1^3x_3^2 + \\ + 845x_2^4x_3^4 - 2366x_1x_2^2x_3^5 + 1690x_1^2x_3^6 = 0.\end{gathered} \tag{32}$$

Hence the factor x_1^2 does not add any non-zero roots to the equation (27) compared to (32). There is also a case when $\{x_1 = 0,\ x_2 = 0,\ x_3 \neq 0\}$. Then

$$\Phi(x, \Theta) = \frac{9}{1625}\Theta^6 - \frac{1}{5}x_3^2\Theta^4 = 0, \tag{33}$$

$$s(x, \Theta) = \Theta^2 x_3 = 0. \tag{34}$$

In this case $\Phi(x, \Theta)$ and $s(x, \Theta)$ have common root $\Theta = 0$ and we do not consider it. The surface that show all other solutions for equation (32) is shown in Figure 1.

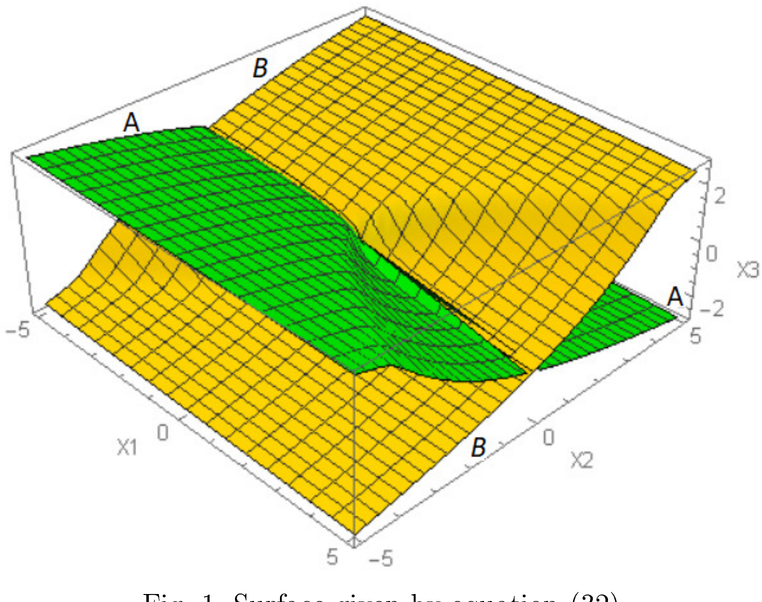

Fig. 1. Surface given by equation (32)

This surface consists of two parts. One of them (Part A) includes points where the common root $\Theta(x)$ of equations (25) and (26) is positive, and the other (Part B) includes points corresponding to the negative root and this part should be excluded.

If we find the switching surface we obtain the system of differential equations with a discontinuous right-hand side. The control $u(x)$ equals -1 above the switching surface and $+1$ below it.

Let us find the switching surface by examining the roots of the polynomial $s(x,\Theta) = 10x_1 + 5\Theta x_2 + \Theta^2 x_3$.

First, we consider the case when $x_3 = 0$. Then

$$\Phi(x,\Theta) = \frac{9\Theta^6}{1625} - \frac{34x_2^2\Theta^2}{5} - \frac{154x_1x_2\Theta}{5} - 38x_1^2, \tag{35}$$

$$s(x,\Theta) = x_2\Theta + 10x_1. \tag{36}$$

Then

$$R(\Phi, s) = x_1^2(160x_1^4 - 1625x_2^6) = 0. \tag{37}$$

The equation (37) has solutions $x_1 = 0$ and $x_1 = \pm\left(\frac{1}{2}\sqrt{5}\left(\frac{13}{2}\right)^{\frac{1}{4}}\sqrt{|x_2|^3}\right)$. Using the fact that $s(x,\Theta)$ has a positive root only when $x_1x_2 < 0$ we obtain the curve:

$$\begin{cases} x_1 + \left(\frac{1}{2}\sqrt{5}\left(\frac{13}{2}\right)^{\frac{1}{4}}\sqrt{|x_2|^3}\right)\operatorname{sign}(x_2) = 0, \\ x_3 = 0. \end{cases} \tag{38}$$

If $x_3 \neq 0$, then $s(x,\Theta)$ is a quadratic polynomial, if $5x_2^2 - 8x_1x_3 > 0$ then it has two roots $\Theta_{1,2} = \frac{-5x_2 \pm \sqrt{5}\sqrt{5x_2^2 - 8x_1x_3}}{2x_3}$. Now we are using the fact that $\Phi(x,\Theta)$ always has exactly one positive root Θ, hence, any point on the switching surface corresponds either to root Θ_1 or to root Θ_2 and we can construct parts of switching surface for this roots separately and then unite them.

