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On two resolvent matrices of the truncated Hausdorff matrix moment problem

We consider the truncated Hausdorff matrix moment problem (THMM) in case of a finite number of even moments to be called non degenerate if two block Hankel matrices constructed via the moments are both positive definite matrices. The set of solutions of the THMM problem in case of a finite number of even moments is given with the help of the block matrices of the so-called resolvent matrix. The resolvent matrix of the THMM problem in the non degenerate case for matrix moments of dimension $q \times q$, is a $2q \times 2q$ matrix polynomial constructed via the given moments.

In 2001, in [Yu.M. Dyukarev, A.E. Choque Rivero, Power moment problem on compact intervals, Mat. Sb.-2001. -69(1-2). -P.175-187], the resolvent matrix $V^{(2n+1)}$ for the mentioned THMM problem was proposed for the first time. In 2006, in [A. E. Choque Rivero, Y. M. Dyukarev, B. Fritzsche and B. Kirstein, A truncated matricial moment problem on a finite interval, Interpolation, Schur Functions and Moment Problems. Oper. Theory: Adv. Appl. -2006. - 165. - P. 121-173], another resolvent matrix $U^{(2n+1)}$ for the same problem was given. In this paper, we prove that there is an explicit relation between these two resolvent matrices of the form $V^{(2n+1)} = AU^{(2n+1)}B$, where A and B are constant matrices. We also focus on the following difference: For the definition of the resolvent matrix $V^{(2n+1)}$, one requires an additional condition when compared with the resolvent matrix $U^{(2n+1)}$ which only requires that two block Hankel matrices be positive definite.

In 2015, in [A. E. Choque Rivero, From the Potapov to the Krein-Nudel'man representation of the resolvent matrix of the truncated Hausdorff matrix moment problem, Bol. Soc. Mat. Mexicana. – 2015. – 21(2). – P. 233–259], a representation of the resolvent matrix of 2006 via matrix orthogonal polynomials was given. In this work, we do not relate the resolvent matrix $V^{(2n+1)}$ with the results of [A. E. Choque Rivero, From the Potapov to the Krein-Nudel'man representation of the resolvent matrix of the truncated Hausdorff matrix moment problem, Bol. Soc. Mat. Mexicana. – 2015. – 21(2). – P. 233–259]. The importance of the relation between $U^{(2n+1)}$ and $V^{(2n+1)}$ is explained by the fact that new relations among orthogonal matrix polynomials, Blaschke-Potapov factors, Dyukarev-Stieltjes parameters, and matrix continued fraction can be found. Although in the present work algebraic identities are used, to prove the relation between $U^{(2n+1)}$ and $V^{(2n+1)}$, the analytic justification of both resolvent matrices relies on the V.P. Potapov method. This approach was successfully developed in a number of works concerning interpolation matrix problems in the Nevanlinna class of functions and matrix moment problems.

Keywords: **Hausdorff matrix moment problem; resolvent matrix.**

2010 Mathematics Subject Classification: 30E05; 47A56.

1. Introduction

We will use $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{N}_0$ to denote the set of complex numbers, real numbers, and nonnegative integers, respectively. We will employ $\mathbb{C}^{p\times q}$, $0_{p\times q}$, 0_q, I_q to denote the $p\times q$ complex-valued matrices, the $p\times q$ zero matrix, the $q\times q$ zero matrix, and the $q\times q$ identity matrix, respectively.

We consider the truncated Hausdorff matrix moment (THMM) problem for an even number of moments, which is stated as follows: Let a and b be real numbers with $a<b$, let $n\in\mathbb{N}_0$, and let $(s_j)_{j=0}^{2n+1}$ be a sequence of complex $q\times q$ matrices. Find the set $\mathcal{M}_{\geq}^{q}[[a,b],\mathfrak{B}\cap[a,b];(s_j)_{j=0}^{2n+1}]$ of all nonnegative Hermitian $q\times q$ measures σ defined on the σ-algebra of all Borel subsets of the interval $[a,b]$ such that

$$s_j=\int\limits_{[a,b]} t^j\sigma(dt)$$

holds true for each integer j with $0\leq j\leq 2n+1$.

We construct the following Hankel matrices:

$$H_{1,n}:=\{s_{l+k}\}_{l,k=0}^{n},\quad \widetilde{H}_{1,n}:=\{s_{l+k+1}\}_{l,k=0}^{n},\tag{1}$$

$$H_{3,n}:=bH_{1,n}-\widetilde{H}_{1,n},\quad H_{4,n}:=-aH_{1,n}+\widetilde{H}_{1,n}.\tag{2}$$

Definition 1. *Let the Hankel matrices $H_{3,n}$ and $H_{4,n}$ be as in (2). The sequence $(s_k)_{k=0}^{2n+1}$ is called Hausdorff positive definite (resp. nonnegative) on $[a,b]$ if the block Hankel matrices $H_{3,n}$ and $H_{4,n}$ are both positive (resp. nonnegative) definite.*

In [1], it is proved that the THMM problem for the even number of moments has a solution if $\{s_j\}_{j=0}^{2n+1}$ is a nonnegative definite sequence.

Once we have verified the existence of solutions for the THMM problem for the even number of moments, instead of determining the set of measures σ, we look for matrix-valued holomorphic functions $s(z)$ defined as

$$s(z) := \int\limits_{[a,b]} \frac{1}{z-t}\sigma(dt), \ z \in \mathbb{C} \setminus [a,b], \quad \sigma \in \mathcal{M}_{\geq}^{q}[[a,b], \mathfrak{B} \cap [a,b]; \{s_j\}_{j=0}^{2n+1}]. \quad (3)$$

In (3), for each constructed measure σ, we have a unique $s(z)$. We use $\mathfrak{S}_{\geq}^{q}[\mathfrak{B} \cap [a,b]; (s_j)_{j=0}^{2n+1}]$ to denote the set of all such $s(z)$. In the case when the sequence $\{s_j\}_{j=0}^{2n+1}$ is a positive definite sequence, called non-degenerate case, there are an infinite number of associated solutions. Each such solution can be written as

$$s(z) = \left(\alpha^{(2n+1)}(z)\,\mathbf{p}(z) + \beta^{(2n+1)}(z)\,\mathbf{q}(z)\right)\left(\gamma^{(2n+1)}(z)\,\mathbf{p}(z) + \delta^{(2n+1)}(z)\,\mathbf{q}(z)\right)^{-1}$$

where $\alpha^{(2n+1)}$, $\beta^{(2n+1)}$, $\gamma^{(2n+1)}$ and $\delta^{(2n+1)}$ are matrix–valued polynomials on the variable z and are determined by the moments $(s_j)_{j=0}^{2n+1}$. The quantities $\mathbf{p}$ and $\mathbf{q}$ denote $q \times q$ matrix-valued functions of z, which do not depend on the moments $(s_j)_{j=0}^{2n+1}$. The set of column pairs column$(\mathbf{p}, \mathbf{q})$ is described in 2006 in [1, Definition 5.2]. The $2q \times 2q$ matrix-valued function

$$U^{(2n+1)} := \begin{bmatrix} \alpha^{(2n+1)} & \beta^{(2n+1)} \\ \gamma^{(2n+1)} & \delta^{(2n+1)} \end{bmatrix} \quad (4)$$

is called the resolvent matrix (RM) of the THMM problem for an even number of moments.

