ISSN 2221-5646(Print) 2523-4641(Online)
Visnyk of V. N. Karazin Kharkiv National University
Ser. “Mathematics, Applied Mathematics
and Mechanics”
2022, Vol. 95, p. 39–48
DOI: 10.26565/2221-5646-2022-95-04
УДК 517.922

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університету імені В. Н. Каразіна
Серія "Математика, прикладна
математика і механіка"
2022, Том 95, с. 39–48

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Cramer’s rule for implicit linear differential equations over a non-Archimedean ring

We consider a linear nonhomogeneous m-th order differential equation in a ring of formal power series with coefficients from some field of characteristic zero. This equation has infinite many solutions in this ring – one for each initial condition of the corresponding Cauchy problem. These solutions can be found using classical methods of differential equation theory.

Let us suppose the coefficients of the equation and the coefficients of nonhomogeneity belong to some integral domain K. We are looking for a solution in the form of a formal power series with coefficients from this integral domain. The methods of classical theory do not allow us to find out whether there exists an initial condition that corresponds to the solution of the coefficients from K and do not allow find this initial condition.

To solve this problem, we use the method proposed by U. Broggi. This method allows to find a formal solution of the linear nonhomogeneous differential equation in the form of some special series.

In previous articles, sufficient conditions for the existence and uniqueness of a solution were found for a certain class of rings K with a non-Archimedean valuation. If these conditions hold, the formal power series obtained using the Broggi’s method is considered. Its coefficients are the sums of series that converge in the non-Archimedean topology considered. It is shown that this series is the solution from $K[[x]]$ of our equation.

Note that this equation over a ring of formal power series can be considered as an infinite linear system of equations with respect to the coefficients of unknown formal power series. In this article it is proved that this system can be solved by some analogue of Cramer’s method, in which the determinants of infinite matrices are found as limits of some finite determinants in the non-Archimedean topology.

Keywords: **differential equation; formal power series; Cramer’s rule.**

2010 Mathematics Subject Classification: 34A30; 13F25; 12J25; 65F05.

1. Introduction

Let us consider m-th order linear differential equation with constant coefficients

$$a_m w^{(m)}(x) + a_{m-1} w^{(m-1)}(x) + \ldots + a_1 w'(x) + a_0 w(x) = f(x), \ a_m \neq 0. \quad (1)$$

Let a_i belong to an integral domain K, $f(x) \in K[[x]]$ be a formal power series. The equation (1) is called implicit if the element a_m is not invertible in K. We are looking for the solution of the equation (1) from $K[[x]]$. In [1] it is proved that for some conditions on K and on coefficients a_i this equation has a unique solution from $K[[x]]$.

Let F denote the quotient field of K. Using the classical differential equations theory methods we are able to solve the Cauchy problem finding the infinitely many solution from $F[[x]]$ – one solution for each initial condition. However, the form of these solutions does not allow us to find the unique solution from $K[[x]]$ [2, Ch.VII].

For this purpose we use the construction proposed by U. Broggi (see [3, §5, 22.1]). He looked for the solution of (1) as a series

$$w(x) = \sum_{k=0}^{\infty} c_k f^{(k)}(x), \quad (2)$$

where coefficients c_k satisfies the equality

$$(a_m s^m + a_{m-1} s^{m-1} + \ldots + a_1 s + a_0)^{-1} = c_0 + c_1 s + c_2 s^2 + c_3 s^3 + \ldots . \quad (3)$$

In [1] we use the non-Archimedean topology construction in some special ring K to formulate conditions for the existence and uniqueness of a solution of equation (1) in $K[[x]]$. We also write the coefficients of formal power series solution in the form of series converged with respect to the non-Archimedean topology. The results from [1] are formulated in the Section 2 of this article.

Differential equation (1) can be regarded as an infinite linear system of equations with respect to the coefficients of the unknown series. In the present article it is shown that the unique solution of the equation (1) from $K[[x]]$ can be found using an analogue to the Cramer's rule. For $m = 1$ and $K = \mathbb{Z}_p$ it was proved in [4]. The proof of the main result of this article is based on using the methods and constructions of [5].

