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Control of wheeled platforms straight motions taking into account jerk restrictions under speeding-up from the state of rest

The generalized mathematical model of wheeled platforms straight motions on the ideal horizontal plane under speeding-up from the state of rest mode is proposed, and the controls satisfying the restrictions of motion jerks are find. The pure mechanical and electromechanical wheeled platforms are considered, as well as the computer simulations of the researched processes are made. The jerks restrictions are reduced to limiting the value of the wheeled platform acceleration time derivative. The proposed approaches are based on the holonomic systems mechanics and on the electromechanical analogies allowing to consider the different kinds of the wheeled platforms taking into account the electric on-board systems like the drive electric motors and the control systems by using the Lagrange equations of second kind. The examples of the proposed approaches using to define the controls satisfying the jerks restrictions under speeding-up from the state of rest are considered for the pure mechanical and electromechanical wheeled platforms. It is obtained the inequality allowing to chose the instantly supplied driving mechanical couple which will provide the admissible jerks of the motion of the wheeled platform under speeding-up from the state of rest. It is shown that the rolling friction and the viscous damping are the principal causes of the wheeled platforms jerks under speeding-up from the state of rest. It is obtained the inequality defining the voltage instantly supplied on the drive electric motors which will provide the admissible jerks of the motion of the electromechanical wheeled platform during speeding-up from the state of rest, and it is shown that the proposed general approaches are suitable for considering the different kinds of wheeled platforms. The computer simulations of the processes of speeding-up from the state of rest for the electromechanical wheeled platform are considered to show results correctness and to illustrate satisfying the restrictions of the motion jerks. The obtained results of the computer simulations are in the full agreement with the well-known fundamental property inherent for the wheeled platforms. The results for the jerks show that the maximum value of the jerk is really at the initial time as was suggested before, and it is shown that the jerks values at the initial time obtained by using the computer simulations are in full agreement with the theoretically defined correspondent exact values. The big jerks of the considered electromechanical wheeled platform are due to the voltage instantly supplying on the drive electric motors at the initial time, and it is understandable that limiting of such instantly supplied voltage value cannot provide any wished small jerks. The smooth time depending for the voltages supplying on the drive electric motors are required to provide any wished small jerks of the electromechanical wheeled platforms.

Keywords: **control; motion; jerk; wheeled platform; mathematical modelling.**

2010 Mathematics Subject Classification: 49K15; 70E60; 70E55.

1. Introduction

Different kinds of wheeled platforms are widely used for human operated transportation systems, but last times it is existed the trend in using them also as the carriers of the different autonomous mobile transportation and technological systems for industrial, military, police, agriculture and house holding purposes. The motions jerks can limit the implementing possibilities of the autonomous wheeled platforms and other robotic systems for automated executing of some kinds of operations. Du to this circumstance, restricting the motions jerks is in current interest problem necessary to increase the operational quality and possibilities of implementing of wheeled platforms [1] and of different kinds of robotic systems. The theme of the proposed research deals with the particular problems about control of wheeled platforms straight motions taking into account jerk restrictions under speeding-up from the state of rest, and this theme is in current interest, because of it is in agreement with the existed general trends in developing the robotic systems directed to extensions of their implementing.

First principal reason for motions jerks limiting is due to the requirements of motion smooth necessary for normal operating of different kinds of robotic systems [2], [3]. The motions smoothness and excluding the jerks can be required for example for delicate or dangerous cargoes transportation [4] as well as for providing the most accurate relative positioning of technological systems parts [5].

It is necessary to note that excluding the motions jerks requires implementing the mechanisms special designs [6], as well as implementing the special control algorithms [4], [5]. So, excluding the wheeled platforms motions jerks is the multidisciplinary problem, and it requires the corresponded developing both the mechanical design both the control systems which must be corresponded with the existed imperfections of the mechanical joints due to the friction and the clearances.

Second principal reason for limiting the motions jerks is due to the motion smooth requirements necessary to provide the normal operation conditions for the sensitive components of on-board measuring systems [7], [8], including the sensors and the complementary electronic devices like analog-to-digital converters and computers for real time processing of the measured signals. Really, motions jerks have influence on on-board sensors like accelerometers or tachometers, and this influence is equivalent to noises disturbing measured signals used for positioning and defining current state parameters like velocities and accelerations [8]. Due to these circumstances, the motions jerks can lead to failures in positioning, in velocities and accelerations defining and in control of planed paths. As the result of all these, normal operation can be broken, and, furthermore, a lot of different dangerous can be created especially in using the fully autonomous wheeled platforms. So, defining the admissible motions jerks providing the normal operation of the wheeled platforms taking into account influencing on on-board measuring systems is the complicated problem required multidisciplinary approaches providing opportunities to consider the interactions between the mechanical, electromechanical and electronic parts [4], [8]. It is naturally that the motions jerks are associated with the accelerations and their changes like was discussed in the research [4] for example, so the quantitative measures of the motions jerks are based on using accelerations and their first and higher derivatives [9]. At the same time, the mechanical motions are represented by the differential equations of second orders, so researching the accelerations derivatives is the special separate problem [10].