In 2001, in [19], another resolvent matrix for solving the same THMM problem was proposed, which we denote by

$$V^{(2n+1)} := \begin{bmatrix} \widehat{\alpha}^{(2n+1)} & \widehat{\beta}^{(2n+1)} \\ \widehat{\gamma}^{(2n+1)} & \widehat{\delta}^{(2n+1)} \end{bmatrix}. \quad (5)$$

The matrices $\widehat{\alpha}^{(2n+1)}$, $\widehat{\beta}^{(2n+1)}$, $\widehat{\gamma}^{(2n+1)}$ and $\widehat{\delta}^{(2n+1)}$ are constructed by using the sequence of moments $(s_j)_{j=0}^{2n+1}$. With the help of the entries of the matrix polynomial (5) and a family of columns pairs column$(\widetilde{\mathbf{p}}, \widetilde{\mathbf{q}})$ [19, Equalities (18)-(20)], an associated solution to the THMM problem is given in [19, Theorem 6].

The goal of this paper is to determine an explicit relation between the resolvent matrices $U^{(2n+1)}$ and $V^{(2n+1)}$ of the form

$$V^{(2n+1)}(z) = AU^{(2n+1)}(z)B \quad (6)$$

for $z \in \mathbb{C}$. Here A and B are constant matrices. The resolvent matrices $U^{(2n+1)}$ and $V^{(2n+1)}$ are both matrix polynomials on the variable z. They differ as described below:

- For the definition of matrix $U^{(2n+1)}$, the positive definiteness of matrices $H_{3,n}$ and $H_{4,n}$ are required. For the definition of matrix $V^{(2n+1)}$, however, the invertibility of matrix $\widetilde{H}_{1,n}$ is additionally required.

The relation between the resolvent matrix $V^{(2n+1)}$ and resolvent matrix for the truncated Hausdorff matrix moment in case of an even number of moments proposed in [8] and the help of orthogonal matrix polynomials will be considered in forthcoming work.

The importance of the relation between $U^{(2n+1)}$ and $V^{(2n+1)}$ is explained by the fact that well-known objects such as orthogonal polynomials [25], [8], Blaschke-Potapov factors [5], [6], Dyukarev-Stieltjes parameters [9],[12], continued fractions [11] and the three-term recurrence relation coefficients [13] related to the THMM problem in the case of an even number of moments can be obtained with new expressions. Additionally, the mentioned relation can be used in the control problem in a similar way as in [4, 3, 10]. In the frame of the V.P. Potapov schema, interpolation problems in the Nevanlinna or Stieltjes class of functions and matrix moment problems are studied in [2], [7], [15], [16], [17], [18], [20], [21] and [23]. The THMM problem was recently studied in [22] via an Schur–Nevanlinna type algorithm. In [26] and [14], the operator approach was applied to solve the THMM problem.

2. Notations and preliminaries

In this section, we reproduce matrices, which allow us to define the entries of the resolvent matrix (5) and (4). Let $R_n : \mathbb{C} \to \mathbb{C}^{(n+1)q\times(n+1)q}$ be given by

$$R_n(z) := (I_{(n+1)q} - zT_n)^{-1}, \quad n \geq 0, \tag{7}$$

with

$$T_0 := 0_q, \quad T_n := \begin{bmatrix} 0_{q\times nq} & 0_q \\ I_{nq} & 0_{nq\times q} \end{bmatrix}, \quad n \geq 1. \tag{8}$$

Let

$$v_0 := I_q, \quad v_n := \begin{bmatrix} I_q \\ 0_{nq\times q} \end{bmatrix}, \quad n \geq 1. \tag{9}$$

Furthermore,

$$u_n := \text{column}\left(-s_0, -s_1, \ldots, -s_n\right), \tag{10}$$

$$u_{3,0} := s_0, \qquad u_{3,n} := -R_n^{-1}(b)u_n, \tag{11}$$

$$u_{4,0} := -s_0, \qquad u_{4,n} := R_n^{-1}(a)u_n. \tag{12}$$

If A is complex matrix, then A^* denotes the conjugate transpose of A.

Lemma 1. *Let $(s_j)_{j=0}^{2n+1}$ be a Hausdorff positive definite sequence on $[a,b]$. Furthermore, let $H_{1,n}$, $\widetilde{H}_{1,n}$, $H_{4,n}$, $H_{3,n}$, T_n, u_n, $u_{3,n}$ and $u_{4,n}$ be as in (1), (2),* *(8), (10), (11) and (12), respectively. Thus, the following identities hold:*

$$H_{1,n} = \frac{1}{b-a}(H_{3,n} + H_{4,n}), \tag{13}$$

$$\widetilde{H}_{1,n} = \frac{1}{b-a}(bH_{4,n} + aH_{3,n}), \tag{14}$$

$$u_n v_n^* = \widetilde{H}_{1,n} T_n^* - H_{1,n}, \tag{15}$$

$$H_{r,n} T_n^* - T_n H_{r,n} = u_{r,n} v_n^* - v_n u_{r,n}^*, \quad r = 3, 4. \tag{16}$$

Proof. *Equality (13) (resp. Equality (14)) can be readily verified from (1) and (2). Equality (13) is considered in [1, Equation (2.1)]. Equality (15) is proved in [9, Equation (6.17)]. Equality (16) is considered in [19, Page 178] and [1, Equation (2.2)].*

Proposition 1. *For $a < b$, let $H_{1,n}$, $\widetilde{H}_{1,n}$, $H_{3,n}$ and $H_{4,n}$ be as in (1) and (2). Moreover, let $H_{3,n}$ and $H_{4,n}$ be positive definite matrices.*
a) Thus, the matrix $H_{1,n}$ is a positive definite matrix.
b) Let $0 < a < b$ (resp. $a < b < 0$), then matrix $\widetilde{H}_{1,n}$ is a positive definite matrix (negative definite matrix).
c) If $a < 0 < b$, the determinant of the matrix $\widetilde{H}_{1,n}$ can be equal to 0.

Proof. *For the proof of part a) and part b) with the condition $0 < a < b$, we use (13), (14) and the fact that the sum of positive definite matrices is a positive definite matrix. To verify the part b) with $a < b < 0$, it is sufficient to multiply equality (14) by -1 and apply part a). Part c) we prove by giving an example. Let $a = -3$, $b = 3(2-\sqrt{3})$, $s_0 = -54(-2+\sqrt{3})$, $s_1 = 81(5-3\sqrt{3})$ and $s_2 = -486(-7+4\sqrt{3})$. Furthermore, let s_3 be such that*

$$-243\left(47\sqrt{3} - 81\right) < s_3 < 243\left(147 - 85\sqrt{3}\right).$$

An approximation of this interval is given by $-98.7522 < s_3 < -54.5094$. Clearly, the matrices $H_{3,1}$ $H_{4,1}$ are positive definite, while the the determinant of the matrix $\widetilde{H}_{1,1}$ on $s_3 = 1458\left(19 - 11\sqrt{3}\right)$ is equal to 0. An approximation of the latter value is $s_3 = -76.6309$.

In the following definition, we use the invertibility of the matrix $\widetilde{H}_{1,n} = \frac{bH_{4,n}+aH_{3,n}}{b-a}$.