2. Preliminaries

Suppose $(F, |\cdot|)$ is a field of characteristic zero with a non-Archimedean valuation $|\cdot|$ and $K = \{s \in F : |s| \leq 1\}$ is its valuation ring ([6, Ch.XII, §4]). Let us consider the differential equation (1) where $a_0, a_1, \ldots, a_m \in K$ and $f(x) \in K[[x]]$. We are looking for the solutions of this equation from $K[[x]]$.

In [1, Section 4, Theorems 4 and 5] sufficient conditions for the existence and uniqueness of a solution in $K[[x]]$ are found. Noticing that an element $a \in K$ is invertible if and only if $|a| = 1$, we can formulate these conditions in the following way:

Theorem 1. *Let in equation* (1) a_0 *be invertible and* a_i *be non-invertible in* K *for any* $1 \le i \le m$. *Then this equation has no more than one solution in* $K[[x]]$.

If, in addition, F *is complete with respect to* $|\cdot|$, *then series* (2) *converges by the topology of coefficientwise convergence (see [7, Chapter 1, Section 3]) in* $K[[x]]$ *and the sum of this series solves* (1).

Let w_k denote the coefficient of x^k of the unknown series $w(x)$. The following statement shows, that we can find the formulas for these coefficients to write the solution in the form $w(x) = \sum_{n=0}^{\infty} w_n x^n$.

Remark 1. *Suppose* (2) *converges by coefficientwise topology. Let* $f(x) = \sum_{n=0}^{\infty} f_n x^n$. *Then* $f^{(k)}(x) = \sum_{n=0}^{\infty} \frac{(n+k)!}{n!} f_{k+n} x^n$. *Thus,*

$$w_n = \sum_{k=0}^{\infty} c_k \frac{(n+k)!}{n!} f_{k+n}, \tag{4}$$

where these series converge with respect to $|\cdot|$.

The following theorem shows that the invertibility of a_0 is an important condition for the existence of a solution of (1).

Theorem 2. *Suppose* a_i *is non-invertible in* K *for all* $1 \le i \le m$. *If equation* (1) *has a solution in* $K[[x]]$ *for every* $f(x) \in K[[x]]$, *then* a_0 *is invertible.*

Proof. Since (1) has a solution for any $f(x)$, then for $f(x) = 1$ also. Writing the degree zero coefficient, we get $\sum_{j=1}^{m} j! a_j w_j + a_0 w_0 = 1$. Since $|a_j| < 1, |w_j| \le 1$ for any $1 \le j \le m$, then $\sum_{j=1}^{m} j! a_j w_j$ is non-invertible element of K. Since valuation is non-Archimedean, then the sum of two non-invertible elements of K is non-invertible, that is K is local. Therefore $1 - \sum_{j=1}^{m} j! a_j w_j = a_0 w_0$ is invertible. Since K is commutative, then a_0 is invertible.

The proof is complete.

Note that in some cases the equation (1) has a unique solution in $K[[x]]$ even if a_0 is not invertible.

Example. Let $K = \mathbb{Z}$ and $m = 1$. For any non-zero coefficients $a_0, a_1 \in \mathbb{Z}$ if $f(x) = a_1 + a_0 x$, then the equation $a_1 w'(x) + a_0 w(x) = f(x)$ has a solution $w(x) = x$. Moreover, this solution is unique, that is the homogeneous equation $a_1 w'(x) + a_0 w(x) = 0$ has only trivial solution. Indeed, if $w(x) = \sum_{n=0}^{\infty} w_n x^n$ is a solution of this equation, then $a_1 n w_n = -a_0 w_{n-1}$. Hence for any $n \ge 1$ we get $n! a_1^n w_n = (-1)^n a_0^n w_0$. Let p denote some prime, that is not divisor of a_0. Then for any j there exists n such that $n!$ is divisible by p^j. Thus w_0 is divisible by p^j for any j. It is impossible for any integer $w_0 \ne 0$.

