

A small gain theorem for finite-time input-to-state stability of infinite networks and its applications

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We prove a small-gain sufficient condition for (global) finite-time input-to-state stability (FTISS) of infinite networks. The network under consideration is composed of a countable set of finite-dimensional subsystems of ordinary differential equations, each of which is interconnected with a finite number of its "neighbors" only and is affected by some external disturbances. We assume that each node (subsystem) of our network is finite-time input-to-state stable (FTISS) with respect to its finite-dimensional inputs produced by this finite set of the neighbors and with respect to the corresponding external disturbance. As an application we obtain a new theorem on decentralized *finite-time* input-to-state stabilization with respect to external disturbances for infinite networks composed of a countable set of strict-feedback form systems of ordinary differential equations. For this we combine our small-gain theorem proposed in the current work with the controllers design developed by S. Pavlichkov and C. K. Pang (NOLCOS-2016) for the gain assignment of the strict-feedback form systems in the case of finite networks.

The current results address the FTISS and decentralized FTISS stabilization and redesign the technique proposed in recent work S. Dashkovskiy and S. Pavlichkov, Stability conditions for infinite networks of nonlinear systems and their application for stabilization, *Automatica*. – 2020. – 112. – 108643, in which the case of ℓ_∞ -ISS of infinite networks was investigated.

Keywords: nonlinear systems; input-to-state stability; small gain conditions.

Павличков С. С. Теорема про мале посилення для стійкості вхід-стан за скінченний час нескінченних мережевих систем і її застосування. Ми доводимо достатню умову стійкості вхід-стан за скінченний час нескінченних мережевих систем в термінах малого посилення (small gain condition). Мережева система, що розглядається, складається зі зліченної множини скінченновимірних систем звичайних диференціальних рівнянь, кожна з яких з'єднана тільки зі скінченною множиною сусідніх підсистем, а також містить зовнішнє збурення. Передбачається, що кожен вузол мережі (кожна підсистема) є стійкою вхід-стан за скінченний час відносно його скінченновимірних входів утворених фазовими змінними сусідніх підсистем і зовнішнім збуренням. Як застосування цього результату (наслідок)

ми отримуємо нову теорему про децентралізовану стабілізацію вхід-стан за скінченний час для нескінченних мережевих систем, які представляють собою злічений набір з'єднаних трикутних систем звичайних диференціальних рівнянь. Для цього ми комбінуємо доведену в даній роботі теорему малого посилення (small gain theorem) з методом побудови децентралізованих стабілізуючих керувань, який отримано в роботі S. Pavlichkov and C. K. Pang (NOLCOS-2016) для кінцевих мережевих систем.

Дана робота переносить результати недавньої роботи S. Dashkovskiy and S. Pavlichkov, Stability conditions for infinite networks of nonlinear systems and their application for stabilization, Automatica. – 2020. – 112. – 108643 на випадок стабілізації за скінченний час.

Ключові слова: нелінійні системи; стійкість вхід-стан; умови малого посилення

Павличков С. С. Теорема о малом усилении для устойчивости вход-состояние за конечное время бесконечных сетевых систем и ее применения. Мы доказываем достаточное условие устойчивости вход-состояние за конечное время бесконечных сетевых систем в терминах малого усиления (small gain condition). Рассматриваемая сетевая система состоит из счетного множества конечномерных систем обыкновенных дифференциальных уравнений, каждая из которых соединена только с конечным множеством соседних систем, а также содержит внешнее возмущение. Предполагается, что каждый узел сети (каждая подсистема) обладает свойством устойчивости вход-состояние за конечное время относительно его конечномерных входов образованных фазовыми переменными соседних подсистем и внешним возмущением. В качестве применения этого результата (следствия) мы получаем новую теорему о децентрализованной стабилизации вход-состояние за конечное время бесконечных сетевых систем представляющих собой счетный набор соединенных треугольных систем обыкновенных дифференциальных уравнений. Для этого мы комбинируем доказанную в настоящей работе теорему малого усиления (small gain theorem) с методом построения децентрализованных стабилизирующих управлений полученным в работе S. Pavlichkov and C. K. Pang (NOLCOS-2016) для конечных сетевых систем.

Настоящая работа переносит результаты недавней работы S. Dashkovskiy and S. Pavlichkov, Stability conditions for infinite networks of nonlinear systems and their application for stabilization, Automatica. – 2020. – 112. – 108643 на случай стабилизации за конечное время.