Definition 2. *[19, Theorem 4] Let $(s_k)_{k=0}^{2n+1}$ be a Hausdorff positive definite sequence on $[a, b]$. Let $H_{r,n}$, $u_{r,n}$ for $r = 3, 4$, R_n and v_n be defined as in (2), (11), (12), (7) and (9), respectively. Furthermore, let $\widetilde{H}_{1,n}$ as in (14) be an invertible matrix.*

The entries of the $2q \times 2q$ matrix polynomial $V^{(2n+1)}(z)$ as in (5) are defined *by*

$$\widehat{\alpha}^{(2n+1)}(z) := I_q + zv_n^* R_n^*(\bar{z}) \left(\frac{bH_{4,n} + aH_{3,n}}{b-a} \right)^{-1} u_n, \tag{17}$$

$$\widehat{\gamma}^{(2n+1)}(z) := u_n^* R_n^*(\bar{z}) \left(\frac{bH_{4,n} + aH_{3,n}}{b-a} \right)^{-1} u_n, \tag{18}$$

$$\widehat{\beta}^{(2n+1)}(z) := (z-b)(z-a) v_n^* R_n^*(\bar{z}) \frac{aH_{4,n}^{-1} + bH_{3,n}^{-1}}{b-a} v_n, \tag{19}$$

$$\widehat{\delta}^{(2n+1)}(z) := I_q + u_n^* R_n^*(\bar{z}) \frac{a(z-b) R_n^{*^{-1}}(a) H_{4,n}^{-1} + b(z-a) R_n^{*^{-1}}(b) H_{3,n}^{-1}}{b-a} v_n. \tag{20}$$

The matrix (5) is called the first resolvent matrix of the THMM problem in the case of an even number of moments.

Matrix $V^{(2n+1)}(z)$ has important properties concerning the matrix

$$J_q := \begin{bmatrix} 0_q & -iI_q \\ iI_q & 0_q \end{bmatrix}. \tag{21}$$

In particular, the inverse matrix of $V^{(2n+1)}(z)$ can be explicitly calculated via two $2q \times 2q$ matrices defined below.

Definition 3. *Let $(s_k)_{k=0}^{2n+1}$ be a Hausdorff positive definite sequence on $[a, b]$. Let $H_{r,n}$, $u_{r,n}$ for $r = 3, 4$, R_n, v_n and J_q be defined as in (2), (11), (12), (7), (9) and (21), respectively. For $r = 3, 4$, let*

$$\widetilde{V}_r^{(2n+1)}(z) := I_{2q} - iz \begin{bmatrix} v_n^* \\ u_{r,n}^* \end{bmatrix} R_n^*(\bar{z}) H_{r,n}^{-1} [v_n \ u_{r,n}] J_q. \tag{22}$$

The matrix (22) is called the first Kovalishina resolvent matrix of the THMM problem in the case of an even number of moments.
Furthermore, let $\widetilde{H}_{1,n}$ as in (14) be an invertible matrix. Let

$$M_{4,n} := -au_n^* \widetilde{H}_{1,n}^{-1} u_n, \qquad M_{3,n} := bu_n^* \widetilde{H}_{1,n}^{-1} u_n,$$
$$N_{4,n} := -bv_n^* H_{4,n}^{-1} \widetilde{H}_{1,n} H_{3,n}^{-1} v_n, \qquad N_{3,n} := av_n^* H_{4,n}^{-1} \widetilde{H}_{1,n} H_{3,n}^{-1} v_n.$$

For $r = 3, 4$, let

$$C_r^{(2n+1)} := \left[\begin{array}{c|c} I_q & 0_q \\ \hline M_{r,n} & I_q \end{array} \right], \qquad D_r^{(2n+1)} := \left[\begin{array}{c|c} I_q & N_{r,n} \\ \hline 0_q & I_q \end{array} \right],$$

and

$$V_r^{(2n+1)}(z) := \widetilde{V}_r^{(2n+1)}(z) C_r^{(2n+1)} D_r^{(2n+1)}. \tag{23}$$

The matrix (23) is called the first auxiliary resolvent matrix of the THMM problem in the case of an even number of moments.

Equality (22) (resp. (23)) appears in [19, Theorem3] (resp. [19, Equality (7)]). Note that the name Kovalishina resolvent matrix for the matrix $\widetilde{V}_r^{(2n+1)}(z)$ was suggested by Yury Dyukarev as the paper [19] was being prepared. Prof. Irina Kovalishina studied matrix interpolation and moment problems in the frame of the V.P. Potapov method. See [23].

Observe that the matrix $\widetilde{V}_r^{(2n+1)}$ can be expressed in the following form

$$\widetilde{V}_r^{(2n+1)}(z) = \begin{bmatrix} \widetilde{\alpha}_r^{(n)}(z) & \widetilde{\beta}_r^{(n)}(z) \\ \widetilde{\gamma}_r^{(n)}(z) & \widetilde{\delta}_r^{(n)}(z), \end{bmatrix},$$

where

$$\begin{aligned}
\widetilde{\alpha}_r^{(n)}(z) :=& I_q + zv_n^* R_n^*(\bar{z}) H_{r,n}^{-1} u_{r,n}, \\
\widetilde{\beta}_r^{(n)}(z) :=& - zv_n^* R_n^*(\bar{z}) H_{r,n}^{-1} v_n, \\
\widetilde{\gamma}_r^{(n)}(z) :=& zu_{4,n}^* R_n^*(\bar{z}) H_{r,n}^{-1} u_{r,n}, \\
\widetilde{\delta}_r^{(n)}(z) :=& I_q - zu_{r,n}^* R_n^*(\bar{z}) H_{r,n}^{-1} v_n.
\end{aligned}$$

In a similar manner, from (23) we write the matrix $V_r^{(2n+1)}$ as follows:

$$V_r^{(2n+1)}(z) = \begin{bmatrix} \breve{\alpha}_r^{(n)}(z) & \breve{\beta}_r^{(n)}(z) \\ \breve{\gamma}_r^{(n)}(z) & \breve{\delta}_r^{(n)}(z) \end{bmatrix}$$

where

$$\begin{aligned}
\breve{\alpha}_r^{(n)}(z) :=& \widetilde{\alpha}_r^{(n)}(z) + \widetilde{\beta}_r^{(n)}(z) M_{r,n}, \\
\breve{\beta}_r^{(n)}(z) :=& \widetilde{\alpha}_r^{(n)}(z) N_{r,n} + \widetilde{\beta}_r^{(n)}(z)(I_q + M_{r,n} N_{r,n}), \\
\breve{\gamma}_r^{(n)}(z) :=& \widetilde{\gamma}_r^{(n)}(z) + \widetilde{\delta}_r^{(n)}(z) M_{r,n}, \\
\breve{\delta}_r^{(n)}(z) :=& \widetilde{\gamma}_r^{(n)}(z) N_{r,n} + \widetilde{\delta}_r^{(n)}(z)(I_q + M_{r,n} N_{r,n}).
\end{aligned}$$

In the next lemma, the explicit representation of the inverse of the matrix $V^{(2n+1)}$ is given.