By substituting the root $\Theta_1 = \frac{-5x_2 + \sqrt{5}\sqrt{5x_2^2 - 8x_1x_3}}{2x_3}$ into (25) we obtain the surface given by equation:

$$\begin{aligned} &1125x_2^6 - 2700x_1x_2^4x_3 + 1620x_1^2x_2^2x_3^2 - 144x_1^3x_3^3 - 585x_2^4x_3^4 + \\ &+ 1170x_1x_2^2x_3^5 - 468x_1^2x_3^6 + \sqrt{5x_2^2 - 8x_1x_3}\Big(-225\sqrt{5} + 360\sqrt{5}x_1x_2^3x_3 - \\ &-108\sqrt{5}x_1^2x_2x_3^2 + 117\sqrt{5}x_2^3x_3^4 - \frac{702x_1x_2x_3^5}{\sqrt{5}}\Big) = 0. \end{aligned} \tag{39}$$

The root Θ_1 is positive when $\frac{-5x_2 + \sqrt{5}\sqrt{5x_2^2 - 8x_1x_3}}{2x_3} > 0$. We can rewrite this as:

$$\begin{aligned} &\text{if } x_3 > 0 \text{ then } \left(\left(x_2 < 0 \text{ and } x_1 < \frac{5x_2^2}{8x_3}\right) \text{ or } x_1 < 0\right), \\ &\text{if } x_3 < 0 \text{ then } \left(x_2 > 0 \text{ and } \frac{5x_2^2}{8x_3} \le x_1 < 0\right). \end{aligned} \tag{40}$$

By constructing (39) only at points where these conditions hold we obtain the part A_1 of the switching surface. Similarly, considering the case of the root $\Theta_2 = \frac{-5x_2-\sqrt{5}\sqrt{5x_2^2-8x_1x_3}}{2x_3} > 0$, with conditions

$$\begin{aligned} &\text{if } x_3 > 0 \text{ then } \left(x_2 < 0 \text{ and } 0 < x_1 \le \frac{5x_2^2}{8x_3}\right), \\ &\text{if } x_3 < 0 \text{ then } \left(\left(x_2 > 0 \text{ and } x_1 > \frac{5x_2^2}{8x_3}\right) \text{ or } x_1 > 0\right), \end{aligned} \tag{41}$$

we obtain the part A_2. By combining the parts A_1, A_2, the curve (38), (purple line in Figure 2) and the point $(0,0,0)$ we get the graph of the switching surface. It also can be shown that in the neighborhood of the curve (38) the root Θ remains continuous, hence we can consider that switching surface consists of two parts, each corresponding to a separate root.