3. Main result

Now suppose a_0 is invertible. Without loss of generality we can assume $a_0 = 1$. Then the sequence $\{c_k\}$ that is found by equality (3) satisfies the following system:

$$\begin{cases} c_0 & = 1 \\ c_0 + \sum_{i=0}^{j-1} a_i c_{j-i} & = 0, \quad j = 1, 2, 3, \ldots, m \\ \sum_{i=0}^{m} a_i c_{j-i} & = 0, \quad j = m+1, m+2, \ldots \end{cases} . \tag{5}$$

Let us rewrite (1) as an infinite linear system of equations. For any $k = 0, 1, 2, \ldots$, extracting coefficients of x^k, we get

$$a_0 w_k + a_1 \frac{(k+1)!}{k!} w_{k+1} + \ldots + a_j \frac{(k+j)!}{k!} w_{k+j} + \ldots + a_m \frac{(k+m)!}{k!} w_{k+m} = f_k, \tag{6}$$

where $w(x) = \sum_{k=0}^{\infty} w_n x^n$.

By definition, put $a_i = 0$ for any $i > m$. Then in the matrix form this system of equations can be written as $Aw = f$, where

$$\mathcal{A} = \begin{pmatrix} 1 & a_1 & 2a_2 & 3!a_3 & 4!a_4 & 5!a_5 & \cdots \\ 0 & 1 & 2a_1 & 3!a_2 & 4!a_3 & 5!a_4 & \cdots \\ 0 & 0 & 1 & \frac{3!}{2}a_1 & \frac{4!}{2}a_2 & \frac{5!}{2}a_3 & \cdots \\ 0 & 0 & 0 & 1 & \frac{4!}{3!}a_1 & \frac{5!}{3!}a_2 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \frac{5!}{4!}a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } f = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ \vdots \end{pmatrix}. \tag{7}$$

That is $\mathcal{A}$ ia an upper triangular matrix that has the terms $\alpha_{ij} = \dfrac{(j-1)!}{(i-1)!} a_{j-i}$ for any $0 \le i \le j$ and $\alpha_{ij} = 0$ for any $0 < j < i$.

Similarly as in [5] for any $n \ge 0$ let $\mathcal{A}_n$ be obtained from the matrix $\mathcal{A}_n$ by replacing the $(n+1)$-th column with the vector f. For any $j \ge 0$ by $\tilde{\Delta}_j$ (respectively, $\tilde{\Delta}_{n,j}$) denote the principal corner minor of the $(j+1)$-th order of the matrix $\mathcal{A}$ (respectively, $\mathcal{A}_n$).

Theorem 3. *Suppose F is a complete field of characteristic zero with a non-Archimedean valuation $|\cdot|$, K is the valuation ring of F, $a_0 = 1$ and $|a_i| < 1$ for any $1 \le i \le m$. Then equation* (1) *has a unique solution*

$$w(x) = \sum_{n=0}^{\infty} w_n x^n$$

from $K[[x]]$. The coefficients of this solution can be found using Cramer's rule:

$$w_n = \frac{\det \mathcal{A}_n}{\det \mathcal{A}} = \det \mathcal{A}_n, n = 0, 1, 2, \ldots, \tag{8}$$

where the determinants are defined as following limits in K with respect to the valuation $|\cdot|$:

$$\det \mathcal{A} = \lim_{r \to \infty} \tilde{\Delta}_r$$

$$\det \mathcal{A}_n = \lim_{r \to \infty} \tilde{\Delta}_{n,r}, n = 1, 2, 3, \ldots$$

Proof. Note that $\tilde{\Delta}_r = 1$ for any r, so $\det \mathcal{A} = 1$. By Theorem 1 equation (1) has a unique solution over K in the form (2). Let us show that $\det \mathcal{A}_n = w_n$.