Lemma 2. *Let $(s_k)_{k=0}^{2n+1}$ be a Hausdorff positive definite sequence on $[a, b]$. Let $H_{r,n}$, $u_{r,n}$, for $r = 3, 4$, R_n, v_n, J_q and $V_r^{(2n+1)}(z)$ be defined as in (2), (11), (12), (7), (9), (21) and Definition 2, respectively. Furthermore, let $\widetilde{H}_{1,n}$ be as in (1). Assume that $\widetilde{H}_{1,n}$ is an invertible matrix. Thus, the following equality holds:*

$$V_r^{(2n+1)^{-1}}(z) = J_q V_r^{(2n+1)^*}(\bar{z}) J_q, \quad r = 3, 4. \tag{24}$$

Moreover,

$$V^{(2n+1)^{-1}}(z) = \begin{bmatrix} \widehat{\delta}^{(2n+1)^*}(\bar{z}) & -\widehat{\beta}^{(2n+1)*}(\bar{z}) \\ -\widehat{\gamma}^{(2n+1)^*}(\bar{z}) & \widehat{\alpha}^{(2n+1)^*}(\bar{z}) \end{bmatrix}. \tag{25}$$

Proof. *The proof of (24) readily follows from the equality*

$$J_q - V_r^{(2n+1)}(x) J_q V_r^{(2n+1)^*}(x) = 0$$

for real x, [19, Equality (8)] and the identity theorem of analytic functions [24, Theorem III.3.2]. Equality (24) appears in [19, Equality (16)]. To prove (25), one uses Equality (24) for $r = 4$ and the equality [19, Equation (10)]

$$V^{(2n+1)}(z) = \begin{bmatrix} I_q & 0_q \\ 0_q & (z-a)^{-1} I_q \end{bmatrix} V_4^{(2n+1)}(z) \begin{bmatrix} I_q & 0_q \\ 0_q & (z-a) I_q \end{bmatrix}, \tag{26}$$

which is valid for $z \in \mathbb{C} \setminus \{a\}$. Finally, because point $z = a$ is a removable discontinuity for the inverse matrix of the right-hand side of (26), we obtain (25).

Definition 4. *[1, Proposition 6.10] Let $(s_k)_{k=0}^{2n+1}$ be a Hausdorff positive definite sequence on $[a, b]$. Let $H_{r,n}$, $u_{r,n}$, for $r = 3, 4$, R_n and v_n be defined as in (2), (11), (12), (7) and (9), respectively.*

The entries of the $2q \times 2q$ matrix polynomial $U^{(2n+1)}(z)$ as in (4) are defined by

$$\alpha^{(2n+1)}(z) := I_q - (z-a) u_{3,n}^* R_n^*(\bar{z}) H_{3,n}^{-1} R_n(a) v_n, \tag{27}$$

$$\beta^{(2n+1)}(z) := u_{4,n}^* R_n^*(\bar{z}) H_{4,n}^{-1} R_n(a) u_{4,n}, \tag{28}$$

$$\gamma^{(2n+1)}(z) := -(b-z)(z-a) v_n^* R_n^*(\bar{z}) H_{3,n}^{-1} R_n(a) v_n, \tag{29}$$

$$\delta^{(2n+1)}(z) := I_q + (z-a) v_n^* R_n^*(\bar{z}) H_{4,n}^{-1} R_n(a) u_{4,n}. \tag{30}$$

The matrix (4) is called the second resolvent matrix of the THMM problem in the case of an even number of moments.

3. Explicit relation between two resolvent matrices

In this section, we prove the explicit relation (6) between the resolvent matrix $V^{(2n+1)}$ presented in [19] and the resolvent matrix $U^{(2n+1)}$ given in [1].

The next Remark can be proved by using (25).

Remark 1. *Let the matrices $\widehat{\alpha}^{(2n+1)}$, $\widehat{\beta}^{(2n+1)}$, $\widehat{\gamma}^{(2n+1)}$ and $\widehat{\delta}^{(2n+1)}$ be as in (17)–(20) Furthermore, let*

$$\mathfrak{J}_q := \begin{bmatrix} 0_q & I_q \\ I_q & 0_q \end{bmatrix}, \tag{31}$$

and the matrix $V^{(2n+1)}$ be as in (5). Thus, the following equality is valid:

$$\mathfrak{J}_q V^{(2n+1)^{-1}}(z) \mathfrak{J}_q = \begin{bmatrix} \widehat{\alpha}^{(2n+1)^*}(\bar{z}) & -\widehat{\gamma}^{(2n+1)^*}(\bar{z}) \\ -\widehat{\beta}^{(2n+1)^*}(\bar{z}) & \widehat{\delta}^{(2n+1)^*}(\bar{z}) \end{bmatrix}. \tag{32}$$

Now we state and prove the main result of this work.

Theorem 1. *Let $(s_k)_{k=0}^{2n+1}$ be a Hausdorff positive definite sequence on $[a,b]$. Let $\widetilde{H}_{1,n}$, $H_{r,n}$, $u_{r,n}$, for $r=3,4$, R_n, v_n, and $\mathfrak{J}_q$ be defined as in (1), (2), (11), (12), (7), (9) and (31), respectively. Furthermore, let $\widetilde{H}_{1,n}$ be an invertible matrix. Moreover, let $U^{(2n+1)}$ and $V^{(2n+1)}$ be the resolvent matrices as in (4) and (5), respectively. In addition, let*

$$\mathfrak{D}^{(2n+1)} := \begin{bmatrix} I_q + au_n^*\widetilde{H}_{1,n}^{-1}R_n(a)v_n & 0_q \\ 0_q & I_q - av_n^*H_{4,n}^{-1}R_n(a)u_{4,n} \end{bmatrix}. \tag{33}$$

Thus, the following equality holds

$$U^{(2n+1)}(z) = \mathfrak{J}_q V^{(2n+1)}(z)\mathfrak{J}_q\mathfrak{D}^{(2n+1)}. \tag{34}$$

The proof of this theorem is provided in the Appendix section.

In the following lemma, to obtain a relation of the form

$$V^{(2n+1)}(z) = AU^{(2n+1)}(z)B, \tag{35}$$

we calculate the inverse of the matrix $\mathfrak{D}^{(2n+1)}$ as in (33).

Lemma 3. *Let $\widetilde{H}_{1,n}$, $H_{4,n}$, T_n, R_n, v_n, u_n and $u_{4,n}$ be as in (1), (2), (8), (7), (10) and (12), respectively. Moreover, let $\widetilde{H}_{1,n}$ be an invertible matrix. Furthermore, let $\mathfrak{D}^{(2n+1)}$ be as in (33). Thus, the following equality holds:*

$$\mathfrak{D}^{(2n+1)^{-1}} = \begin{pmatrix} I_q - au_{4,n}^*R_n^*(a)H_{4,n}^{-1}v_n & 0_q \\ 0_q & I_q + av_n^*R_n^*(a)\widetilde{H}_{1,n}^{-1}u_n \end{pmatrix}. \tag{36}$$

Proof. *Let $\eta_n := I_q + au_n^*\widetilde{H}_{1,n}^{-1}R_n(a)v_n$, $\kappa_n := I_q - av_n^*H_{4,n}^{-1}R_n(a)u_{4,n}$, $\nu_n := I_q - au_{4,n}^*R_n^*(a)H_{4,n}^{-1}v_n$ and $\tau_n := I_q + av_n^*R_n^*(a)\widetilde{H}_{1,n}^{-1}u_n$. Thus, Equality (36) is equivalent to the following two equalities:*

$$\eta_n\nu_n = I_q, \tag{37}$$

$$\kappa_n\tau_n = I_q. \tag{38}$$

We prove (37). Using (12), we have

$$\begin{aligned} \eta_n\nu_n &= (I_q + au_n^*\widetilde{H}_{1,n}^{-1}R_n(a)v_n)(I_q - au_{4,n}^*R_n^*(a)H_{4,n}^{-1}v_n) \\ &= I_q + au_n^*\widetilde{H}_{1,n}^{-1}R_n(a)[-(I_q - aT_n)\widetilde{H}_{1,n} + H_{4,n} - av_nu_n^*]H_{4,n}^{-1}v_n \\ &= I_q + au_n^*\widetilde{H}_{1,n}^{-1}R_n(a)[-\widetilde{H}_{1,n} + H_{4,n} + aH_{1,n}]H_{4,n}^{-1}v_n \\ &= I_q. \end{aligned}$$

In the third equality, we used (15). The last equality follows from the second equality of (2). In a similar manner, Equality (38) can be proved using the Equalities (12), (15) and (2). Thus, Equality (36) is valid.