Ключевые слова: нелинейные системы; устойчивость вход-состояние; условия малого усиления

2010 Mathematics Subject Classification: 93C10; 93A15; 93D25; 93B70; 93D40; 93A14.

1. Introduction

The definition of input-to-state stability (ISS) was introduced in 1989 in [36] as a natural generalization of the classical global asymptotic stability for the case when the dynamics of the system under consideration is affected by some external disturbance. Very soon, the concept of ISS became very fruitful. First, it

appeared that the classical converse Lyapunov theorems (on the existence of the Lyapunov functions for globally asymptotically stable systems) can be extended to the case of ISS. More specifically a systems of ordinary differential equations (ODE) with external disturbance input is ISS *if and only if* it has an ISS Lyapunov function [37]. Second, deep results devoted to various characterizations of ISS including its relationship with the classical Lyapunov stability and asymptotic stability properties were obtained [38]; in particular one of results from [38] states that a system of ODE with external disturbance inputs of class L_∞ is ISS if and only if it is globally asymptotically stable whenever the disturbance is identically equal to zero, and the system possesses the so-called asymptotic gain (AG) property, which means that, for any disturbance input which is different from zero, each trajectory of the system eventually converges to a ball with its center at origin and the radius of this ball is a \mathcal{K} -function of the L_∞ -norm of the corresponding disturbance regardless of the initial condition. Third, this theory led to the so-called small gain theorems firstly for two interconnected systems [17],[16], which later was extended to the general case of $N \geq 2$ interconnected subsystems [8], [18], [9]. Later these classical results devoted to systems of ODE were extended to networks of impulsive and delayed systems [10], interconnections of partial differential equations (PDE)[27], etc. In general, the purport of the small gain theorems is to provide sufficient conditions for (ISS) stability of entire interconnection of several ISS subsystems. This, in turn, provided many applications such as nonlinear stabilization in presence of dynamic uncertainties [17],[39] or decentralized(or distributed or cooperative control) of multi-agent systems, see e.g. [24],[32],[33], [31].

The problem of finite-time stabilization was raised and solved in 1979 for linear control systems in [21, 22] by means of the controllability function method, which was later developed in many works such as [2, 23, 4, 1]. This area enjoyed a renaissance after 2000, see, for instance, [3, 15, 40, 13, 14, 34]. In contrast to the above-mentioned papers based on the controllability function method [21, 22, 2, 23, 4, 1], in which the corresponding finite-time control Lyapunov function (i.e., the controllability function) is defined as an implicit function, works [15, 40, 13] are based on a certain revision of the backstepping approach for the case of finite-time stabilization, where the Lyapunov functions and controls are designed explicitly. The latter allowed to obtain an extension of the ISS framework and small-gain approaches to the case of finite-time stability and further applications in design of nonlinear finite-time stabilizers in presence of dynamic uncertainties similarly to the classical work [17]. Such a generalization was proposed in [14].

Since stability and stabilization of large-scale networks has many meaningful applications [11, 25, 26], another recent popular topic has become infinite networks [5, 35, 6, 41]. The main focus was the infinite networks of finite-dimensional linear control systems with linear interconnections. In work [7], a new small-gain theorem for infinite networks of nonlinear ODE systems interconnected nonlinearly was proved and its applications were demonstrated by solving the decentralized stabilization problem for infinite networks of nonlinear control systems with

uncontrollable linearizations and power integrators. Also there are new recent results devoted to characterizations of ISS of infinite-dimensional systems [28] as well as further extensions of ISS small gain conditions to the case of infinite networks of nonlinear ODE systems, see, for instance, [19].

The goal of the current work is to extend the results of [7] to the case of *finite-time stability and stabilization* of infinite networks, i.e., to prove the corresponding small gain theorem and to show how it can be applied to the decentralized *finite-time* stabilization of infinite networks composed of nonlinear control systems of ordinary differential equations. The current paper extends and generalizes its conference predecessor [30] to the case of finite-time ISS stability and decentralized stabilization in presence of external disturbance inputs and with respect to these disturbance inputs. In the special case when all these external disturbances are zeroes (i.e. are absent), we just obtain finite-time stability and finite-time decentralized stabilization of infinite networks as a corollary.

