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On relation between statistical ideal and ideal generated by a modulus function

Ideal on an arbitrary non-empty set Ω it's a non-empty family of subset $\mathfrak{I}$ of the set Ω which satisfies the following axioms: $\Omega \notin \mathfrak{I}$, if $A, B \in \mathfrak{I}$, then $A \cup B \in \mathfrak{I}$, if $A \in \mathfrak{I}$ and $D \subset A$, then $D \in \mathfrak{I}$. The ideal theory is a very popular branch of modern mathematical research. In our paper we study some classes of ideals on the set of all positive integers $\mathbb{N}$, namely the ideal of statistical convergence $\mathfrak{I}_s$ and the ideal $\mathfrak{I}_f$ generated by a modular function f. Statistical ideal it's a family of subsets of $\mathbb{N}$ whose natural density is equal to 0, i.e. $A \in \mathfrak{I}_s$ if and only if $\lim\limits_{n\to\infty} \dfrac{\#\{k \le n : k \in A\}}{n} = 0$. A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is called a modular function, if $f(x) = 0$ only if $x = 0$, $f(x+y) \le f(x) + f(y)$ for all $x, y \in \mathbb{R}^+$, $f(x) \le f(y)$ whenever $x \le y$, f is continuous from the right 0, and finally $\lim\limits_{n\to\infty} f(n) = \infty$. Ideal, generated by the modular function f it's a family of subsets of $\mathbb{N}$ with zero f-density, in other words, $A \in \mathfrak{I}_f$ if and only if $\lim\limits_{n\to\infty} \dfrac{f(\#\{k \le n : k \in A\})}{f(n)} = 0$. It is known that for an arbitrary modular function f the following is true: $\mathfrak{I}_f \subset \mathfrak{I}_s$. In our research we give the complete description of those modular functions f for which $\mathfrak{I}_f = \mathfrak{I}_s$. Then we analyse obtained result, give some partial cases of it and prove one simple sufficient condition for the equality $\mathfrak{I}_f = \mathfrak{I}_s$. The last section of this article is devoted to examples of some modulus functions f, g for which $\mathfrak{I}_f = \mathfrak{I}_s$ and $\mathfrak{I}_g \ne \mathfrak{I}_s$. Namely, if $f(x) = x^p$ where $p \in (0, 1]$ we have $\mathfrak{I}_f = \mathfrak{I}_s$; for $g(x) = \log(1+x)$, we obtain $\mathfrak{I}_g \ne \mathfrak{I}_s$. Then we consider more complicated function f which is given recursively to demonstrate that the conditions of the main theorem of our paper can't be reduced to the sufficient condition mentioned above.

Keywords: **ideal, statistical ideal, modulus function.**

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1. Introduction

Let Ω be a non-empty set. Let us remind that a non-empty family $\mathfrak{I} \subset 2^\Omega$ is called *an ideal* on Ω if $\mathfrak{I}$ satisfies:

1. $\Omega \notin \mathfrak{I}$;

2. if $A, B \in \mathfrak{I}$ then $A \cup B \in \mathfrak{I}$;

3. if $A \in \mathfrak{I}$ and $D \subset A$ then $D \in \mathfrak{I}$.

In our article we consider those ideals $\mathfrak{I}$ which contain the family of finite sets $\mathfrak{Fin}$.

For a subset $A \subset \mathbb{N}$ denote $\alpha_A(n) := |A \cap [1, n]|$, where $|M|$ stands for a number of elements in the set $M \subset \mathbb{N}$. Let $A \subset \mathbb{N}$. *The natural density* of A is $d(A) := \lim\limits_{n\to\infty} \dfrac{\alpha_A(n)}{n}$.

The ideal of sets $A \subset \mathbb{N}$ having $d(A) = 0$ is called *the statistical ideal.* We denote this ideal $\mathfrak{I}_s$.

The statistical ideal is related to the statistical convergence and is a very popular branch of research.