In the next corollary of Theorem 1, the relation (35) is proven.

Corollary 1. *Let $\widetilde{H}_{1,n}$, $H_{4,n}$, T_n, R_n, v_n, u_n and $u_{4,n}$ be as in (1), (2), (8), (7), (10) and (12), respectively. Furthermore, let $\mathfrak{D}^{(2n+1)}$ be as in (33), and let $\widetilde{H}_{1,n}$ be an invertible matrix. Moreover, let $V^{(2n+1)}$ and $U^{(2n+1)}$ be as in (5) and (4). Thus, the following equality is valid:*

$$V^{(2n+1)}(z) = \mathfrak{J}_q U^{(2n+1)}(z) \mathfrak{D}^{(2n+1)^{-1}} \mathfrak{J}_q. \tag{39}$$

Proof. The proof of Equality (39) readily follows from (34), equality $\mathfrak{J}_q^{-1} = \mathfrak{J}_q$ and (36).

4. Auxiliary identities

In this section, we consider auxiliary identities that we will use in the main theorem of this work.

Lemma 4. *Let T_n, R_n, v_n, $H_{1,n}$, $\widetilde{H}_{1,n}$, $H_{3,n}$, $H_{4,n}$, u_n, $u_{3,n}$ and $u_{4,n}$ be as in (8), (7), (9), (1), (2) (10), (11) and (12), respectively. Moreover, let $z \in \mathbb{C}$. Thus, the following equalities are valid.*

$$(b-a)I_{(n+1)q} = bR_n^{*^{-1}}(a) - aR_n^{*^{-1}}(b), \tag{40}$$

$$-R_n^{-1}(a)H_{3,n}R_n^{*^{-1}}(\bar{z}) + R_n^{-1}(z)H_{3,n}R_n^{*^{-1}}(a) = (z-a)(H_{3,n}T_n^* - T_nH_{3,n}), \tag{41}$$

$$R_n^{-1}(z)\widetilde{H}_{1,n}R_n^{*^{-1}}(a) + zv_nu_{4,n}^* - H_{4,n}R_n^{*^{-1}}(\bar{z}) - (z-a)u_nv_n^* = 0_q, \tag{42}$$

$$(z-a)(v_nu_{3,n}^* - u_{3,n}v_n^*) - R_n^{-1}(a)H_{3,n}R_n^{*^{-1}}(\bar{z}) + R_n^{-1}(z)H_{3,n}R_n^{*^{-1}}(a) = 0_q, \tag{43}$$

$$(z-a)v_nu_{3,n}^* + (z-b)u_{4,n}v_n^* - R_n^{-1}(a)H_{3,n}R_n^{*^{-1}}(\bar{z}) - R_n^{-1}(z)H_{4,n}R_n^{*^{-1}}(b) = 0_q. \tag{44}$$

Proof. The proof of the identities (40) and (41) readily follows from (7), (8) and (2). To verify Equality (42), we use (7), (12), (15) and (2) to obtain:

$$\begin{aligned}
&R_n^{-1}(z)\widetilde{H}_{1,n}R_n^{*^{-1}}(a) + zv_nu_{4,n}^* - H_{4,n}R_n^{*^{-1}}(\bar{z}) - (z-a)u_nv_n^* \\
&= (I_{(n+1)q} - zT_n)\widetilde{H}_{1,n}(I_{(n+1)q} - aT_n^*) + z(T_n\widetilde{H}_{1,n} - H_{1,n})(I_{(n+1)q} - aT_n^*) \\
&\quad - (-aH_{1,n} + \widetilde{H}_{1,n})(I_{(n+1)q} - zT_n^*) - (z-a)(\widetilde{H}_{1,n}T_n^* - H_{1,n}) \\
&= \widetilde{H}_{1,n} - a\widetilde{H}_{1,n}T_n^* - zT_n\widetilde{H}_{1,n} + azT_n\widetilde{H}_{1,n}T_n^* + zT_n\widetilde{H}_{1,n} - zH_{1,n} \\
&\quad - azT_n\widetilde{H}_{1,n}T_n^* + azH_{1,n}T_n^* + aH_{1,n} - \widetilde{H}_{1,n} - azH_{1,n}T_n^* + z\widetilde{H}_{1,n}T_n^* \\
&\quad - z\widetilde{H}_{1,n}T_n^* + zH_{1,n} + a\widetilde{H}_{1,n}T_n^* - aH_{1,n} \\
&= 0_q.
\end{aligned}$$

Equality (43) follows from (16) for $r = 3$ and (41).

We now we prove (44). We use (7), the identities (40) and (15):

$$
\begin{aligned}
&- R_n^{-1}(a)H_{3,n}R_n^{*^{-1}}(\bar{z}) - R_n^{-1}(z)H_{4,n}R_n^{*^{-1}}(b)\\
&= -(I_{(n+1)q} - aT_n)H_{3,n}(I_{(n+1)q} - zT_n^*) - (I_{(n+1)q} - zT_n)H_{4,n}(I_{(n+1)q} - bT_n^*)\\
&= -bH_{1,n} + bzH_{1,n}T_n^* + \widetilde{H}_{1,n} - z\widetilde{H}_{1,n}T_n^* + abT_nH_{1,n} - abzT_nH_{1,n}T_n^*\\
&\quad - aT_n\widetilde{H}_{1,n} + azT_n\widetilde{H}_{1,n}T_n^* + aH_{1,n} - abH_{1,n}T_n^* - \widetilde{H}_{1,n} + b\widetilde{H}_{1,n}T_n^*\\
&\quad - azT_nH_{1,n} + abzT_nH_{1,n}T_n^* + zT_n\widetilde{H}_{1,n} - bzT_n\widetilde{H}_{1,n}T_n^*\\
&= -a(T_n\widetilde{H}_{1,n} - H_{1,n}) - bz(T_n\widetilde{H}_{1,n} - H_{1,n})T_n^* + z(T_n\widetilde{H}_{1,n} - H_{1,n})\\
&\quad + ab(T_n\widetilde{H}_{1,n} - H_{1,n})T_n^* + b(\widetilde{H}_{1,n}T_n^* - H_{1,n}) - abT_n(\widetilde{H}_{1,n}T_n^* - H_{1,n})\\
&\quad + azT_n(\widetilde{H}_{1,n}T_n^* - H_{1,n}) - z(\widetilde{H}_{1,n}T_n^* - H_{1,n})\\
&= -av_nu_n^* - bzv_nu_n^*T_n^* + zv_nu_n^* + abv_nu_n^*T_n^*\\
&\quad + bu_nv_n^* - abT_nu_nv_n^* + azT_nu_nv_n^* - zu_nv_n^*\\
&= -(z-a)v_nu_n^*(-I_{(n+1)q} + bT_n^*) - (z-b)(I_{(n+1)q} - aT_n)u_nv_n^*\\
&= -(z-a)v_nu_{3,n}^* - (z-b)u_{4,n}v_n^*.
\end{aligned}
$$

In the last equality, we used (7), (11) and (12).