2. Preliminaries

A function $\alpha : [0, +\infty[\rightarrow [0, +\infty[$ is said to be of class \mathcal{K} , if it is continuous, strictly increasing and $\alpha(0) = 0$, and $\alpha : [0, +\infty[\rightarrow [0, +\infty[$ is said to be of class \mathcal{K}_∞ if it is of class \mathcal{K} and unbounded. In compliance with [14], we say that $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a generalized \mathcal{K} -function, or a $\mathcal{G}\mathcal{K}$ -function, if it is continuous, with $\alpha(0) = 0$ and satisfies $\alpha(s) = \max\{0, \bar{\alpha}(s) - \bar{\alpha}(s_0)\}$, where $\bar{\alpha}(\cdot)$ is a \mathcal{K} -function and $s_0 \geq 0$ is a given parameter. A continuous function $\beta : [0, +\infty[\times [0, +\infty[\rightarrow [0, +\infty[$ is said to be of class \mathcal{KL} if for each fixed $t \geq 0$ the function $\beta(\cdot, t)$ is of class \mathcal{K} and for each fixed $s \geq 0$, we have $\beta(s, t) \rightarrow 0$ as $t \rightarrow +\infty$ and $t \mapsto \beta(s, t)$ is decreasing. A continuous function $\beta : [0, +\infty[\times [0, +\infty[\rightarrow [0, +\infty[$ is said to be a generalized \mathcal{KL} -function, or a $\mathcal{G}\mathcal{KL}$ -function if for each $t \geq 0$ the function $\beta(\cdot, t)$ is a $\mathcal{G}\mathcal{K}$ -function and for each $s \geq 0$ the function $\beta(s, \cdot)$ is decreasing with $\beta(s, t) \rightarrow 0$ as $t \rightarrow +\infty$ with some $T(s) \leq +\infty$ and $t \mapsto \beta(s, t)$ is decreasing.

For any finite-dimensional vector $\xi \in \mathbb{R}^N$, by $|\xi|$, $|\xi|_\infty$, and $|\xi|_1$ we denote its Euclidean norm, max-norm, and Manhattan Taxicab norm respectively, i.e., $|\xi| := \langle \xi, \xi \rangle^{\frac{1}{2}}$, $|\xi|_\infty := \max_{1 \leq i \leq N} |\xi_i|$, and $|\xi|_1 := \sum_{i=1}^N |\xi_i|$. If \mathcal{N} is a *finite* set, i.e., it has a *finite* number of elements, then we denote the number of its elements by $|\mathcal{N}|$.

Let (\mathcal{M}, d) be a metric space. A map $\mathbb{R} \supset [a, b] \ni t \mapsto \mathcal{X}(t) \in \mathcal{M}$ is said to be absolutely continuous on the segment $[a, b]$ if and only if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every finite sequence of pairwise disjoint subintervals $]\tau_m, s_m[$ of $[a, b]$, we have:

$$\sum_m (s_m - \tau_m) < \delta \Rightarrow \sum_m d(\mathcal{X}(s_m), \mathcal{X}(\tau_m)) < \varepsilon. \tag{1}$$

Throughout the paper, by $AC([a, b]; \mathcal{M})$ we denote the class of absolutely continuous maps $\mathbb{R} \supset [a, b] \ni t \mapsto \mathcal{X}(t) \in \mathcal{M}$.

Next, instead of $i = \overline{1, n}$ we can also write $i \in \{1, \dots, n\}$ to make some formulae shorter.

3. A finite-time small gain theorem for infinite networks

In this Section, we deal with the following infinite network

$$\dot{X}_i(t) = \Phi_i(X_i(t), \{X_j(t)\}_{j \in J(i)}, D_i(t)), \quad i \in \mathbb{N}, \quad (2)$$

where $X_i = [X_{i,1}, \dots, X_{i,N_i}]^\top \in \mathbb{R}^{N_i}$ is the state vector of the i -th subsystem, $D_i = [D_{i,1}, \dots, D_{i,n_i}]^\top \in \mathbb{R}^{n_i}$ is the disturbance input of the i -th subsystem, every Φ_i is of class $C(\mathbb{R}^{N_i + \sum_{j \in J(i)} N_j}; \mathbb{R}^{N_i})$, and, for every $i \in \mathbb{N}$, the set $J(i) \subset \mathbb{N}$ of the neighbors of the i -th subsystem is a *finite* set of the corresponding indices from \mathbb{N} . According to our notation, $i \notin J(i)$ for all $i \in \mathbb{N}$, i.e. i -th subsystem is not treated as a neighbor of itself. We suppose that the state vector $X = \{X_i\}_{i=1}^\infty$, and the disturbance input vector $D = \{D_i\}_{i=1}^\infty$ of the entire network (2) are elements of ℓ_∞ . Furthermore, it is assumed that the dynamics of (2) is locally uniformly bounded in the following sense:

$$\sup_{i \in \mathbb{N}} \max_{j \in J(i)} \max_{|X_i|_\infty \leq R, |X_j|_\infty \leq R, |D_i|_\infty \leq R} |\Phi_i(X_i, \{X_j\}_{j \in J(i)}, D_i)|_\infty < \infty \quad (3)$$