In [1] authors introduced a generalization of the natural density of subset in $\mathbb{N}$. They called it *f-density*, where f is a modulus function.

Recall that a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is called *an unbounded modulus function* (modulus function for short) if:

1. $f(x) = 0$ if and only if $x = 0$;

2. $f(x + y) \leq f(x) + f(y)$ for all $x, y \in \mathbb{R}^+$;

3. $f(x) \leq f(y)$ if $x \leq y$;

4. f is continuous from the right at 0;

5. $\lim\limits_{n\to\infty} f(n) = \infty$.

Let f be a modulus function. The quantity $d_f(A) := \lim\limits_{n\to\infty} \dfrac{f(\alpha_A(n))}{f(n)}$ is called *the f-density* of $A \subset \mathbb{N}$. The ideal $\mathfrak{I}_f := \{A \subset \mathbb{N} : \ d_f(A) = 0\}$ is called *the f-ideal.* $\mathfrak{I}_f$ appears implicitly in [1] where the convergence of sequences with respect to $\mathfrak{I}_f$ was studied, and appears explicitly in [3].

In [1, p. 527] it is noted that for an arbitrary modulus function f and $A \subset \mathbb{N}$ if $d_f(A) = 0$ then $d(A) = 0$. In other words, $\mathfrak{I}_f \subset \mathfrak{I}_s$. The ideals $\mathfrak{I}_f$ and $\mathfrak{I}_s$ and the corresponding ideal convergences have some similarities and some differences. The aim of the paper is to present a complete description of those modulus functions f for which $\mathfrak{I}_f = \mathfrak{I}_s$. We do this in Theorem 1. After that in Theorem 2 we give a handy sufficient condition for the equality $\mathfrak{I}_s = \mathfrak{I}_f$, and finally we present some illustrative examples.

2. Main results

Let f be a modulus function, $t \in [1, +\infty)$, $k \in \mathbb{N}$. Denote

$$h_f(t) := \limsup_{n\to\infty} \frac{f(n)}{f(tn)}, \quad g_f(k) := h_f(2^k) = \limsup_{n\to\infty} \frac{f(n)}{f(2^k n)}.$$

The following Theorem is the promised main result of our paper.

Theorem 1. *Let f be a modulus. The following statements are equivalent:*

1. $\mathfrak{I}_s = \mathfrak{I}_f$;
2. $\lim\limits_{k\to\infty} g_f(k) = 0$.
3. $\lim\limits_{t\to\infty} h_f(t) = 0$.

Proof. Remark that the equivalence of conditions (2) and (3) follows evidently from the monotonicity of $h_f(t)$ in the variable t. We include both of the conditions because of some conveniences in the proof and for the future applications. So, it is sufficient to demonstrate implications (3) $\Rightarrow$ (1) and (1) $\Rightarrow$ (2).

(**3**) $\Rightarrow$ (**1**). As we remarked in the Introduction, the inclusion $\mathfrak{I}_s \supset \mathfrak{I}_f$ is known, so we need to show that $\mathfrak{I}_s \subset \mathfrak{I}_f$. Denote $\delta_t^f := h_f(t) + \frac{1}{t}$. The quantity δ_t^f is decreasing in t and $\lim\limits_{t\to\infty} \delta_t^f = 0$. We know that $\limsup\limits_{n\to\infty} \frac{f(n)}{f(tn)} < \delta_t^f$ for all $t \in [1, +\infty)$, in particular we have that for every $k \in \mathbb{N}$ there exists $N_1(k) \in \mathbb{N}$ such that for all $n \geq N_1(k)$ the following holds true: $f\left(\frac{n}{k}\right) < \delta_k^f f(n)$.