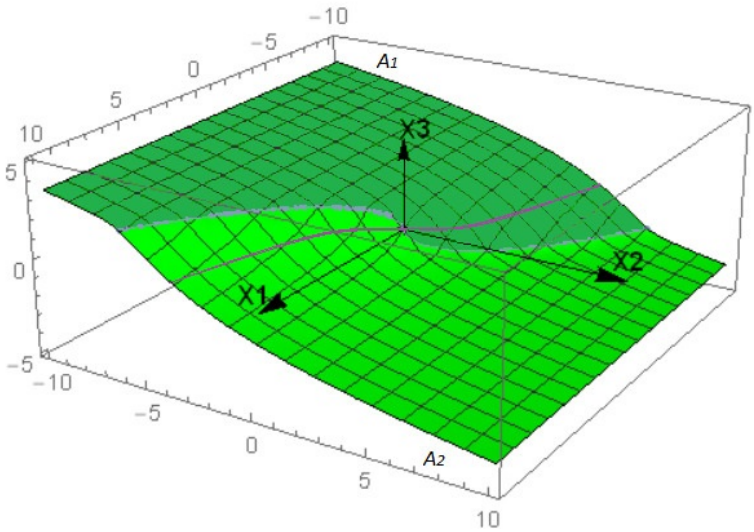

Fig. 2. Switching surface

The line separating these parts (blue line in Figure 2) consists of points where $\Theta_1 = \Theta_2 = \frac{-5x_2}{2x_3}$ and can be found explicitly. By substituting root $\Theta = \frac{-5x_2}{2x_3}$ into $\Phi(x,\Theta)$ and by using the fact that in this case $5x_2^2 - 8x_1x_3 = 0$, we can write as follows:

$$\begin{cases} 1125x_2^6 - 15860x_2^4x_3^4 + 43264x_1x_2^2x_3^5 - 31616x_1^2x_3^6 = 0, \\ 5x_2^2 - 8x_1x_3 = 0. \end{cases}$$

The solutions of the form $x_1 = 0, x_2 = 0, x_3 \neq 0$ belong to case when the common root $\Theta = 0$, all other solutions can be written as

$$x_1 = -\text{sign}(x_2)\sqrt[4]{\frac{325}{2048}}\sqrt{|x_2|^3}, \quad x_3 = -\text{sign}(x_2)\sqrt[4]{\frac{25}{26}}\sqrt{|x_2|}. \tag{42}$$

Now let us denote:

$$P_1(x_1,x_2,x_3) = 1125x_2^6 - 2700x_1x_2^4x_3 + 1620x_1^2x_2^2x_3^2 - 144x_1^3x_3^3 - 585x_2^4x_3^4 + \\ + 1170x_1x_2^2x_3^5 - 468x_1^2x_3^6,$$

$$P_2(x_1,x_2,x_3) = -225\sqrt{5} + 360\sqrt{5}x_1x_2^3x_3 - 108\sqrt{5}x_1^2x_2x_3^2 + 117\sqrt{5}x_2^3x_3^4 - \\ - \frac{702x_1x_2x_3^5}{\sqrt{5}}.$$

Hence, the switching surface is written in the form $s(x_1,x_2,x_3) = 0$, where:

$$s(x_1,x_2,x_3) = x_1 + \left(\tfrac{1}{2}\sqrt{5}\left(\tfrac{13}{2}\right)^{\frac{1}{4}}\sqrt{|x_2|^3}\right)\operatorname{sign}(x_2), \text{ if } x_3 = 0,$$

$$\begin{gathered} s(x_1,x_2,x_3) = P_1(x_1,x_2,x_3) + \sqrt{5x_2^2 - 8x_1x_3}P_2(x_1,x_2,x_3), \\ \text{if } x_1 < -\operatorname{sign}(x_2)\sqrt[4]{\tfrac{325}{2048}}\sqrt{|x_2|^3} \text{ and } (40), \end{gathered} \tag{43}$$

$$\begin{gathered} s(x_1,x_2,x_3) = P_1(x_1,x_2,x_3) - \sqrt{5x_2^2 - 8x_1x_3}P_2(x_1,x_2,x_3), \\ \text{if } x_1 \geq -\operatorname{sign}(x_2)\sqrt[4]{\tfrac{325}{2048}}\sqrt{|x_2|^3} \text{ and } (41). \end{gathered}$$

Now we show graphically that S is a sliding surface [8]. Consider an arbitrary point x on the surface S and its velocity vectors f^+ and f^- when it approaches the switching surface from above and from below respectively. And let α be a tangent plane to the surface S at the point x (Fig. 3).