Let $\tilde{B}_r$ denote the determinant

$$\tilde{B}_r = \begin{vmatrix} a_1 & 2a_2 & 3!a_3 & 4!a_4 & 5!a_5 & \cdots & r!a_r \\ 1 & 2a_1 & 3!a_2 & 4!a_3 & 5!a_4 & \cdots & r!a_{r-1} \\ 0 & 1 & \frac{3!}{2}a_1 & \frac{4!}{2}a_2 & \frac{5!}{2}a_3 & \cdots & \frac{r!}{2}a_{r-2} \\ 0 & 0 & 1 & \frac{4!}{3!}a_1 & \frac{5!}{3!}a_2 & \cdots & \frac{r!}{3!}a_{r-3} \\ 0 & 0 & 0 & 1 & \frac{5!}{4!}a_1 & \cdots & \frac{r!}{4!}a_{r-4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & \frac{r!}{(r-1)!}a_1 \end{vmatrix}. \tag{9}$$

Let us consider

$$\tilde{\Delta}_{0,r} = \begin{vmatrix} f_0 & a_1 & 2a_2 & 3!a_3 & 4!a_4 & 5!a_5 & \cdots & r!a_r \\ f_1 & 1 & 2a_1 & 3!a_2 & 4!a_3 & 5!a_4 & \cdots & r!a_{r-1} \\ f_2 & 0 & 1 & \frac{3!}{2}a_1 & \frac{4!}{2}a_2 & \frac{5!}{2}a_3 & \cdots & \frac{r!}{2}a_{r-2} \\ f_3 & 0 & 0 & 1 & \frac{4!}{3!}a_1 & \frac{5!}{3!}a_2 & \cdots & \frac{r!}{3!}a_{r-3} \\ f_4 & 0 & 0 & 0 & 1 & \frac{5!}{4!}a_1 & \cdots & \frac{r!}{4!}a_{r-4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{r-1} & 0 & 0 & 0 & 0 & 0 & \ldots & \frac{r!}{(r-1)!}a_1 \\ f_r & 0 & 0 & 0 & 0 & 0 & \ldots & 1 \end{vmatrix}.$$

Decomposing it relative to the first column, we get

$$\tilde{\Delta}_{0,r} = f_0 - f_1 a_1 + f_2 \begin{vmatrix} a_1 & 2a_2 \\ 1 & 2a_1 \end{vmatrix} - f_3 \begin{vmatrix} a_1 & 2a_2 & 3!a_3 \\ 1 & 2a_1 & 3!a_2 \\ 0 & 1 & \frac{3!}{2}a_1 \end{vmatrix} + f_4 \begin{vmatrix} a_1 & 2a_2 & 3!a_3 & 4!a_4 \\ 1 & 2a_1 & 3!a_2 & 4!a_3 \\ 0 & 1 & \frac{3!}{2}a_1 & \frac{4!}{2}a_2 \\ 0 & 0 & 1 & \frac{4!}{3!}a_1 \end{vmatrix} + \ldots =$$

$$= f_0 - f_1 a_1 + f_2 \tilde{B}_2 - f_3 \tilde{B}_3 + f_4 \tilde{B}_4 - \ldots = f_0 - \sum_{i=1}^{r} (-1)^{i-1} f_i \tilde{B}_i. \tag{10}$$

In [5] the determinants

$$B_r = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_r \\ 1 & a_1 & a_2 & a_3 & \cdots & a_{r-1} \\ 0 & 1 & a_1 & a_2 & \cdots & a_{r-2} \\ 0 & 0 & 1 & a_1 & \cdots & a_{r-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & a_1 \end{vmatrix} \tag{11}$$

are consedered and there is proved that $B_r = (-1)^r c_r$, where the secuence $\{c_r\}$ is a solution of (5).