Let $A \in \mathfrak{I}_s$. By the definition of $\mathfrak{I}_s$ his means that $\lim\limits_{n\to\infty} \frac{\alpha_A(n)}{n} = 0$. In other words, for each $k \in \mathbb{N}$ there exists $N_2(k) \in \mathbb{N}$ such that $\alpha_A(n) < \frac{n}{k}$ for each $n > N_2(k)$. Denote $N_k := \max\{N_1(k), N_2(k)\}$. Then for each $n > N_k$

$$f(\alpha_A(n)) < f\left(\frac{n}{k}\right) < \delta_k^f f(n).$$

From the previous inequality we have:

$$\limsup_{n\to\infty} \frac{f(\alpha_A(n))}{f(n)} \leq \delta_k^f \underset{k\to\infty}{\longrightarrow} 0,$$

which completes the proof of the implication (3) $\Rightarrow$ (1).

(**1**) $\Rightarrow$ (**2**). Assume that $\lim\limits_{k\to 0} g_f(k) \neq 0$. By monotonicity, this implies the existence of $\xi > 0$ such that $g_f(k) > \xi$ for every $k \in \mathbb{N}$. Consequently, $\limsup\limits_{n\to\infty} \frac{f(n)}{f(2^k n)} > \xi$ for each $k \in \mathbb{N}$. Then for every $k \in \mathbb{N}$ there exists an infinite subset $N_k \subset \mathbb{N}$ such that for each $n \in N_k$

$$f(n) > \xi f(2^k n). \tag{1}$$

Choose $0 = n_0 < n_1 < n_2 < n_3 < \ldots$ such that $n_j \in N_j$ for each $j \in \mathbb{N}$. So for each $k \in \mathbb{N}$

$$f(n_k) > \xi f(2^k n_k).$$

Denote $m_k := 2^k n_k$, $k = 1, 2, \ldots$. Let us consider the following set A:

$$A = \{m_1 - n_1 + 1, m_1 - n_1 + 2, ..., m_1 - 1,$$
$$m_2 - n_2 + n_1, m_2 - n_2 + n_1 + 1, m_2 - n_2 + n_1 + 2, ..., m_2 - 1, ...\},$$

that is from each block of naturals in $(m_{k-1}, m_k]$ we chose the $n_k - n_{k-1}$ biggest ones. For the correctness of the definition of A we have to check that $m_k - m_{k-1} > n_k - n_{k-1}$ for every $k \in \mathbb{N}$. Indeed, for all $k \in \mathbb{N}$ we have:

$$m_k - n_k + n_{k-1} - m_{k-1} = 2^k n_k - n_k + n_{k-1} - 2^{k-1} n_{k-1} =$$
$$= n_k(2^k - 1) - n_{k-1}(2^{k-1} - 1) > 0,$$

because $n_k > n_{k-1}$ and $2^k - 1 > 2^{k-1} - 1$.

Denote $\alpha_n := \alpha_A(n)$. Let us show that $A \notin \mathfrak{I}_f$. By our construction, $\alpha_{m_k} = n_k$ for all $k \in \mathbb{N}$. Using the inequality (1) we obtain that $\dfrac{f(\alpha_{m_k})}{f(m_k)} > \xi > 0$, so $\dfrac{f(\alpha_n)}{f(n)} \not\to 0$ as $n \to \infty$, that's why $A \notin \mathfrak{I}_f$.

Let us finally show that $A \in \mathfrak{I}_s$. For every $k \in \mathbb{N}$ we can split the block of naturals $[m_k + 1, m_{k+1}] \cap \mathbb{N}$ as follows:

$$[m_k + 1, m_{k+1}] \cap \mathbb{N} = [m_k + 1, m_{k+1} - n_{k+1} + n_k - 1] \cap \mathbb{N}$$
$$\cup [m_{k+1} - n_{k+1} + n_k, m_{k+1} - 1] \cap \mathbb{N} \cup \{m_{k+1}\}.$$