To research the wheeled platforms motions jerks it is necessary to have some general methodology which will allow considering different causes leading to the jerks. There are a lot of causes leading to the wheeled platforms motions jerks [1], and it is necessary to research all of them, but it is the complicated problem not for one research. It is well-known [1] that the jerks are inherent especially for transient modes of wheeled platforms motions. Thus, the purpose of this research is in considering the particular problem about control of wheeled platforms straight motions on the ideal horizontal plane taking into account jerk restrictions under speeding-up from the state of rest. It is understood that the speeding-up is the particular case of transient modes of wheeled platforms, and jerks will be necessarily presented on this mode. Choosing the state of rest as the initial state is to simplify formulating the initial conditions, and such simplification is suitable for obtaining the primary results for planning the further researches in the field of the motion control under jerks restrictions. To realize the purpose of the research the follows tasks will be considered:

• the generalized approaches to define the controls satisfying the straight motions jerks restrictions of wheeled platforms will be developed for the speeding-up from the state of rest modes;

• the examples of the proposed approaches using to define the permissible controls satisfying the jerks restrictions under speeding-up from the state of rest will be considered for the pure mechanical and electromechanical wheeled platforms;

• computer simulations of the processes of speeding-up from the state of rest will be executed for the electromechanical wheeled platforms to show the results correctness and to illustrate satisfying the restrictions of the motions jerks.

Developing all noted above tasks will allow giving the clear imaginations about the proposed generalized approaches and their using in the important particular cases, as well as it will allow illustrating the influence of the researched control processes on the motions jerks for the wheeled platforms under speeding-up from the state of rest.

2. Generalized approaches

Developing the generalized approaches is more suitable than developing the particular approaches for each particular task. The generalized approaches to define the controls satisfying the wheeled platforms jerks restrictions under speeding-up modes from the state of rest are reduced to mathematical modelling of the researched modes and to resolving the formulated restrictions. The mathematical modelling of the wheeled platforms speeding-up modes will be considered under the most generalized assumptions that the researched wheeled platforms can be reduced to the holonomic systems. It is really the serious simplification because of the nonholonomic constraints are inherent for the wheeled platforms in general, but we have the hope that considering the particular case of the straight motions under speeding-up modes from the state of rest allow reducing to the holonomic systems.

It is well-known [11], [12] that the state of the holonomic systems can be defined by using the generalized coordinates:

$$q_k = q_k(t), k = 1, 2, .., N, \tag{1}$$

where $q_k, k = 1, 2, .., N$ are the generalized coordinates; N is the number of the freedom degrees of the holonomic system; $t \geq 0$ is the time.

It is necessary to note that not all generalized coordinates (1) will have the mechanical sense like linear displacements or angles, and some of these coordinates (1) can have the electrical sense like the electrical charges in the case of the electromechanical wheeled platforms. The translational straight motions of the wheeled platform can be imagined as the motions of its mass center, and it can be represented in the natural coordinates, so that we will have for the holonomic system the follows relation:

$$s = s(q_1, q_2, ..., q_N), \tag{2}$$

where s is the length of the arc of the trajectory of the mass center of the considered wheeled platform.

It is not unexpectedly to define the jerk as the time derivative of the acceleration and as the time third derivative of the coordinate:

$$j = \frac{d^3 s}{dt^3}, \tag{3}$$

where j is the estimation of the jerk of the motion of the considered wheeled platform.

Taking into account the used estimation of the motion jerk (3) and the relations (2), (1), we will have the follows:

$$j(t) = \sum_{k=1}^{N} \left(\sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^3 s}{\partial q_k \partial q_i \partial q_j} \frac{dq_k}{dt} \frac{dq_i}{dt} \frac{dq_j}{dt} + 3 \sum_{i=1}^{N} \frac{\partial^2 s}{\partial q_k \partial q_i} \frac{dq_k}{dt} \frac{d^2 q_i}{dt^2} + \frac{\partial s}{\partial q_k} \frac{d^3 q_k}{dt^3} \right). \tag{4}$$

Relation (4) shows that the jerks of the translational motions of the wheeled platforms are depended on the generalized velocities, generalized accelerations and the generalized accelerations time derivatives as well as on the building of the wheeled platform.

The Lagrange equations of second kind give us one of the most general form of the differential equations of dynamics of holonomic systems representing the different kinds of wheeled platform under the different operational modes:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = -\frac{\partial \mathcal{R}}{\partial \dot{q}_k} + Q_k, \;\; k = 1, 2, .., N, \tag{5}$$

where $\mathcal{L}$ is the Lagrange function defined as difference between the kinetic and potential energies of the considered wheeled platform; $\dot{q}_k \equiv dq_k/dt$; $\mathcal{R}$ is the generalized Raleigh function defining all the dissipation for the considered wheeled platform; Q_k are the generalized forces corresponding with the relevant generalized coordinates and defining all the driving forces and couples of the considered wheeled platform.

The equations (5) are the differential equations of second order, so the assumption about the initial state of rest for the considered wheeled platform allows formulating the initial conditions:

$$q_k(0) = 0, \;\; \dot{q}_k = 0, \;\; k = 1, 2, .., N. \tag{6}$$

Thus, the differential equations (5) with the initial conditions (6) generally represent the mathematical model of motion from the state of rest of the wheeled platform considered under the restrictions leading to the correspondent holonomic system with the generalized coordinates (1).

Taking into account the purpose of the research, we will consider further the transient modes from the initial state (6) to some state of uniform motion with the relative small velocity allowing the linearization of the differential equations (5) of the dynamic of the wheeled platform which is considered as the holonomic system.

Such linearization will allow represent the Lagrange $\mathcal{L}$ and Raleigh functions $\mathcal{R}$ in the follows form:

$$\mathcal{L} = \frac{1}{2}\sum_{k=1}^{N}\sum_{i=1}^{N} m_{ki}\dot{q}_k\dot{q}_i - \frac{1}{2}\sum_{k=1}^{N}\sum_{i=1}^{N} c_{ki}q_k q_i, \tag{7}$$

$$\mathcal{R} = \sum_{k=1}^{N} f_k\dot{q}_k + \frac{1}{2}\sum_{k=1}^{N}\sum_{i=1}^{N}\beta_{ki}\dot{q}_k\dot{q}_i, \tag{8}$$

where m_{ki} and c_{ki} are the generalized inertia and stiffness constant parameters of the considered wheeled platform parts; f_k are the parameters defining the non-viscous frictions not depending on the velocities; β_{ki} are the generalized damping parameters satisfying the conditions: $\beta_k \geq 0$, $\beta_{ki} = \beta_{ik}$ and $\sum_{k=1}^{N}\sum_{i=1}^{N}\beta_{ki}\dot{q}_k\dot{q}_i \geq 0$ and defining the linearized viscous damping.