Lemma 5. *Let the conditions of Lemma 4 be satisfied. Thus, the following identities are satisfied:*

$$
\begin{aligned}
&zR_n^{-1}(a)H_{3,n}R_n^{*^{-1}}(\bar{z}) - z(z-a)v_nu_{3,n}^* + (b-z)(z-a)u_nv_n^*\\
&= R_n^{-1}(z)((az + bz - ab)\widetilde{H}_{1,n}T_n^* - z\widetilde{H}_{1,n} + abH_{1,n}R_n^{*^{-1}}(\bar{z}),
\end{aligned} \tag{45}
$$

$$
\begin{aligned}
&(z-a)(z-b)H_{4,n}T_n^* - (z-a)(z-b)T_nH_{4,n}\\
&\quad + (z-b)R_n^{-1}(a)H_{4,n}R_n^{*^{-1}}(\bar{z}) - (z-a)R_n^{-1}(z)H_{4,n}R_n^{*^{-1}}(b)\\
&= -(b-a)R_n^{-1}(z)H_{4,n}R_n^*(\bar{z}),
\end{aligned} \tag{46}
$$

$$
\begin{aligned}
&R_n^{*^{-1}}(\bar{z}) + \frac{(z-b)}{(b-a)}R_n^{*^{-1}}(a) + b\frac{(z-a)}{(b-a)}R_n^{*^{-1}}(b)H_{3,n}^{-1}v_nu_n^*\\
&= -\frac{(z-a)}{(b-a)}R_n^{*^{-1}}(b)H_{3,n}^{-1}R_n^{-1}(b)\widetilde{H}_{1,n}.
\end{aligned} \tag{47}
$$

P r o o f. To prove (45), we use (2), (7) and (15):

$$\begin{aligned}
&zR_n^{-1}(a)H_{3,n}R_n^{*^{-1}}(\bar z) - z(z-a)v_nu_{3,n}^* + (b-z)(z-a)u_nv_n^* \\
&= z(I_{(n+1)q} - aT_n)H_{3,n}(I_{(n+1)q} - zT_n^*) - z(z-a)v_nu_{3,n}^* + (b-z)(z-a)u_nv_n^* \\
&= zH_{3,n} - z^2H_{3,n}T_n^* - azT_nH_{3,n} + az^2T_nH_{3,n}T_n^* - azT_n\widetilde H_{1,n} + azH_{1,n} \\
&\quad + abzT_n\widetilde H_{1,n}T_n^* - abzH_{1,n}T_n^* + z^2T_n\widetilde H_{1,n} - z^2H_{1,n} - z^2bT_n\widetilde H_{1,n}T_n^* \\
&\quad + bz^2H_{1,n}T_n^* + bz\widetilde H_{1,n}T_n^* - bzH_{1,n} - ab\widetilde H_{1,n}T_n^* + abH_{1,n} - z^2\widetilde H_{1,n}T_n^* \\
&\quad + z^2H_{1,n} + az\widetilde H_{1,n}T_n^* - azH_{1,n} \\
&= zbH_{1,n} - z\widetilde H_{1,n} - bz^2H_{1,n}T_n^* + z^2\widetilde H_{1,n}T_n^* - azbT_nH_{1,n} + azT_n\widetilde H_{1,n} \\
&\quad + abz^2T_nH_{1,n}T_n^* - az^2T_n\widetilde H_{1,n}T_n^* - azT_n\widetilde H_{1,n} - ab(I_{(n+1)q} - zT_n)\widetilde H_{1,n}T_n^* \\
&\quad - abzH_{1,n}T_n^* + z^2T_n\widetilde H_{1,n} + bz^2H_{1,n}T_n^* + bz(I_{(n+1)q} - zT_n)\widetilde H_{1,n}T_n^* - bzH_{1,n} \\
&\quad + abH_{1,n} - z^2\widetilde H_{1,n}T_n^* + az\widetilde H_{1,n}T_n^* \\
&= ab(I_{(n+1)q} - zT_n)H_{1,n} - abz(I_{(n+1)q} - zT_n)H_{1,n}T_n^* - ab(I_{(n+1)q} - zT_n)\widetilde H_{1,n}T_n^* \\
&\quad + bz(I_{(n+1)q} - zT_n)\widetilde H_{1,n}T_n^* - z(I_{(n+1)q} - zT_n)\widetilde H_{1,n} + az(I_{(n+1)q} - zT_n)\widetilde H_{1,n}T_n^* \\
&= (I_{(n+1)q} - zT_n)((az + bz - ab)\widetilde H_{1,n}T_n^* - z\widetilde H_{1,n} + abH_{1,n}(I_{(n+1)q} - zT_n^*)) \\
&= R_n^{-1}(z)((az + bz - ab)\widetilde H_{1,n}T_n^* - z\widetilde H_{1,n} + abH_{1,n}R_n^{*^{-1}}(\bar z).
\end{aligned}$$

In the second equality, we used the first equality of (2). The last equality follows from (7).

Now we prove Equality (46). We use (7) and obtain

$$\begin{aligned}
&(z-a)(z-b)H_{4,n}T_n^* - (z-a)(z-b)T_nH_{4,n} + (z-b)R_n^{-1}(a)H_{4,n}R_n^{*^{-1}}(\bar z) \\
&\quad - (z-a)R_n^{-1}(z)H_{4,n}R_n^{*^{-1}}(b) \\
&= z^2H_{4,n}T_n^* - bzH_{4,n}T_n^* - azH_{4,n}T_n^* + abH_{4,n}T_n^* - z^2T_nH_{4,n} + bzT_nH_{4,n} \\
&\quad + azT_nH_{4,n} - abT_nH_{4,n} + zH_{4,n} - z^2H_{4,n}T_n^* - azT_nH_{4,n} + az^2T_nH_{4,n}T_n^* \\
&\quad - bH_{4,n} + bzH_{4,n}T_n^* + abT_nH_{4,n} - azbT_nH_{4,n}T_n^* - zH_{4,n} + bzH_{4,n}T_n^* \\
&\quad + z^2T_nH_{4,n} - bz^2T_nH_{4,n}T_n^* + aH_{4,n} - abH_{4,n}T_n^* - azT_nH_{4,n} + abzT_nH_{4,n}T_n^* \\
&= (b-a)zT_nH_{4,n} + (b-a)zH_{4,n}T_n^* - (b-a)z^2T_nH_{4,n}T_n^* - (b-a)H_{4,n} \\
&= (b-a)(-(I_{(n+1)q} - zT_n)H_{4,n} + z(I_{(n+1)q} - zT_n)H_{4,n}T_n^*) \\
&= -(b-a)(I_{(n+1)q} - zT_n)H_{4,n}(I_{(n+1)q} - zT_n^*) \\
&= -(b-a)R_n^{-1}(z)H_{4,n}R_n^*(\bar z).
\end{aligned}$$

In the last equality, we used (7).

Finally, we prove the identity of (47). We perform the left–hand side of (47).