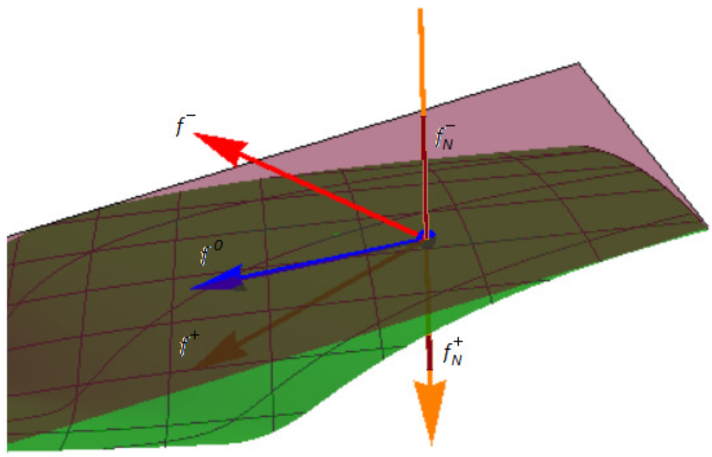

Fig. 3. Velocity vector on the switching surface

We consider

$$f_N^- = \frac{\langle \nabla s, f^- \rangle}{|\nabla s|}, \quad f_N^+ = \frac{\langle \nabla s, f^+ \rangle}{|\nabla s|}, \tag{44}$$

and build the graphs of $\tilde{f}_N^- = \langle \nabla s, f^- \rangle$ and $\tilde{f}_N^+ = \langle \nabla s, f^+ \rangle$ (Figures 4 and 5 respectively). We see that $f_N^- \leq 0$ and $f_N^+ \geq 0$ (and $f_N^- = 0$ if and only if $f_N^+ = 0$) for an arbitrary point $x \in S$. This means that at any point the velocity vectors are located on different sides of the plane α and, therefore, the resulting vector always lies in this plane.

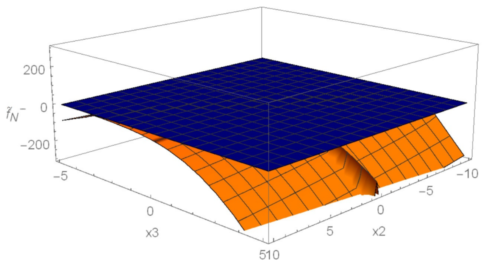

Fig. 4. Graph of $\tilde{f}_N^-$

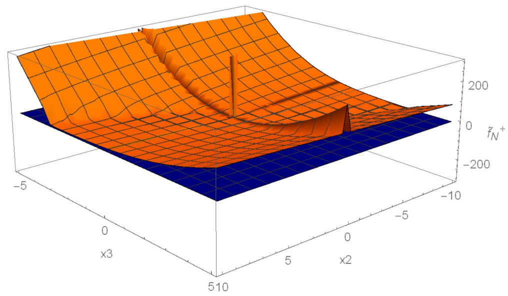

Fig. 5. Graph of $\tilde{f}_N^+$

3. Approximation of the surface

To find specific trajectories, we propose to use an approximate surface that has a simpler shape. One of the methods can be a construction with an interpolation polynomial in the form $x_3 = L(x_1, x_2)$. By substituting numbers instead of x_1, x_2 in equation $s(x_1, x_2, x_3) = 0$ and finding the solution for x_3, we can get any number of points on the switching surface. For interpolation, we select the points in such a way that they form a rectangular grid in the x_1x_2 plane. Then the interpolation polynomial is given by the formula

$$L(x_1, x_2) = \sum_{n=1}^{N} \sum_{m=1}^{M} \left(x_3(x_{1i}, x_{2j}) \prod_{i=1, i \neq n}^{N} \frac{x_1 - x_{1i}}{x_{1n} - x_{1i}} \prod_{j=1, j \neq m}^{M} \frac{x_2 - x_{2j}}{x_{2m} - x_{2j}} \right). \tag{45}$$

The approximated control $u(x)$ is given in the form: $u(x) = -\text{sign}(x_3 - L(x_1, x_2))$. The surface obtained by interpolation and the trajectory of the point $(-1, 2.5, 1)$ are shown in Figure 6.

Another method of approximation that can be used is the least-squares approximation. As an example, we choose multiples with maximal power 3 for x_1, x_2 and construct the approximating surface in the following form:

$$x_3 = w(x_1, x_2) = a_1 x_1 + a_2 x_1^2 + a_3 x_1^3 + a_4 x_2 + a_5 x_1 x_2 + \dots + a_{15} x_1^3 x_2^3, \tag{46}$$

where $a_1, a_2, \dots, a_{15}$ are unknown coefficients.