It is naturally to imagine that motion control of the wheeled platforms is realized thru the driving generalized forces. We will assume that the control of the wheeled platform can be reduced to one time depended function:

$$u = u(t), \tag{9}$$

where u is the parameter defining the control influence on the considered wheeled platform.

The assumption (9) limits the possible class of the considered wheeled platforms, but this theoretically limited class can represent the most of actually existed and widely used wheeled platforms. Really, each wheeled platform has the energy source, the transmission as well as the drive and supporting wheels, so that the state of the energy source naturally defines the state of the wheeled platform. Although, the physical essentials of the power produced by the energy source is significantly depended on the type and on the design of the energy source, but it is more principally for us to define the state of the energy source by the power supplied to the transmission to move the drive wheels of the wheeled platform. Due to the noted here circumstances, the assumption (9) seems as the natural because of we have only one principal parameter defining the state of the considered wheeled platform and this parameter is the power supplied from the energy source to the transmission. Of course, the supplied power can be defined by other parameters like the torque, the position of the fuel valve or the voltage supplied to drive electric motors. Exactly, the noted case is the typical for the most of existed and used wheeled platforms. Considering the transient modes from the initial state (6) to some state of the motion with the relative small velocity is in agreement with the purposes of this research, and it allows linearization of the differential equations (5) of the dynamics of the wheeled platform which is considered as some holonomic system. Thus, the driving generalized forces can be represented taking into account the assumption (9) in the follows linearized view:

$$Q_k = \sum_{i=1}^{N}\alpha_{ki}\dot{q}_i + b_k u(t), k = 1, 2, .., N, \tag{10}$$

where α_{ki} are the parameters defining the linearized velocity depending of the driving generalized forces, but b_k are the constant parameters characterizing the sensitivity of the control of the considered wheeled platform.

Taking into account the relations (7), (8) and (10) in the Lagrange equations of second kind (5), we will have the follows linearized differential equations representing the dynamics of the considered wheeled platform:

$$\sum_{i=1}^{N} m_{ki}\ddot{q}_i = -\sum_{i=1}^{N} c_{ki}q_i - \sum_{i=1}^{N} d_{ki}\dot{q}_i - f_k + b_k u(t), k = 1, 2, .., N, \tag{11}$$

where $d_{ki} = \beta_{ki} - \alpha_{ki}$.

Further, it will be suitable to have the vector-matrix representation of the differential equations (11), and to have this representation, we will introduce the follows vectors and matrices:

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{pmatrix}, \bar{\mathbf{f}} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{pmatrix}, \bar{\mathbf{b}} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix},$$
$$\mathbf{M} = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1N} \\ m_{21} & m_{22} & \cdots & m_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ m_{N1} & m_{N2} & \cdots & m_{NN} \end{pmatrix}, \mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1N} \\ c_{21} & c_{22} & \cdots & c_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ c_{N1} & c_{N2} & \cdots & c_{NN} \end{pmatrix}, \tag{12}$$
$$\mathbf{D} = \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1N} \\ d_{21} & d_{22} & \cdots & d_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ d_{N1} & d_{N2} & \cdots & d_{NN} \end{pmatrix}.$$

The introduced above vectors and matrices (12) allow representing the differential equations (11) and the initial conditions (6) in the suitable vector-matrix form:

$$\mathbf{M}\ddot{\mathbf{q}} = -\mathbf{C}\mathbf{q} - \mathbf{D}\dot{\mathbf{q}} - \bar{\mathbf{f}} + \bar{\mathbf{b}}u(t), \ \mathbf{q}(0) = \mathbf{0}, \ \dot{\mathbf{q}}(0) = \mathbf{0}, \tag{13}$$

where $\mathbf{0}$ is the zero vector having the correspondent dimension.

Solving the initial-value problem (13) will give the opportunities to find the jerks (4) corresponded to the given control (9), so in the form (13) we have the mathematical model of the considered wheeled platform representing its dynamical properties which must be taken into account to design the controls satisfying the motions jerks restrictions. We will consider further one of the principal kinds of the control (9) defined by the constant:

$$u(t) = u_c, \tag{14}$$

where $u_c > 0$ is the given constant corresponded to some quasi-stationary mode of the motions of the considered wheeled platform characterized by the constant velocity of its mass center.

Considering the particular case (14) of the control (9) is really very important from the point of view on designing the control of wheeled platforms speeding-up from the state of rest taking account the motion jerks restriction. Really, the motions defined by the differential equations and the initial conditions (13) for the control (14) represent the transient characteristics of the considered wheeled platform, and exactly these transient characteristics define the transient processes including the jerks during speeding-up of the wheeled platform from the state of rest. It is naturally to assume that the maximum jerks of the wheeled platform are at the beginning of the motions, because of exactly in this moment the motion is created from the state of rest, and further we will have only increasing of the velocity of the already existed motion, until this velocity will achieve the steady value, corresponded to the control (14). Taking into account the initial conditions (6), the relation (4) allows defining the wheeled platform jerk at the initial time of the speeding-up process:

$$j(0) = j_0,\ j_0 = \sum_{k=1}^{N} j_k \tfrac{d^3 q_k}{dt^3}(0), \tag{15}$$

where j_0 is the jerk at the initial time; $j_k = \frac{\partial s}{\partial q_k}\Big|_{\substack{q_i=0\\ i=1,2,\ldots,N}}$.