On the initial part of this set for $j \in [m_k + 1, m_{k+1} - n_{k+1} + n_k - 1]$ we have: $\alpha_j = n_k$ and $\dfrac{\alpha_j}{j} = \dfrac{n_k}{j} \le \dfrac{n_k}{m_k + 1} \le \dfrac{n_k}{m_k} = \dfrac{1}{2^k}$. On the next part, for $j \in [m_{k+1} - n_{k+1} + n_k, m_{k+1} - 1] = [n_{k+1}(2^{k+1} - 1) + n_k, 2^{k+1} n_{k+1}]$ we have $\alpha_j = n_k + x_j$, where $1 \le x_j \le n_{k+1} - n_k$. Using this, we obtain that $\dfrac{1}{j} \le \dfrac{x_j}{j} \le \dfrac{n_{k+1} - n_k}{j}$, and so

$$\frac{\alpha_j}{j} = \frac{n_k + x_j}{j} \le \frac{n_{k+1}}{n_{k+1}(2^{k+1} - 1) + n_k} \le \frac{n_{k+1}}{n_{k+1}(2^{k+1} - 1)} = \frac{1}{2^{k+1} - 1} < \frac{1}{2^k}.$$

At the last point $j = m_{k+1}$ we have: $\alpha_j = n_{k+1}$ and

$$\frac{\alpha_j}{j} = \frac{n_{k+1}}{m_{k+1}} = \frac{1}{2^{k+1}} < \frac{1}{2^k}.$$

So for an arbitrary $k \in \mathbb{N}$ and for $j \in [m_k + 1, m_{k+1}]$ we have $\dfrac{\alpha_j}{j} < \dfrac{1}{2^k}$, in other words $A \in \mathfrak{I}_s$.

Now let us discuss a particular case of Theorem 1 in which the condition for $\mathfrak{I}_f = \mathfrak{I}_s$ can be substantially simplified. First, a simple technical lemma.

Lemma 1. *Let f be a modulus function. Let there exists* $\lim_{n\to\infty}\frac{f(n)}{f(2n)} = a$. *Then* $g_f(k) = \lim_{n\to\infty}\frac{f(n)}{f(2^k n)} = a^k$ *for all* $k \in \mathbb{N}$.

Proof. Let us use method of mathematical induction.

The base of induction: $k = 2$.

$$\lim_{n\to\infty}\frac{f(n)}{f(4n)} = \lim_{n\to\infty}\frac{f(n)\cdot f(2n)}{f(2n)\cdot f(4n)} = \lim_{n\to\infty}\frac{f(n)}{f(2n)}\cdot\lim_{n\to\infty}\frac{f(2n)}{f(4n)} = a^2.$$

The inductive transition: $k \to k+1$.

$$\lim_{n\to\infty}\frac{f(n)}{f(2^{k+1}n)} = \lim_{n\to\infty}\frac{f(n)}{f(2^k n)}\cdot\lim_{n\to\infty}\frac{f(2^k n)}{f(2^{k+1}n)} = a^k\cdot a = a^{k+1}.$$

Theorem 2. *Let f be a modulus. Suppose that there exists* $\lim_{n\to\infty}\frac{f(n)}{f(2n)}$. *Then the following statements are equivalent:*

1. $\mathfrak{I}_s = \mathfrak{I}_f$;
2. $\lim_{n\to\infty}\frac{f(n)}{f(2n)} < 1$.

Proof. Under the assumption of existence of $\lim_{n\to\infty}\frac{f(n)}{f(2n)}$, Lemma 1 gives the equivalence of our condition (2) and the condition (2) of Theorem 1.

3. Examples

At first, let us show that among the very elementary modulus functions f the both possibilities $\mathfrak{I}_f = \mathfrak{I}_s$ and $\mathfrak{I}_f \neq \mathfrak{I}_s$ may happen.

Example 1. $f(x) = x^p$, $p \in (0,1]$. For this kind of functions $\mathfrak{I}_f = \mathfrak{I}_s$. Indeed, $\lim_{n\to\infty}\frac{f(n)}{f(2n)} = \lim_{n\to\infty}\frac{n^p}{(2n)^p} = \left(\frac{1}{2}\right)^p < 1$.