Note that

$$\tilde{B}_r = \begin{vmatrix} a_1 & 2a_2 & 3!a_3 & 4!a_4 & \cdots & k!a_r \\ 1 & 2a_1 & 3!a_2 & 4!a_3 & \cdots & r!a_{r-1} \\ 0 & 1 & \frac{3!}{2}a_1 & \frac{4!}{2}a_2 & \cdots & \frac{r!}{2}a_{r-2} \\ 0 & 0 & 1 & \frac{4!}{3!}a_1 & \cdots & \frac{r!}{3!}a_{r-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \frac{r!}{(r-1)!}a_1 \end{vmatrix} =$$

$$= 2 \cdot 3! \cdot \ldots \cdot r! \cdot \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_r \\ 1 & a_1 & a_2 & a_3 & \cdots & a_{r-1} \\ 0 & \frac{1}{2} & \frac{1}{2}a_1 & \frac{1}{2}a_2 & \cdots & \frac{1}{2}a_{r-2} \\ 0 & 0 & \frac{1}{3!} & \frac{1}{3!}a_1 & \cdots & \frac{1}{3!}a_{r-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{(r-1)!}a_1 \end{vmatrix} =$$

$$= \frac{2 \cdot 3! \cdot \ldots \cdot r!}{2 \cdot 3! \cdot \ldots \cdot (r-1)!} B_r = r! B_r. \quad (12)$$

Then $\tilde{B}_r = r!(-1)^r c_r$. Hence

$$\tilde{\Delta}_{1,r} = f_0 - \sum_{i=1}^{r} (-1)^{i-1} f_i \tilde{B}_i = f_0 + \sum_{i=1}^{r} f_i i! c_i = \sum_{i=0}^{r} f_i i! c_i. \quad (13)$$

Thus

$$\frac{\det \mathcal{A}_0}{\det \mathcal{A}} = \frac{\lim_{r\to\infty} \tilde{\Delta}_{0,r}}{\lim_{r\to\infty} \tilde{\Delta}_r} = \lim_{r\to\infty} \sum_{i=0}^{r} i! f_i c_i = \sum_{i=0}^{\infty} i! f_i c_i. \quad (14)$$

It coincides to the w_0 found in Remark 1.

Let us now consider $\mathcal{A}_n$ for any $n \ge 1$ and its minor of i-th order $\tilde{\Delta}_{n,i-1}$. We are interested in the limit by i, so it is enough to consider i such that $i > n$ and $i > m$. Then the principal corner minor $\tilde{\Delta}_{n,i-1}$ is equal to

$$\left|\begin{array}{c|ccccccc}
 & f_0 & (n+1)!a_{n+1} & \cdots & m!a_m & 0 & \cdots & * \\
 & f_1 & (n+1)!a_n & \cdots & m!a_{m-1} & (m+1)!a_m & \cdots & * \\
 & f_2 & \frac{(n+1)!}{2}a_{n-1} & \cdots & \frac{m!}{2}a_{m-2} & \frac{(m+1)!}{2}a_{m-1} & \cdots & * \\
{}^n\overline{\mathcal{A}} & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
 & f_{n-2} & \frac{(n+1)!}{(n-2)!}a_3 & \cdots & \frac{m!}{(n-2)!}a_{m-n+2} & \frac{(m+1)!}{(n-2)!}a_{m-n+3} & \cdots & * \\
 & f_{n-1} & \frac{(n+1)!}{(n-1)!}a_2 & \cdots & \frac{m!}{(n-1)!}a_{m-n+1} & \frac{(m+1)!}{(n-1)!}a_{m-n+2} & \cdots & * \\
\hline
 & f_n & (n+1)a_1 & \cdots & \frac{m!}{n!}a_{m-n} & \frac{(m+1)!}{n!}a_{m-n+1} & \cdots & * \\
\ddots \quad \iddots & f_{n+1} & 1 & \cdots & \frac{m!}{(n+1)!}a_{m-n-1} & \frac{(m+1)!}{(n+1)!}a_{m-n} & \cdots & * \\
0 & f_{n+2} & 0 & \cdots & \frac{m!}{(n+2)!}a_{m-n-2} & \frac{(m+1)!}{(n+2)!}a_{m-n-1} & \cdots & * \\
\iddots \quad \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
 & f_{i-1} & 0 & \cdots & 0 & 0 & \cdots & 1
\end{array}\right|,$$

where the last i-th column is $(0, \cdots, 0, \frac{(i-1)!}{(i-m-1)!}a_m, \cdots, \frac{(i-1)!}{(i-3)!}a_2, \frac{(i-1)!}{(i-2)!}a_1, 1)^\top$ and ${}^n\overline{\mathcal{A}}$ is the square submatrix of $\mathcal{A}$ formed by deleting all rows an columns except the first $n-1$ ones.