To restrict the jerks of the considered wheeled platform it is naturally to limit the initial jerk (15):

$$|j_0| \leq [j]\,, \tag{16}$$

where $[j] \geq 0$ is the admissible jerk of the considered wheeled platform.

Considering the transient process (13) during the wheeled platform speeding-up for the control (14) will allow defining the control satisfying the jerk restriction (16), but to do this it is principally more suitable to represent the mathematical model (13) representing the considered wheeled platform in the form of the system of first ordered differential equations. To represent the second ordered differential equations (13) as the system of the first ordered differential equations we will introduce the follows phase state space:

$$x_1 = q_1,\ x_2 = q_2, \ldots,\ x_N = q_n,\ x_{N+1} = \dot{q}_1,\ x_{N+2} = \dot{q}_2, \ldots,\ x_{2N} = \dot{q}_n, \tag{17}$$

where $x_k, k = 1, 2, ...2N$ are the phase coordinates.

It is suitable to represent the phase coordinate (17) as the vector:

$$\mathbf{x} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^T, \tag{18}$$

where $n = 2N$ is the dimension of the state phase space and T is the transpose operation symbol.

The introduced vector (18) and the assumption (14) about the control allow representing the differential equations and the initial conditions (13) in the follows suitable form:

$$\tfrac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} - \mathbf{f} + \mathbf{b}u_c,\ \mathbf{x}(0) = \mathbf{0}, \tag{19}$$

where $\mathbf{A}$ is some matrix, $\mathbf{f}$ and $\mathbf{b}$ are some vectors; $\mathbf{0}$ is the zero vector.

Comparing the equation (19) and the equation (13) allow us to write the matrix $\mathbf{A}$ and the vectors $\mathbf{f}$ and $\mathbf{b}$ included in the equation (19):

$$\mathbf{A} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{C} & -\mathbf{M}^{-1}\mathbf{D} \end{pmatrix}, \mathbf{f} = \begin{pmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\bar{\mathbf{f}} \end{pmatrix}, \mathbf{b} = \begin{pmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\bar{\mathbf{b}} \end{pmatrix}, \tag{20}$$

where $\mathbf{O}$ and $\mathbf{I}$ are the zero and unit matrix, but $\mathbf{0}$ is the zero vector of the correspondent dimensions.

Taking into account the introduced above vector (17), (18), the initial jerk (15) of the wheeled platform can be represented in the follows view:

$$j_0 = \mathbf{j}\frac{d^2\mathbf{x}}{dt^2}(0),\ \mathbf{j} = \left(\underbrace{0 \quad 0 \quad \cdots \quad 0}_{N} \quad j_1 \quad j_2 \quad \cdots \quad j_N \right). \tag{21}$$

Solution of the initial-value problem (19), (20) and the relation (21) allow finding the initial jerk j_0 required for the jerk restriction (16) of the considered wheeled platform. Really, the solution of the problem (19) can be represented in the follows form:

$$\mathbf{x}(t) = \left(e^{\mathbf{A}t} - \mathbf{I}\right)\left(\mathbf{A}^{-1}\left(\mathbf{b}u_c - \mathbf{f}\right)\right). \tag{22}$$

The solution (22) and the relation (21) allow finding the initial jerk of the motion for the considered wheeled platform:

$$j_0 = \mathbf{j}\mathbf{A}\left(\mathbf{b}u_c - \mathbf{f}\right). \tag{23}$$

Relation (23) and the the restriction (16) will allow defining the control (14) and representing this control thru the primary linearized differential equations (13). To do this, it is necessary to take into account the relations (20) and (21), so the result of all these will lead to the restriction of the control (14) in the follow view:

$$\left|\bar{\mathbf{j}}\mathbf{M}^{-1}\mathbf{D}\mathbf{M}^{-1}\bar{\mathbf{f}} - \left(\bar{\mathbf{j}}\mathbf{M}^{-1}\mathbf{D}\mathbf{M}^{-1}\bar{\mathbf{b}}\right)u_c\right| \leq [j], \tag{24}$$

where $\bar{\mathbf{j}} = \begin{pmatrix} j_1 & j_2 & \cdots & j_N \end{pmatrix}$

The relation (24) is actually gave the restriction of the considered wheeled platform control (14) providing speeding-up from the state of rest under the limited motion jerks. We can see from the relation (24) that the jerks can be only due to existing the linear dissipative and gyroscopic generalized forces, because the zero matrix $\mathbf{D}$ allows satisfying the jerk restriction (24) for any control (14). These dissipative forces are usually the result of the aerodynamic and hydrodynamic frictions; the Coriolis forces are the example of the gyroscopic forces.

The constant generalized forces of the wheeled platform are represented by the vector $\bar{\mathbf{f}}$ and are had the significant influencing on the motions jerks. These constant generalized forces are usually for example the gravity forces acting on the wheeled platforms moved on the inclining road or the rolling friction couples of the wheels interacting with the soil.

3. Examples

The developed approaches reduced to the inequality (24) for control of the straight motion under speeding-up from the state of rest mode taking into account the jerks restrictions can be used for different kinds of the wheeled platforms. Further, we will illustrate the mechanics foundations of the developed approaches as well as we will consider the particular application of these developed approaches deals with the control of autonomous electromechanical wheeled platform.

Example 1. The simple schematization (fig. 1a) of the four-wheeled platform will be considered firstly to illustrate the mechanical foundations of the proposed approaches reduced to the inequality (24). This schematization (fig. 1a) is based on the assumption (1) about the generalized coordinates, and in this particular case it will be assumed that the straight motion of the considered four-wheeled platform can be defined by one generalized coordinate q_1 representing the rotation angle of its wheels, so the straight motion can be defined as follows (fig. 1a):

$$s = q_1 r, \tag{25}$$

where s is the linear coordinate defining the straight motion; q_1 is the rotation angle and r is the radius of the wheels of the considered platform.