Example 2. $f(x) = \log(1+x)$. In this case $\mathfrak{I}_f \neq \mathfrak{I}_s$, because $\lim_{n\to\infty}\frac{f(n)}{f(2n)} = \lim_{n\to\infty}\frac{\log(1+n)}{\log(1+2n)} = 1$.

Our next goal is to show that Theorem 1 does not reduce to its particular case given in Theorem 2, i.e. that there is a modulus functions f for which the limit of $\frac{f(n)}{f(2n)}$ does not exist.

Example 3. Put $f(0) = 0$, $f(1) = 1$, $f(2) = 2$. The values of f in the remaining natural numbers we define recurrently: if for some $n \in \mathbb{N}$ the values $f(k)$ are already defined for $k \in [1, 2^n]$, we define $f(k)$ for $k = 2^n + \alpha \in [2^n + 1, 2^{n+1}]$, $\alpha \in [1, 2^n]$, by means of the formula

$$f(2^n+\alpha) = \begin{cases} f(2^n), \text{if } n \in \{1,3,5,..\} \\ f(2^n) + f(\alpha), \text{if } n \in \{2,4,6,...\}. \end{cases} \tag{2}$$

This defines $f(n)$ for each $n \in \mathbb{N} \cup \{0\}$. In the intermediate points let us define f by means of linear interpolation. Such f is defined for each $x \in \mathbb{R}^+$, is monotonic, continuous, and $f(2^k) := 2^{\lceil \frac{k}{2} \rceil}$ for $k \in \mathbb{N} \cup \{0\}$.

Let us verify that f is a modulus function. For every $w, z \in \mathbb{N}$ (without lost of generality we consider $w > z$) we have to demonstrate that

$$f(w+z) \leq f(w) + f(z). \tag{3}$$

This can be done by induction in n, where n is the smallest natural for which $w + z \leq 2^n$.

The base $n = 1$ is straightforward. Suppose now that we already proved (3) for $0 \leq w+z \leq 2^n$ and let us prove it for $2^n < w+z \leq 2^{n+1}$. Denote $w+z = 2^n+\alpha$, where $\alpha \in [1, 2^n]$.

1. Let n be an odd number. It is clear that there are numbers $\tilde{w}, \tilde{z} \in \mathbb{N}$, $\tilde{w} < w$, $\tilde{z} < z$ such that $\tilde{w} + \tilde{z} = 2^n$. Then $f(w+z) = f(2^n) = f(\tilde{w}+\tilde{z}) \leq f(\tilde{w}) + f(\tilde{z}) \leq f(w) + f(z)$.

2. Let n be an even number. Then $f(w+z) = f(2^n+\alpha) = f(2^n) + f(\alpha)$.

   (a) Let $w \geq 2^n$, then $z \leq \alpha$. Represent w in the form of $w = 2^n + \beta$. In this case $f(w+z) = f(2^n) + f(\alpha)$ and $f(w) = f(2^n) + f(\beta)$. Then the inequality (3) rewrites as $f(\alpha) \leq f(z) + f(\beta)$ which is true by the inductive assumption.

   (б) Let $w < 2^n$, which means that $2^{n-1} < w < 2^n$ and $z > \alpha$. Then $f(w) = f(2^{n-1}) = f(2^n)$, because $n-1$ is odd. Again, in this case the inequality (3) is equivalent to a simpler one: $f(\alpha) \leq f(w)$ which is true since $z > \alpha$.

So, we proved that the function, defined by (2) is a modulus function. Consider now the sequence $\dfrac{f(2^n)}{f(2^{n+1})}$, $n = 0, 1, 2, \ldots$. When n is odd we have $\dfrac{f(2^n)}{f(2^{n+1})} = 1$ and if n is even we have $\dfrac{f(2^n)}{f(2^{n+1})} = \dfrac{1}{2}$. This means that the sequence $\dfrac{f(n)}{f(2n)}$ has no limit.

By the way, in this example $g_f(k) = \limsup\limits_{n\to\infty} \dfrac{f(n)}{f(2^k n)} = \dfrac{1}{2^{k-1}}$, so $\mathfrak{I}_f = \mathfrak{I}_s$.