We see that its determinant $\tilde{\Delta}_{n,i-1}$ equals

$$\begin{vmatrix}
f_n & (n+1)a_1 & \frac{(n+2)!}{n!}a_2 & \cdots & \frac{m!}{n!}a_{m-n} & \cdots & \frac{(i-1)!}{n!}a_{i-n-1} \\
f_{n+1} & 1 & (n+2)a_1 & \cdots & \frac{m!}{(n+1)!}a_{m-n-1} & \cdots & \frac{(i-1)!}{(n+1)!}a_{i-n-2} \\
f_{n+2} & 0 & 1 & \cdots & \frac{m!}{(n+2)!}a_{m-n-2} & \cdots & \frac{(i-1)!}{(n+2)!}a_{i-n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
f_{i-1} & 0 & 0 & \cdots & 0 & \cdots & 1
\end{vmatrix}. \tag{15}$$

Set $\check{B}_0 = 1$ and

$$\check{B}_r = \begin{vmatrix}
(n+1)a_1 & \frac{(n+2)!}{n!}a_2 & \frac{(n+3)!}{n!}a_3 & \frac{(n+4)!}{n!}a_4 & \cdots & \frac{(n+r)!}{n!}a_r \\
1 & (n+2)a_1 & \frac{(n+3)!}{(n+1)!}a_2 & \frac{(n+4)!}{(n+1)!}a_3 & \cdots & \frac{(n+r)!}{(n+1)!}a_{r-1} \\
0 & 1 & (n+3)a_1 & \frac{(n+4)!}{(n+2)!}a_2 & \cdots & \frac{(n+r)!}{(n+2)!}a_{r-2} \\
0 & 0 & 1 & (n+4)a_1 & \cdots & \frac{(n+r)!}{(n+3)!}a_{r-3} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \dots & (n+r)a_1
\end{vmatrix}. \tag{16}$$

Then, decomposing $\tilde{\Delta}_{n,i-1}$ relative to the first column, we obtain

$$\tilde{\Delta}_{n,i-1} = \sum_{s=1}^{i-n} (-1)^{s-1} f_{n+s-1} \breve{B}_{s-1}. \qquad (17)$$

Note that $\breve{B}_r = \frac{(n+1)!\cdot(n+2)!\cdot\ldots\cdot(n+r)!}{n!\cdot(n+1)!\cdot\ldots\cdot(n+r-1)!} B_r = \frac{(n+r)!}{n!} B_r$. Therefore

$$\tilde{\Delta}_{n,i-1} = \sum_{s=1}^{i-n} (-1)^{s-1} f_{n+s-1} \frac{(n+s-1)!}{n!} B_{s-1} = \sum_{s=0}^{i-n-1} f_{n+s} \frac{(n+s)!}{n!} c_s.$$

Then

$$\lim_{i\to\infty} \tilde{\Delta}_{n,i} = \sum_{s=0}^{\infty} f_{n+s} \frac{(n+s)!}{n!} c_s = w_n.$$

The proof is complete.