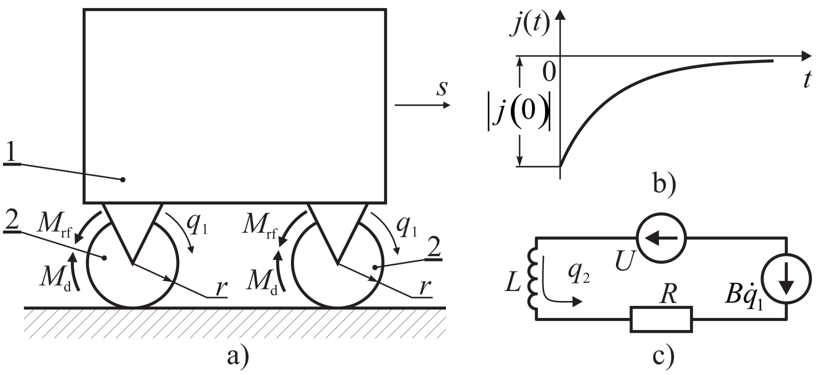

Fig. 1. Schematizing of the four-wheeled platform (a) with the housing-1 and the wheels-2, as well as the result for the jerk (b) of this platform and the equivalent scheme of the drive electric motors (c)

The relation (25) actually is the particular case of the generalized form relation (2), so the relation (4) defining the jerk (3) will have the more simple view:

$$j(t) = r\frac{d^3 q_1}{dt^3}. \tag{26}$$

For the assumed schematization (fig. 1a) of the considered four-wheeled platform we will have the follows Lagrange function $\mathcal{L}$, the Raleigh function $\mathcal{R}$ and the driving generalized force Q_1:

$$\mathcal{L} = \tfrac{1}{2} J \dot{q}_1^2, J = m_p r^2 + 4J_w, \tag{27}$$

$$\mathcal{R} = 4M_{rf}\dot{q}_1 + \tfrac{1}{2}\beta\dot{q}_1^2, \tag{28}$$

$$Q_1 = 4M_d,\ u_c = M_d, \tag{29}$$

where m_p is the total mass and J_w is the inertia moment of the wheel of the considered platform (fig. 1a); $M_{rf} = const$ is the rolling friction couple; β is the parameter defining the viscous linear damping; M_d is the driving couple acting to each of the wheels.

The relations (27)-(29) and the Lagrange equations (5) with the assumed initial conditions (6) in the considered case of the system with one freedom degree ($N = 1$) allow writing the follows differential equation and the initial conditions:

$$J\ddot{q}_1 + \beta\dot{q}_1 = 4\left(M_d - M_{rf}\right),\ q_1(0) = 0,\ \dot{q}_1(0) = 0. \tag{30}$$

Solution of the Cauchy linear problem (30) can be represented in the follows view:

$$q_1(t) = \tfrac{4}{\beta}\left(M_d - M_{rf}\right)\left(t - \tfrac{J}{\beta}\left(1 - e^{-\frac{\beta}{J}t}\right)\right). \tag{31}$$

The solution (32) allows finding the jerk of the considered wheeled platform using the relation (26):

$$j(t) = -\tfrac{4\beta r}{J^2}\left(M_d - M_{rf}\right)e^{-\frac{\beta}{J}t}. \tag{32}$$

Solution (32) shows (fig. 1b) that the maximal jerk of the motion is in the initial time moment corresponding to the beginning of speeding-up of the considered wheeled platform from the state of rest, and this circumstance in the full agreement with the previously used limitation of the jerks which was represented by the inequality (16). Thus, the maximal jerk of the considered wheeled platform (fig. 1b) can be defined by the relation (32) at the initial time moment $t = 0$:

$$j(0) = -\tfrac{4\beta r}{J^2}\left(M_d - M_{rf}\right). \tag{33}$$

Due to the relation (33), it is possible to have the particular representation of the generalized inequality (16):

$$\tfrac{4\beta r}{J^2}\left|M_d - M_{rf}\right| \leq [j]. \tag{34}$$

To provide the motion of the considered wheeled platform it is necessary to satisfy the follows relation:

$$M_d \geq M_{rf}. \tag{35}$$

Due to the inequalities (34) and (35), it is possible to have the condition on the driving couple:

$$M_d \leq M_{rf} + \tfrac{J^2}{4\beta r}[j]. \tag{36}$$

The inequality (36) allows choosing the driving couple which will provide the admissible jerks of the motion of the wheeled platform speeding up from the state of rest. The inequality (36) shows that the rolling friction and the viscous damping are the principal causes of the jerks of the wheeled platforms under speeding up from the state of rest. Besides, the obtained result (34) is the illustration of the generalized approaches reduced to the inequality (24). Really, the result (34) can be obtained by using the generalized inequality (24), if are will be assumed the follows:

$$\bar{\mathbf{j}} = (r),\ \mathbf{M} = (J),\ \mathbf{D} = (\beta),\ \bar{\mathbf{f}} = (4M_{rf}),\ \bar{\mathbf{b}} = (4). \tag{37}$$