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Conclusion

In our paper we studied the ideal of statistical convergence $\mathfrak{I}_s$ and the ideal $\mathfrak{I}_f$ generated by a modular function f. In our research we gave the complete description of those modular functions f for which $\mathfrak{I}_f = \mathfrak{I}_s$. Then we analysed obtained result, gave some partial cases of it and proved one simple sufficient condition for the equality $\mathfrak{I}_f = \mathfrak{I}_s$. At the end of this article we gave some examples of some modulus functions f, g for which $\mathfrak{I}_f = \mathfrak{I}_s$ and $\mathfrak{I}_g \neq \mathfrak{I}_s$.

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Про зв'язок між статистичним ідеалом та ідеалом, породженим модульною функцією

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Ідеал на довільній непорожній множині Ω – це непорожня сім'я підмножин $\mathfrak{I}$ множини Ω, яка задовольняє наступним умовам: $\Omega \notin \mathfrak{I}$, якщо $A, B \in \mathfrak{I}$, то $A \cup B \in \mathfrak{I}$, якщо $A \in \mathfrak{I}$ і $D \subset A$, то $D \in \mathfrak{I}$. Теорія ідеалів є дуже популярною областю сучасних математичних досліджень. В даній роботі досліджено деякі спеціальні класи ідеалів на множині натуральних чисел $\mathbb{N}$, а саме ідеал статистичної збіжності $\mathfrak{I}_s$, або статистичний ідеал, та ідеал $\mathfrak{I}_f$, який задано модульною функцією $\mathfrak{I}_f$. Статистичний ідеал – це сім'я підмножин множини $\mathbb{N}$, які мають нульову натуральну щільність, тобто $A \in \mathfrak{I}_s$ тоді і тільки тоді, коли $\lim\limits_{n\to\infty} \dfrac{\#\{k \le n : k \in A\}}{n} = 0$. Функцію $f : \mathbb{R}^+ \to \mathbb{R}^+$ називають модульною функцією, якщо $f(x) = 0$ тільки при $x = 0$, $f(x+y) \le f(x) + f(y)$ для будь-яких $x, y \in \mathbb{R}^+$, $f(x) \le f(y)$ якщо $x \le y$, f неперервна справа в 0, і $\lim\limits_{n\to\infty} f(n) = \infty$. Ідеал, який задано модульною функцією – це сім'я підмножин множини $\mathbb{N}$, які мають нульову f-щільність, тобто $A \in \mathfrak{I}_f$ тоді і тільки тоді, коли $\lim\limits_{n\to\infty} \dfrac{f(\#\{k \le n : k \in A\})}{f(n)} = 0$. Відомо, що для довільної модульної функції f ми маємо наступне включення: $\mathfrak{I}_f \subset \mathfrak{I}_s$. В нашій статті ми даємо повний опис таких модульних функцій f, що $\mathfrak{I}_f = \mathfrak{I}_s$. Далі ми досліджуємо отриманий результат, наводимо деякі часткові випадки основного результату та доводимо просту достатню умову для рівності $\mathfrak{I}_f = \mathfrak{I}_s$. Останній розділ нашої роботи присвячено розгляду прикладів конкретних модульних функцій f, для котрих $\mathfrak{I}_f = \mathfrak{I}_s$ і $\mathfrak{I}_f \ne \mathfrak{I}_s$. А саме, у випадку $f(x) = x^p$, при $p \in (0, 1]$ маємо $\mathfrak{I}_f = \mathfrak{I}_s$; якщо $f(x) = \log(1 + x)$, маємо $\mathfrak{I}_f \ne \mathfrak{I}_s$. Далі в якості прикладу ми розглядаємо більш складну функцію f, яка має рекурентну побудову, і яка показує, що умови основного результату даної роботи не можна послабити до одного часткового випадку.

Ключові слова: **ідеал, статистичний ідеал, модульні функції.**

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