Remark 2. *Equality* (12) *shows the connection between the determinants considered and the determinants constructed in the similar situation for the m-th order implicit linear difference equation considered in [5].*

Remark 3. *By [1, Corollary 3] under the conditions of Theorem 3 for the second order equation $a_2 w''(x) + a_1 w'(x) + w(x) = f(x)$ the solution can be found by the following explicit formula:*

$$w(x) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{\infty} \frac{(j+n)!}{n!} f_{n+j} \sum_{i=0}^{[\frac{j}{2}]} (-1)^{j-i} \binom{j-i}{i} a_1^{j-2i} a_2^i \right) x^n.$$

By Theorem 3 the solution of this equation can be found using the Cramer's rule

$$w_n = \det \mathcal{A}_n = \lim_{r\to\infty} \tilde{\Delta}_{n,r},\, n = 0, 1, 2, \ldots$$

It means that for

$$\mathcal{A} = \begin{pmatrix} 1 & a_1 & 2a_2 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 2a_1 & 6a_2 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 3a_1 & 12a_2 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 4a_1 & 20a_2 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 5a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad (18)$$

the determinant of the matrix formed by replacing the $(n+1)$-th column of $\mathcal{A}$ by the column vector f we can find in the following form:

$$\det \mathcal{A}_n = \sum_{j=0}^{\infty} \frac{(j+n)!}{n!} f_{n+j} \sum_{i=0}^{[\frac{j}{2}]} (-1)^{j-i} \binom{j-i}{j} a_1^{j-2i} a_2^i.$$

Acknowledgement. The research was supported by the National Research Foundation of Ukraine funded by Ukrainian State budget in frames of project 2020.02/0096 "Operators in infinite-dimensional spaces: the interplay between geometry, algebra and topology".

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Article history: Received: 15 June 2022; Final form: 22 June 2022
Accepted: 26 June 2022.

How to cite this article:

A. Goncharuk, Cramer's rule for implicit linear differential equations over a non-Archimedean ring, Visnyk of V. N. Karazin Kharkiv National University. Ser. Mathematics, Applied Mathematics and Mechanics, Vol. 95, 2022, p. 39–48. DOI: 10.26565/2221-5646-2022-95-04

Правило Крамера для неявного лінійного диференціального рівняння над неархімедовим кільцем

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Розглянемо лінійне неоднорідне диференціальне рівняння m-го порядку у кільці формальних степеневих рядів з коефіцієнтами з деякого поля нульової характеристики. Таке рівняння має нескінченно багато розв' язків у цьому кільці – єдиний розв' язок для кожної початкової умови відповідної задачі Коші. Ці розв' язки можуть бути знайдені за допомогою класичних методів теорії диференціальних рівнянь.

Розглянемо таке рівняння у випадку, коли коефіцієнти рівняння і коефіцієнти неоднорідності належать до деякої області цілісності K і будемо шукати розв' язок у вигляді формального степеневого ряду з коефіцієнтами з цієї області цілісності. Методи класичної теорії не дають нам змоги з'ясувати, чи існуватиме початкова умова, що відповідає розв' язку з коефіцієнтами із K і яка саме.

Для роз' вязання цієї задачи ми користуємося методом, що був запропонований у роботі У. Бродджі, який знаходить формальний розв' язок лінійного неоднорідного диференціального рівняння у вигляді деякого спеціального ряду.

У попередніх роботах знайдено достатні умови існування та єдиності такого розв'язку для деякого класу кілець K з неархімедевим нормуванням. У випадку виконання цих умов розглянуто формальний степеневий ряд, отриманий за допомогою методу Бродджі. Коефіцієнтами цього ряду є суми рядів, які збігаються у розглянутій неархімедовій топології до елементів із кільця K. Показано, що цей ряд є розв' язком нашого рівняння у кільці $K[[x]]$.

Варто відмітити, що таке рівняння у кільці формальних степеневих рядів можна розглядати як нескінченну лінійну систему рівнянь відносно коефіцієнтів невідомого формального степеневого ряду. В цій статті доведено, що цю систему можна розв'язувати за допомогою деякого аналогу методу Крамера, в якому визначники нескінченних матриць знаходяться як границі скінченних визначників у неархімедовій топології.

Ключові слова: **диференціальне рівняння; формальні степеневі ряди; правило Крамера.**

Історія статті: отримана: 15 червня 2022; останній варіант: 22 червня 2022
прийнята: 26 червня 2022.