Example 2. In the previously considered example, the control was reduced to the drive couple (29) immediately acting on the wheel. At the same time, the drive couples are often the results of some power source operating, and it is possible only the indirect control of the drive couples due to the controlling of the power source state. This circumstance make more difficult the wheeled platforms control under the motions jerks restrictions because of the power sources have the own inherent properties and can have additional influence on the wheeled platforms. To show this, we will consider the same four-wheeled platform (fig. 1a), but driving by means the direct current electric motors schematized as shown on the fig. 1c. In this case the generalized coordinate q_2 representing the electric charge in the equivalent electric circuits of the electric motors actually defines the state of the drive electric motors, and the voltage $U = U(t)$ supplied to the each of these drive electric motor actually controls the drive couple M_d on the wheels. So, the Lagrange function, the generalized Raleigh function and the generalized forces representing the four-wheeled platform (fig. 1a) with the driving electric couples (fig. 1c) on each of the wheels will have the follows view:

$$\mathcal{L} = \tfrac{1}{2}J\dot{q}_1^2 + \tfrac{1}{2}4L\dot{q}_2^2, \tag{38}$$

$$\mathcal{R} = 4M_{rf}\dot{q}_1 + \tfrac{1}{2}\beta\dot{q}_1^2 + \tfrac{1}{2}4R\dot{q}_2^2, \tag{39}$$

$$Q_1 = 4M_d,\ M_d = B\dot{q}_2,\ Q_2 = 4(U - B\dot{q}_1),\ u_c = U, \tag{40}$$

where L is the inductance, R is the resistance of the equivalent electric circuit and B is the electromechanical parameter of the drive direct current electric motor; U is the supplied voltage on the drive electric motors.

The relations (38)-(40) and the Lagrange equations (5) with the assumed initial conditions (6) in the considered case of the system with two freedoms degree ($N = 2$) allow writing the follows differential equations and the initial conditions:

$$J\ddot{q}_1 = -\beta\dot{q}_1 + 4B\dot{q}_2 - 4M_{rf}, 4L\ddot{q}_2 = -4B\dot{q}_1 - 4R\dot{q}_2 + 4U, \tag{41}$$

$$q_1(0) = 0,\ q_2(0) = 0,\ \dot{q}_1(0) = 0,\ \dot{q}_2(0) = 0. \tag{42}$$

The differential equations (41) with the initial conditions (42) can be represented in the generalized form (13) in which we will have the follows vectors and matrices:

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \bar{\mathbf{f}} = \begin{pmatrix} 4M_{rf} \\ 0 \end{pmatrix}, \bar{\mathbf{b}} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \tag{43}$$

$$\mathbf{M} = \begin{pmatrix} J & 0 \\ 0 & 4L \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} \beta & -4B \\ 4B & 4R \end{pmatrix}. \tag{44}$$

Substituting the vectors (43) and matrices (44) to the inequality (24) allows obtaining the limitation on the supplied voltage on the drive electric motors providing the required restriction of the jerk of the straight motion under the speeding-up of the four-wheeled electromechanical platform:

$$j_0 = \tfrac{4r}{J}\left(\tfrac{\beta}{J}M_{rf} + \tfrac{B}{L}U\right), \ \tfrac{4r}{J}\left|\tfrac{\beta}{J}M_{rf} + \tfrac{B}{L}U\right| \leq [j]. \tag{45}$$

The results (34) and (45) allow showing that increasing the inertia of the wheeled platform represented by the generalized inertia moment J leads to decreasing the straight motion jerks under speeding-up from the state of rest. So, in the case of importance of limiting the jerks it is necessary to increase the mass of the wheeled platform. The results (34) and (45) also showing that decreasing the radius of the wheels of the platform leads to decreasing the jerks of the straight motion under speeding-up from the state of rest. Both the results (34) and (45) show that the rolling friction will necessarily lead to the jerks. At the same time, the result (35) shows that choosing the drive couple allows provide any wished small jerk, even if the rolling friction is presented, but the result (45) shows that it is impossible to have any wished small jerks of the electromechanical wheeled platform, if the rolling friction is presented, and it is only possible to minimize the jerks. This difference in the results (34) and (45) is due to that the properties of the sources of the drive mechanical torque of the wheels are not considered in the result (34), but this was considered in the result (45). So, properties of the the power source have the significant influence on the control providing the jerks restrictions of the straight motion under speeding-up from the state of rest of the wheeled platform.

4. Computer simulations

Further, we will consider the computer simulation of the wheeled electromechanical platform defined by the mathematical model (41), (42). This computer simulation will be reduced to the numerical solving of the initial value problem (41), (42), which will be represented as the system of the first ordered differential equations with the initial conditions (19). To have the required representation (19) of the initial value problem (41), (42) we will use new variables (17) with the $N = 2$ generalized coordinates and the control $u_c = U$, as it was defined in the last relation (40). Thus, taking into account the relations (20), (43) and (44), we will have the vector $\mathbf{x}$, the matrix $\mathbf{A}$ as well as the vectors $\mathbf{f}$ and $\mathbf{b}$ defining the linear differential equations (19) in the follows view:

$$\mathbf{x} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^T, \tag{46}$$

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\beta/J & 4B/J \\ 0 & 0 & -B/L & -R/J \end{pmatrix}, \ \mathbf{f} = \begin{pmatrix} 0 \\ 0 \\ 4M_{rf}/J \\ 0 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/L \end{pmatrix}. \tag{47}$$

The involved in the differential equations (41) numerical parameters representing the characteristics of the wheeled electromechanical platform will be considered as follows:

$$J = 80\ \mathrm{kg}\cdot\mathrm{m}^2,\ r = 0,15\ \mathrm{m},\ \beta = 2,5\ \tfrac{\mathrm{kg}\cdot\mathrm{m}^2}{\mathrm{s}},\ M_{rf} = 515\ \mathrm{N}\cdot\mathrm{m}, \tag{48}$$

$$L = 2,6\mathrm{mH},\ R = 1,18\ \Omega,\ B = 4\ \tfrac{\mathrm{N}\cdot\mathrm{m}}{\mathrm{A}}. \tag{49}$$