We use the Equalities (7) and (15) and obtain

$$\begin{aligned}
&R_n^{*^{-1}}(\bar{z}) + \frac{(z-b)}{(b-a)}R_n^{*^{-1}}(a) + b\frac{(z-a)}{(b-a)}R_n^{*^{-1}}(b)H_{3,n}^{-1}v_nu_n^* \\
&= \frac{(z-a)}{(b-a)}[I_{(n+1)q} - bT_n^* + (I_{(n+1)q} - bT_n^*)bH_{3,n}^{-1}v_nu_n^*] \\
&= \frac{(z-a)}{(b-a)}R_n^{*^{-1}}(b)H_{3,n}^{-1}[H_{3,n} + bv_nu_n^*] \\
&= \frac{(z-a)}{(b-a)}R_n^{*^{-1}}(b)H_{3,n}^{-1}[H_{3,n} + b(T_n\widetilde{H}_{1,n} - H_{1,n})] \\
&= -\frac{(z-a)}{(b-a)}R_n^{*^{-1}}(b)H_{3,n}^{-1}R_n^{-1}(b)\widetilde{H}_{1,n}.
\end{aligned}$$

We now give the proof of Theorem 1.
Proof of Theorem 1

Proof. *Since $\mathfrak{J}_q^{-1} = \mathfrak{J}_q$ and the inverse of the matrix $V^{(2n+1)}(z)$ is well defined, we prove the equality*

$$(\mathfrak{J}_qV^{(2n+1)}(z)\mathfrak{J}_q)^{-1}U^{(2n+1)}(z) = \mathfrak{D}^{(2n+1)}, \tag{48}$$

which is equivalent to (34). Using the equality $\mathfrak{J}_q^{-1} = \mathfrak{J}_q$, Equality (32) and the representation (4), we denote the left-hand side of (48) as follows:

$$\begin{pmatrix} W_{11;n} & W_{12;n} \\ W_{21;n} & W_{22;n} \end{pmatrix} := (\mathfrak{J}_qV^{(2n+1)}(z)\mathfrak{J}_q)^{-1}U^{(2n+1)}(z), \tag{49}$$

where

$$W_{11;n} := \widehat{\alpha}^{(2n+1)^*}(\bar{z})\alpha^{(2n+1)}(z) - \widehat{\gamma}^{(2n+1)^*}(\bar{z})\gamma^{(2n+1)}(z), \tag{50}$$
$$W_{12;n} := \widehat{\alpha}^{(2n+1)^*}(\bar{z})\beta^{(2n+1)}(z) - \widehat{\gamma}^{(2n+1)^*}(\bar{z})\delta^{(2n+1)}(z), \tag{51}$$
$$W_{21;n} := -\widehat{\beta}^{(2n+1)^*}(\bar{z})\alpha^{(2n+1)}(z) + \widehat{\delta}^{(2n+1)^*}(\bar{z})\gamma^{(2n+1)}(z), \tag{52}$$
$$W_{22;n} := -\widehat{\beta}^{(2n+1)^*}(\bar{z})\beta^{(2n+1)}(z) + \widehat{\delta}^{(2n+1)^*}(\bar{z})\delta^{(2n+1)}(z). \tag{53}$$

We now perform the expression (50)–(53). For the expression (50), by using (17), *(27), (18) and (29), we have*

$$\begin{aligned}
&\widehat{\alpha}^{(2n+1)^*}(\bar{z})\alpha^{(2n+1)}(z) - \widehat{\gamma}^{(2n+1)^*}(\bar{z})\gamma^{(2n+1)}(z) \\
&= I_q - (z-a)u_{3,n}^* R_n^*(\bar{z})H_{3,n}^{-1}R_n(a)v_n + zu_n^*\widetilde{H}_{1,n}^{-1}R_n(z)v_n \\
&\quad - z(z-a)u_n^*\widetilde{H}_{1,n}^{-1}R_n(z)v_n u_{3,n}^* R_n^*(\bar{z})H_{3,n}^{-1}R_n(a)v_n \\
&\quad + (b-z)(z-a)u_n^*\widetilde{H}_{1,n}^{-1}R_n(z)u_n v_n^* R_n^*(\bar{z})H_{3,n}^{-1}R_n(a)v_n \\
&= I_q + zu_n^* R_n^*(\bar{z})H_{3,n}^{-1}R_n(a)v_n - bzu_n^*T_n^*R_n^*(\bar{z})H_{3,n}^{-1}R_n(a)v_n \\
&\quad - au_n^*R_n^*(\bar{z})H_{3,n}^{-1}R_n(a)v_n + abu_n^*T_n^*R_n^*(\bar{z})H_{3,n}^{-1}R_n(a)v_n \\
&\quad + azu_n^*T_n^*R_n^*(\bar{z})H_{3,n}^{-1}R_n(a)v_n + bzu_n^*T_n^*R_n^*(\bar{z})H_{3,n}^{-1}R_n(a)v_n \\
&\quad - abu_n^*T_n^*R_n^*(\bar{z})H_{3,n}^{-1}R_n(a)v_n - zu_n^*R_n^*(\bar{z})H_{3,n}^{-1}R_n(a)v_n \\
&\quad + abu_n^*\widetilde{H}_{1,n}^{-1}H_{1,n}H_{3,n}^{-1}R_n(a)v_n \\
&= I_q - au_n^*(I_{(n+1)q} - zT_n^*)R_n^*(\bar{z})H_{3,n}^{-1}R_n(a)v_n + abu_n^*\widetilde{H}_{1,n}^{-1}H_{1,n}H_{3,n}^{-1}R_n(a)v_n \\
&= I_q + au_n^*\widetilde{H}_{1,n}^{-1}[-\widetilde{H}_{1,n} + bH_{1,n}]H_{3,n}^{-1}R_n(a)v_n \\
&= I_q + au_n^*\widetilde{H}_{1,n}^{-1}R_n(a)v_n.
\end{aligned}$$

In the second equality, we used (11) and (15). In the fourth equality, we used the first equality of (2). Thus, Equality (48) is proved for the (1,1) block matrix.

Let us now prove Equality (48) for the (1,2) block matrix. Using (51), (17), (18), (28), (30) and the identity (14), we have

$$\begin{aligned}
&\widehat{\alpha}^{(2n+1)^*}(\bar{z})\beta^{(2n+1)}(z) - \widehat{\gamma}^{(2n+1)^*}(\bar{z})\delta^{(2n+1)}(z) \\
&= u_{4,n}^*R_n^*(\bar{z})H_{4,n}^{-1}R_n(a)u_{4,n} + zu_n^*\widetilde{H}_{1,n}^{-1}R_n(z)v_n u_{4,n}^*R_n^*(\bar{z})H_{4,n}^{-1}R_n(a)u_{4,n} \\
&\quad - u_n^*\widetilde{H}_{1,n}^{-1}R_n(z)u_n - (z-a)u_n^*\widetilde{H}_{1,n}^{-1}R_n(z)u_n v_n^*R_n^*(\bar{z})H_{4,n}^{-1}R_n(a)u_{4,n} \\
&= u_n^*\widetilde{H}_{1,n}^{-1}R_n(z)[R_n^{-1}(z)\widetilde{H}_{1,n}R_n^{*^{-1}}(a) + zv_n u_{4,n}^* - H_{4,n}R_n^{*^{-1}}(\bar{z}) \\
&\quad - (z-a)u_n v_n^*]R_n^*(\bar{z})H_{4,n}^{-1}u_n \\
&= 0_q.
\end{aligned}$$

In the last equality, we used (42). Thus, Equality (48) for the (1,2) block matrix is proved.