In this case, the points do not necessarily have to form a rectangular grid, so the interpolating surface can be constructed for both parts of the surface S separately (Fig. 7). In addition, if we take symmetrically located points, then the resulting parts will also be symmetrical.

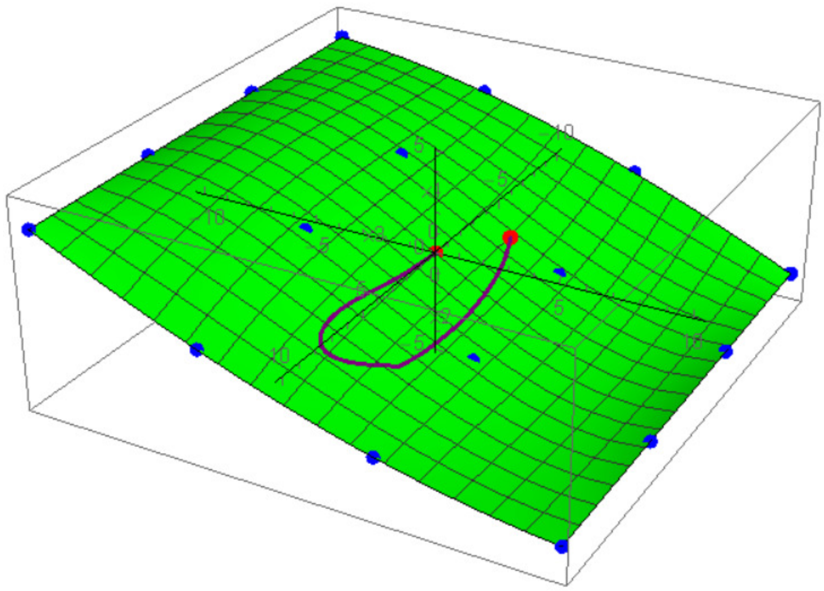

Fig. 6. Interpolating surface and the trajectory

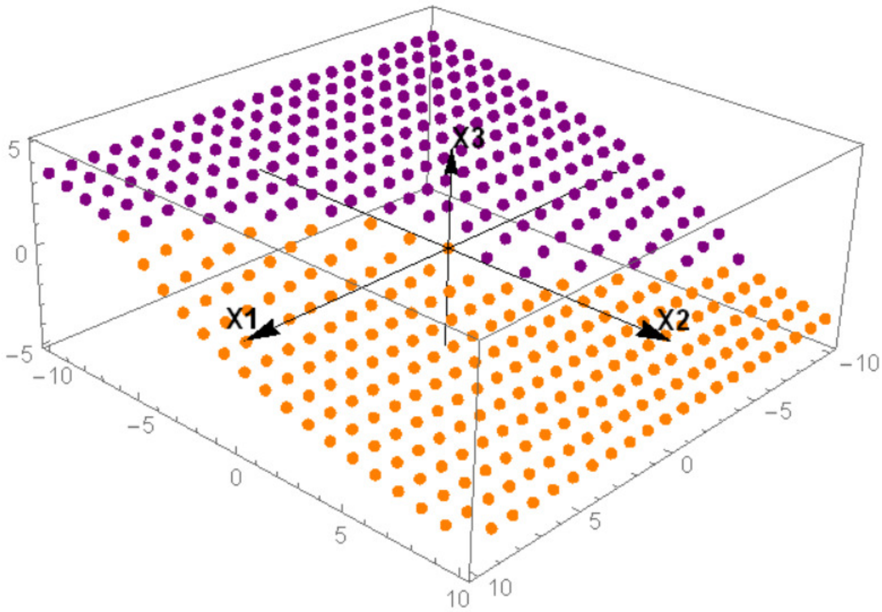

Fig. 7. Points for approximation

Numbers $a_1, a_2, \ldots, a_{15}$ are chosen to minimize the function

$$J(a_1, a_2, \ldots, a_{15}) = \sum_{i=1}^{k} \left(x_{3i} - L(x_{1i}, x_{2i})\right)^2 . \tag{47}$$