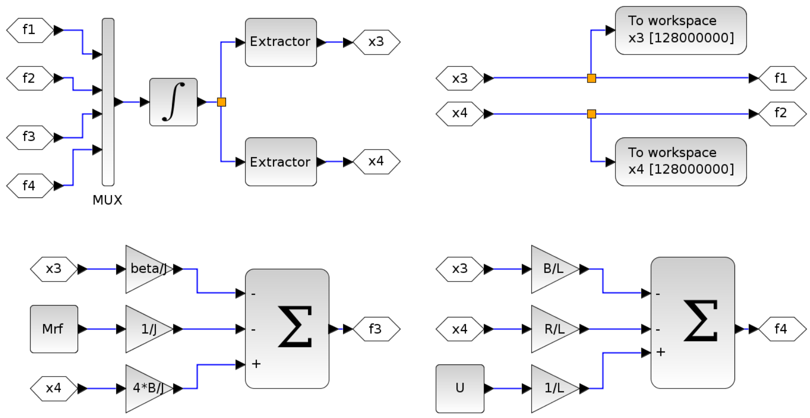

Fig. 2. Graphical representation of the model of the electromechanical wheeled platform in the Scilab free open source software

To solve the initial value problem (41), (42), (46)-(49) we will use the Scilab free open source scientific software in which we will use the especially designed graphical representation of the model of the considered electromechanical wheeled platform as shown on the fig. 2. This computer model (fig. 2) allows having different results, but further, we will consider only the follows:

$$v\,(t) = r x_3\,(t)\,, \tag{50}$$

$$j\,(t) = r \tfrac{d^2 x_3}{dt^2}\,(t)\,, \tag{51}$$

where v is the velocity and j is the jerk of the motion of the considered wheeled electromechanical platform.

Numerical solving of the initial value problem (41), (42), (46)-(49) allows having only the approximate solution for the $x_3\,(t)$, but this approximate solution will be close to the exact solution of this problem, so we can have the correct results for the velocity (50) of the considered wheeled platform. At the same time, it is well known that differentiation of the approximate solution $x_3\,(t)$ is incorrect in the Hadamard sense, and due to this we cannot have the correct results for the jerk of the considered electromechanical wheeled platform, if the formula (51) will be used directly. To exclude the Hadamard incorrectness to have the correct results for the jerk (51) it is necessary to represent the derivative d^2x_3/dt^2 thru the $\mathbf{x}$

vector. It is not difficult in the considered example; really, taking into account the relation $x_3 = \dot{q}_1$ and first differential equation (41), we will have the follows relation:

$$\frac{d^2x_3}{dt^2} = -\frac{\beta}{J}\ddot{q}_1 + \frac{4B}{J}\ddot{q}_2. \quad (52)$$

Further, it is necessary to exclude the second derivatives of the generalized coordinates from the obtained relation (52) using the differential equations (41). All these and the definitions (17) will allow having the follows:

$$\frac{d^2x_3}{dt^2} = \left(\frac{\beta^2}{J^2} - \frac{4B^2}{JL}\right)x_3 - \left(\frac{4\beta B}{J^2} + \frac{4BR}{JL}\right)x_4 + \frac{4\beta M_{rf}}{J^2} + \frac{4B}{JL}U. \quad (53)$$

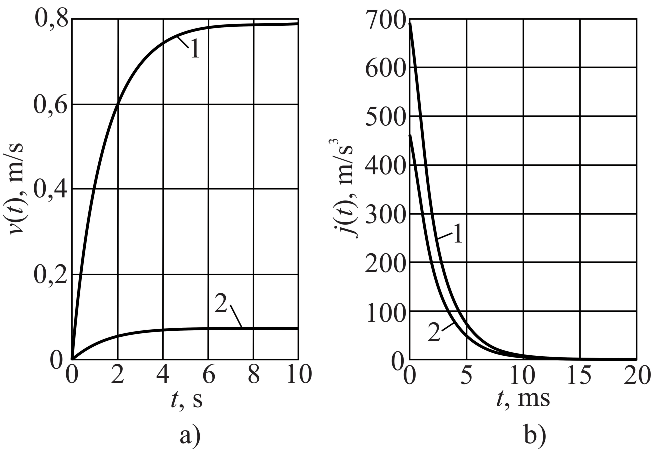

Fig. 3. Velocity (a) and jerk (b) of the electromechanical wheeled platform corresponded to the voltages $U = 60$ V (curve 1) and $U = 40$ V (curve 2) supplied on the drive electric motors

It is necessary to note, that instead the particular result (53) it is possible to use the generalized result obtained from the differential equations (19):

$$\frac{d^2x}{dt^2} = (\mathbf{AA})\,\mathbf{x} - \mathbf{Af} + (\mathbf{A})\,\mathbf{b}u_c. \quad (54)$$

The opportunities of representing the jerk thru the vector $\mathbf{x}$ in the general form (54) for the linearized problem (19) are really very important to exclude the differentiation of the $\mathbf{x}$ vector leading to the Hadamard incorrectness in the case of using the numerical methods for finding the $\mathbf{x}$ vector. The most interested quantitative results obtained by using the computer simulations (fig. 2) for the velocity (50) and for the jerk (51), (53) of the considered wheeled electromechanical platform are presented on the fig. 3. We can see (fig. 3a) that the velocity of the wheeled platform is directed to the maximum value corresponding to equilibrium between the viscous damping and the driving couples which are depended on the voltage supplied to the drive electric motors. This is in the full agreement with the well-known fundamental property inherent for the wheeled platforms. The results for the jerk (fig. 3b) show that the maximum value of the jerk is really at the initial time moment as was suggested before in the relations (16) and (24). The jerks values at the initial time moment (fig. 3b) obtained by using the computer simulations are in full agreement with the correspondent exact values defined theoretically by using first relation (45). Aspiration of the jerk's value to zero value during the time is in the agreement with aspiration of the acceleration value to zero. We can see (fig. 3b) the significant values of the jerks of the considered electromechanical wheeled platform due to instant voltage supplying on the drive electric motors at the initial time moment, and it is understandable that limiting of the value of the instantly supplied voltage cannot provide any given small jerks. Thus, to provide any small given jerks of the electromechanical wheeled platforms the smooth time's depending for the voltages supplying on the drive electric motors is required, and it is looked understandable.