We now prove the Equality (48) for the (2,1) block matrix. Using (52), (19), *(20), (27) and (29), we obtain*

$$
\begin{aligned}
&-\widehat{\beta}^{(2n+1)^*}(\bar z)\alpha^{(2n+1)}(z)+\widehat{\delta}^{(2n+1)^*}(\bar z)\gamma^{(2n+1)}(z)\\
&=(z-b)\frac{(z-a)}{(b-a)}v_n^*[-(aH_{4,n}^{-1}+bH_{3,n}^{-1})R_n(z)R_n^{-1}(a)H_{3,n}R_n^{*^{-1}}(\bar z)\\
&\quad+(z-a)(aH_{4,n}^{-1}+bH_{3,n}^{-1})R_n(z)v_nu_{3,n}^*+(b-a)I_q+(a(z-b)H_{4,n}^{-1}R_n^{-1}(a)\\
&\quad+b(z-a)H_{3,n}^{-1}R_n^{-1}(b))R_n(z)u_nv_n^*]R_n^*(\bar z)H_{3,n}^{-1}R_n(a)v_n\\
&=(z-b)\frac{(z-a)}{(b-a)}v_n^*\{aH_{4,n}^{-1}R_n(z)[-R_n^{-1}(a)H_{3,n}R_n^{*^{-1}}(\bar z)+(z-a)v_nu_{3,n}^*\\
&\quad+(z-a)v_nu_{3,n}^*+(z-b)R_n^{-1}(a)u_nv_n^*-R_n^{-1}(z)H_{4,n}R_n^{*^{-1}}(b)]+bH_{3,n}^{-1}R_n(z)\\
&\quad\cdot[-R_n^{-1}(a)H_{3,n}R_n^{*^{-1}}(\bar z)+(z-a)R_n^{-1}(b)u_nv_n^*+R_n^{-1}(z)H_{3,n}R_n^{*^{-1}}(a)]\}\\
&\quad\cdot R_n^*(\bar z)H_{3,n}^{-1}R_n(a)v_n\\
&=0_q.
\end{aligned}
$$

In the last equality, we employed Equality (44). Thus, Equality (48) for the (2,1) matrix–block is proved.

We now prove the Equality (48) for the (2,2) block matrix. Using (19), (20), (28) and (30), we have

$$
\begin{aligned}
&-\widehat{\beta}^{(2n+1)^*}(\bar z)\beta^{(2n+1)}(z)+\widehat{\delta}^{(2n+1)^*}(\bar z)\delta^{(2n+1)}(z)\\
&=I_q+\frac{(z-a)}{(b-a)}v_n^*[-(z-b)(aH_{4,n}^{-1}+bH_{3,n}^{-1})R_n(z)v_nu_{4,n}^*+(b-a)I_q\\
&\quad+(a\frac{(z-b)}{(z-a)}H_{4,n}^{-1}R_n^{-1}(a)+bH_{3,n}^{-1}R_n^{-1}(b))R_n(z)H_{4,n}R_n^{*^{-1}}(\bar z)\\
&\quad+(a(z-b)H_{4,n}^{-1}R_n^{-1}(a)+b(z-a)H_{3,n}^{-1}R_n^{-1}(b))R_n(z)u_nv_n^*]R_n^*(\bar z)H_{4,n}^{-1}\\
&\quad\cdot R_n(a)u_{4,n}\\
&=I_q+\frac{(z-a)}{(b-a)}v_n^*\{a(z-b)H_{4,n}^{-1}R_n(z)[H_{4,n}T_n^*-T_nH_{4,n}\\
&\quad+\frac{1}{(z-a)}R_n^{-1}(a)H_{4,n}R_n^{*^{-1}}(\bar z)-\frac{1}{(z-b)}R_n^{-1}(z)H_{4,n}R_n^{*^{-1}}(b)]\\
&\quad+bH_{3,n}^{-1}R_n(z)[-(z-b)v_nu_{4,n}^*+R_n^{-1}(b)H_{4,n}R_n^{*^{-1}}(\bar z)-(z-a)u_{3,n}v_n^*\\
&\quad+R_n^{-1}(z)H_{3,n}R_n^{*^{-1}}(a)]\}R_n^*(\bar z)H_{4,n}^{-1}R_n(a)u_{4,n}\\
&=I_q+\frac{a}{(b-a)}v_n^*H_{4,n}^{-1}R_n(z)[-(b-a)(I_{(n+1)q}-zT_n)H_{4,n}(I_{(n+1)q}-zT_n^*)]R_n^*(\bar z)\\
&\quad\cdot H_{4,n}^{-1}R_n(a)u_{4,n}\\
&=I_q-av_n^*H_{4,n}^{-1}R_n(a)u_{4,n}.
\end{aligned}
$$

In the second equality, we used (11), (12), (16) for $r=3$*, (40), and (42). In the third equality, we use Equality (47).*

Thus, Equality (48) for the block matrix (2,2) is proved. Consequently, having proven Equality (48), we have proved Equality (34).

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Про дві матриці розв'язання задачі усіченого матричного моменту Хаусдорфа.

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Ми розглядаємо усічену матричну проблему моментів Хаусдорфа у випадку скінченної парної кількості моментів, яка називається невиродженою, якщо дві блочні Ганкелеві матриці, побудовані за допомогою моментів, є додатно визначеними. Множина розв'язків усіченої проблеми моментів Хаусдорфа у випадку скінченної парної кількості моментів задається за допомогою так званої резольвентної матриці. Резольвентна матриця усіченої проблеми моментів Хаусдорфа у невиродженому випадку для матричних моментів вимірности $q \times q$ є $2q \times 2q$ матричним поліномом, який будується за допомогою заданих моментів.

У 2001 р., в роботі [Yu.M. Dyukarev, A.E. Choque Rivero, Power moment problem on compact intervals, Mat. Sb.-2001.-69(1-2).-P.175-187], для згаданої вище усіченої проблеми моментів Хаусдорфа вперше була запропонована резольвентна матриця $V^{(2n+1)}$. У 2006 р., в роботі [A. E. Choque Rivero, Y. M. Dyukarev, B. Fritzsche and B. Kirstein, A truncated matricial moment problem on a finite interval, Interpolation, Schur Functions and Moment Problems. Oper. Theory: Adv. Appl. -2006. - 165. - P. 121-173], була дана інша резольвентна матриця $U^{(2n+1)}$ для тієї самої проблеми. В даній роботі ми доводимо, що існує явне співвідношення між цими двома резольвентними матрицями вигляду $V^{(2n+1)} = AU^{(2n+1)}B$, де A і B – сталі матриці. Ми також фокусуємось на наступній відмінності: для визначення резольвентної матриці $V^{(2n+1)}$ має виконуватися додаткова умова, у порівнянні з визначенням резольвентної матриці $U^{(2n+1)}$, для якої вимагається лише щоб дві блочні Ганкелеві матриці були додатно визначені.

У 2015 р., в роботі [A. E. Choque Rivero, From the Potapov to the Krein-Nudel'man representation of the resolvent matrix of the truncated Hausdorff matrix moment problem, Bol. Soc. Mat. Mexicana. – 2015. – 21(2). – P. 233–259], було дане зображення резольвентної матриці, отриманої в 2006 р., через матричні ортогональні поліноми. В даній роботі ми не пов'язуємо резольвентну матрицю $V^{(2n+1)}$ з результатами [A.E. Choque Rivero, From the Potapov to the Krein-Nudel'man representation of the resolvent matrix of the truncated Hausdorff matrix moment problem, Bol. Soc. Mat. Mexicana. – 2015. – 21(2). – P. 233–259]. Важливість співвідношення між $U^{(2n+1)}$ і $V^{(2n+1)}$ пояснюється тим, що можуть бути знайдені нові співвідношення між ортогональними матричними поліномами, множниками Бляшке-Потапова, параметрами Дюкарева-Стілтьєса і матричними неперервними дробами. Хоча в даній роботі використовуються алгебраїчні тотожності для доведення співвідношення між $U^{(2n+1)}$ і $V^{(2n+1)}$, аналітичне обґрунтування обох резольвентних матриць спирається на метод В.П. Потапова. Цей підхід був успішно розвинений в багатьох роботах, пов'язаних з матричними проблемами інтерполяції в класі функцій Неванлінни і матричною проблемою моментів.

Ключові слова: **Задача матричного моменту Хаусдорфа, матриця розв'язання.**

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