Then in our case we have:

$$w(x_1,x_2) \approx \begin{cases} -0.433897x_1 - 0.05253x_1^2 - 0.00240945x_1^3 - \\ -0.994791x_2 + 0.170404x_1x_2 - 0.0174874x_1^2x_2 - \\ -0.000655178x_1^3x_2 - 0.118976x_2^2 - 0.0222263x_1x_2^2 - \\ -0.00191042x_1^2x_2^2 - 0.0000592797x_1^3x_2^2 - 0.00572649x_2^3 - \\ -0.000934956x_1x_2^3 - 0.000068384x_1^2x_2^3 - 1.8162\cdot 10^{-6}x_1^3x_2^3 \\ \text{ if } x_1 \geq -\operatorname{sign}(x_2)\sqrt[4]{\frac{325}{2048}}\sqrt{|x_2|^3}, \\ \\ -0.433897x_1 + 0.05253x_1^2 - 0.00240945x_1^3 - \\ -0.994791x_2 - 0.170404x_1x_2 - 0.0174874x_1^2x_2 + \\ +0.000655178x_1^3x_2 + 0.118976x_2^2 - 0.0222263x_1x_2^2 + \\ +0.00191042x_1^2x_2^2 - 0.0000592797x_1^3x_2^2 - 0.00572649x_2^3 + \\ +0.000934956x_1x_2^3 - 0.000068384x_1^2x_2^3 + 1.8162\cdot 10^{-6}x_1^3x_2^3 \\ \text{ if } x_1 < -\operatorname{sign}(x_2)\sqrt[4]{\frac{325}{2048}}\sqrt{|x_2|^3}. \end{cases} \tag{48}$$

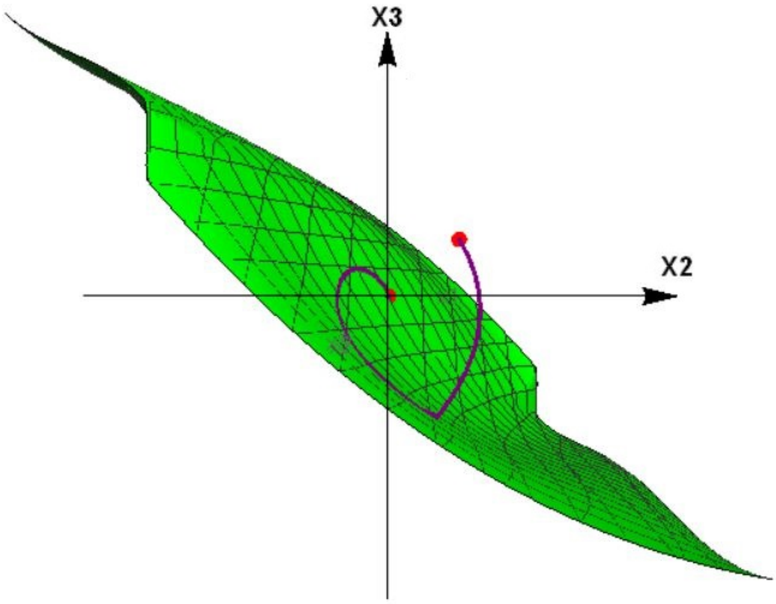

Fig. 8. Approximating surface and the trajectory

The trajectory starting at the point $(-1, 2.5, 1)$ is shown in Figure 8. We note that the question whether the concrete obtained approximating or interpolating surface is a sliding surface can be checked in the same way as for the surface S and in general this can be not true. The problem which can be considered is how to choose the interpolation nodes to obtain the sliding surface and to ensure that the trajectories reach the origin in a finite time, and if so, how much can time increase comparing to the original surface.

REFERENCES

1. A.M. Lyapunov, The General Problem of the Stability of Motion. - 1892. Kharkov Mathematical Society, Kharkov. English translation: The general problem of the stability of motion (trans. A.T. Fuller), in International Journal of Control, 55:3, 531-573. 1992 DOI: 10.1080/00207179208934253

2. L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, E. F. Mishechenko, The Mathematical Theory of Optimal Processes. VIII + 360 S. New York/London 1962. John Wiley & Sons. Preis 90/-. DOI: 10.1002/zamm.19630431023

3. R. Kalman, Y. Ho, K. Narendra, Controllability of linear dynamical systems. Contributions to Differential Equations 1. - 1963, 189-213.