Conclusion

The researches of the particular problem about control of wheeled platforms straight motions on the ideal horizontal plane taking into account jerk restrictions under speeding-up from the state of rest allowed obtaining some results, and due to these results it is possible to have the follows conclusions.

First of all, the generalized approaches to define the controls satisfying the straight motions jerks restrictions of wheeled platforms are developed for the modes of speeding-up from the state of rest. The jerks restrictions are reduced to limiting of the time derivative value of the wheeled platform acceleration. These generalized approaches based on the holonomic systems mechanics and on the electromechanical analogies allow considering the different kinds of the wheeled platforms taking into account the electric on-board systems like the drive electric motors and the control systems by using the Lagrange equations of second kind. Although, holonomic systems can represent only some particular motions of the wheeled platforms, but such particular cases are really important for solving the problems about the speeding-up and slowing-up straight motions of wheeled platforms. Considering the nongolonomic systems which can represent all the modes of the motions of wheeled platforms is planned for the future researches.

Secondly, the examples of the proposed approaches using to define the controls satisfying the jerks restrictions under speeding-up from the state of rest are considered for the pure mechanical and electromechanical wheeled platforms. It is obtained the inequality which allows choosing the instantly supplied driving mechanical couple which will provide the admissible motion jerks of the wheeled platform under speeding-up from the state of rest. It is shown, the rolling friction and the viscous damping are the principal causes of the motion jerks of the wheeled platforms under speeding-up from the state of rest. It is obtained the inequality defining the voltage instantly supplied on the drive electric motors which will provide the admissible motion jerks of the electromechanical wheeled platform under speeding-up from the state of rest, and it is shown that the proposed general approaches are suitable also for considering the jerks of different kinds of wheeled platforms.

Thirdly, the computer simulations of the processes of speeding-up from the state of rest for the electromechanical wheeled platform are considered to show the results correctness and to illustrate satisfying the motions jerks restrictions. The obtained results of the computer simulations are in the full agreement with the well-known fundamental property inherent for the wheeled platforms. The results for the jerk show that the maximum value of the jerk is really at the initial time moment as was suggested before, and it is noted that the jerks values at the initial time moment obtained by using the computer simulations are in full agreement with the correspondent exact values defined theoretically. The big values obtained for the jerks of the considered electromechanical wheeled platform are due to instant voltage supplying on the drive electric motors at the initial time moment, and it is understandable that limiting of the value of the instantly supplied voltage cannot provide any wished small jerks. To provide any wished jerks of the electromechanical wheeled platforms it is required to have the smooth time depending for the voltages supplying on the drive electric motors.

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Керування прямолінійним рухом колісних платформ з урахуванням обмежень на ривки при розганянні зі стану спокою

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Запропоновано узагальнену математичну модель процесу розганяння колісних платформ на ідеальній горизонтальній площині зі стану спокою та одержано керування, що задовольняє обмеження на ривки відповідних прямолінійних рухів. Розглянуті чисто механічна та електромеханічна колісні платформи, виконано комп'ютерне моделювання досліджуваних процесів. Узагальнені підходи засновані на механіці голономних систем та електромеханічних аналогіях, що дозволяють за допомогою

рівнянь Лагранжа другого роду розглядати різні типи колісних платформ з урахуванням електричних бортових систем, таких як приводні електродвигуни та системи керування. Хоча голономні системи відображають лише деякі окремі рухи колісних платформ, але такі окремі випадки дійсно важливі для розв'язування задач про прискорення та уповільнення рухів колісних платформ з урахуванням обмежень на ривки. Для суто механічних та електромеханічних колісних платформ розглянуто приклади використання запропонованих підходів для визначення допустимих керувань, що задовольняють обмеження на ривки при розганянні зі стану спокою. Отримано нерівність щодо визначення миттєво поданої ведучої механічної пари, яка забезпечить допустимі ривки руху колісної платформи, що прискорюється зі стану спокою. Показано, що тертя кочення та в'язкий опір є основними причинами ривків колісних платформ при розганянні зі стану спокою. Отримано нерівність, яка визначає електричну напругу, що миттєво подається на приводні електродвигуни та забезпечує допустимі ривки руху електромеханічної колісної платформи, що прискорюються зі стану спокою. Завдяки цьому показано, що запропоновані загальні підходи підходять також для дослідження колісних платформ різного типу. Розглядається комп'ютерне моделювання процесів розганяння зі стану спокою електромеханічних колісних платформ щоб мати підтвердження можливості використання запропонованих моделей та проілюструвати виконання обмежень на ривки під час рухів. Отримані результати комп'ютерного моделювання повністю узгоджуються з відомою фундаментальною властивістю, притаманною колісним платформам. Результати для ривків показують, що максимальне значення ривка дійсно є в початковий момент часу, як було запропоновано раніше, і показано, що значення ривків у початковий момент часу, отримані за допомогою комп'ютерного моделювання, повністю узгоджуються з відповідними значеннями, точно визначеними теоретично. Великі значення, отримані для ривків розглянутої електромеханічної колісної платформи, зумовлені миттєвою подачею напруги на приводні електродвигуни в початковий момент часу, і, зрозуміло, що обмеження величини миттєво поданої напруги не може забезпечити будь-яких бажаних невеликих ривків. Для забезпечення будь-яких невеликих бажаних ривків електромеханічних колісних платформ необхідно мати плавну залежить від часу напруг, що подають на електродвигуни приводу.
Ключові слова: **керування; рух; ривок; колісна платформа; математичне моделювання.**

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