4. V. I. Korobov. A general approach to the solution of the bounded control synthesis problem in a controllability problem, Math. USSR Sb. 37. - 1980. - No. **4**, 535-557. DOI: 10.1070/SM1980v037n04ABEH002094

5. V. I. Korobov, G. M. Sklyar. Time optimality and the power moment problem. Mat. Sb., Nov. Ser. 134(176). - 1987. no. 2(10), 186-206; Math. USSR, Sb. 62. - 1989. No. 1, 185-206. DOI: 10.1070/SM1989v062n01ABEH003235

6. V. I. Korobov, G. M. Sklyar. Methods for constructing of positional controls and an admissible maximum principle, Differ. Equations 26(1990), Vol. 11, 1422-1431.

7. V.I. Korobov, Method of controllability function, R&C Dynamics, Moskow-Ijevsk, 2007. ISBN 978-5-93972-610-8.

8. A. F. Filippov, Differential equations with discontinuous right-hand side, Mat. Sb. (N.S.), 51(93):1 (1960), 99-128.

Article history: Received: 28 October 2022; Final form: 19 December 2022
Accepted: 24 December 2022

How to cite this article:

V. I. Korobov, O. S. Vozniak, The explicit form of the switching surface in admissible synthesis problem, Visnyk of V. N. Karazin Kharkiv National University. Ser. Mathematics, Applied Mathematics and Mechanics, Vol. 96, 2022, p. 40–55. DOI: 10.26565/2221-5646-2022-96-03

Явний вигляд поверхні перемикання в задачі допустимого позиційного синтезу

Коробов В. І., Возняк О. С.

Харківський національний університет імені В. Н. Каразіна

61022, м. Харків, майд. Свободи, 4

В цій статті розглядається проблема, пов'язана із задачею допустимого позиційного синтезу та методом функції керованості, а саме, з допустимим принципом максимуму. На відміну від більш звичого підходу, допустимий принцип максимуму дає розривний розв'язок задачі синтезу. Нехай задана канонічна керована система $\dot{x}_i = x_{i+1}, i = \overline{1, n-1}, \dot{x}_n = u$ з обмеженнями на керування вигляду $|u| \leq d$. Задача синтезу для довільної лінійної системи вигляду $\dot{x} = Ax + bu$ може бути зведена до канонічної. Функція керованості $\Theta(x)$ задана як єдиний додатний розв'язок деякого рівняння $\Phi(x, \Theta) = 0$. Керування обирається таким чином, щоб мінімізувати похідну функції $\Theta(x)$ за часом в кожній точці, і воно може бути записано у вигляді $u(x) = -d\ \mathrm{sign}(s(x, \Theta(x)))$. Множина точок, що задовольняє рівності $s(x, \Theta(x)) = 0$, називається поверхнею перемикання і визначає точки, де керування змінює свій знак. Зазвичай вона включає змінну Θ, що є неявним розв'язком рівняння $\Phi(x, \Theta) = 0$. В цій роботі ми шукаємо явне представлення поверхні перемикання, тобто таке, що не включає зміної Θ. В нашому випадку вирази $\Phi(x, \Theta)$ та $s(x, \Theta)$ є поліномами відносно Θ, тому задача пов'язана з задачею знаходження умов при яких два поліноми мають спільний додатний корінь. Раніше було відомо рішення для 2-вимірного випадку. Але в ході дослідження з'ясувалося, що для систем більшої розмірності існують певні труднощі. У цій статті представлено та досліджено поверхню перемикання для тривимірного випадку. Також показано, що ця поверхня перемикання є поверхнею ковзання (згідно з визначенням Філіппова). В роботі також запропоновані інші способи побудови поверхні перемикання за допомогою інтерполяції та апроксимації. Ці способи застосовано для знаходження траєкторій конкретних початкових точок.

Ключові слова: **керованість; метод функції керованості; допустимий принцип максимуму; поверхня перемикання.**

Історія статті: отримана: 28 жовтня 2022; останній варіант: 19 грудня 2022
прийнята: 24 грудня 2022.