for every $R \in]0, +\infty[$. Throughout the paper, we assume that external disturbances $\mathbb{R} \ni t \mapsto D(t) = \{D_i(t)\}_{i=1}^\infty$ are such that for each $i \in \mathbb{N}$ we have $D_i(\cdot) \in L_\infty(\mathbb{R}; \mathbb{R}^{n_i})$ and $D(t) = \{D_i(t)\}_{i=1}^\infty \in \ell_\infty$ a.e. on $t \in \mathbb{R}$, and such that $\sup_{i \in \mathbb{N}} \max_{j=1, n_i} \|D_{i,j}(\cdot)\|_{L_\infty(\mathbb{R}; \mathbb{R})} < +\infty$. This class of disturbances

$D(\cdot) = \{D_i(\cdot)\}_{i=1}^\infty$ is denoted by $L_\infty(\mathbb{R}; \ell_\infty)$ throughout the paper, and, by definition, we denote $\|D(\cdot)\|_{L_\infty(\mathbb{R}; \ell_\infty)} := \sup_{i \in \mathbb{N}} \max_{j=1, n_i} \|D_{i,j}(\cdot)\|_{L_\infty(\mathbb{R}; \mathbb{R})}$. Also, by definition, we put $\|X_i(\cdot)\|_{C([t_0, T]; \mathbb{R}^{N_i})} := \max_{t \in [t_0, T]} |X_i(t)|_\infty$ for every $i \in \mathbb{N}$, and every $X_i(\cdot) \in C([t_0, T]; \mathbb{R}^{N_i})$.

The following two definitions of a solution (trajectory) of (2) are the same as in [7].

Definition 1. *Take any $D(\cdot) = \{D_i(\cdot)\}_{i=1}^\infty \in L_\infty(\mathbb{R}; \ell_\infty)$ and any nonempty $\mathcal{T} \subset \mathbb{R}$ of the form $\mathcal{T} =]a, b[$, or $\mathcal{T} = [a, b]$, or $\mathcal{T} = [a, b[$, or $\mathcal{T} =]a, b]$. A map $\mathcal{T} \ni t \mapsto X(t) = \{X_i(t)\}_{i=1}^\infty \in \ell_\infty$ is said to be a solution to (2) on \mathcal{T} , if and only if for each $[a', b'] \subset \mathcal{T}$ and each $i \in \mathbb{N}$ the map $t \mapsto X_i(t)$ is of class $AC([a', b']; \mathbb{R}^{N_i})$, and (2) holds a.e. on $t \in \mathcal{T}$, or, which is the same, for each $t_0 \in \mathcal{T}$, we have*

$$\forall i \in \mathbb{N} \quad \forall t \in \mathcal{T} \quad X_i(t) = X_i(t_0) + \int_{t_0}^t \Phi_i(X_i(s), \{X_j(s)\}_{j \in J(i)}, D_i(s)) ds. \quad (4)$$

Definition 2. *Given any nonempty (open, half-open, or closed) interval $\mathcal{T} \subset \mathbb{R}$, any $t_0 \in \mathcal{T}$, any $X^0 = \{X_i^0\}_{i=1}^\infty$ in ℓ_∞ , and any $D(\cdot) = \{D_i(\cdot)\}_{i=1}^\infty$ in $L_\infty(\mathbb{R}; \ell_\infty)$, let $Y(t_0, X^0, D(\cdot), \mathcal{T})$ denote the set of all solutions $\mathcal{T} \ni t \mapsto X(t)$ to (2) on \mathcal{T} in the sense of Definition 1 such that $X(t_0) = X^0$ and $D_i = D_i(t)$.*

Let us note that, according to Lemma 1 from [7], if $\mathcal{T} \ni t \mapsto X(t) = \{X_i(t)\}_{i=1}^\infty \in \ell_\infty$ is a solution to (2) on \mathcal{T} in the sense of Definition 2, then the entire map $\mathcal{T} \ni t \mapsto X(t) = \{X_i(t)\}_{i=1}^\infty \in \ell_\infty$ is also absolutely continuous on each $[a', b'] \subset \mathcal{T}$ (although the latter statement seems to be stronger at the first glance).

In contrast to [7], we are now interested in the problem of ℓ_∞ -finite-time input-to-state stability (ℓ_∞ -FTISS) of (2) in the current paper; as a corollary we will obtain global ℓ_∞ -finite-time stability (ℓ_∞ -FTS) of (2) in the special case $D(\cdot) = 0 \in L_\infty(\mathbb{R}; \ell_\infty)$, which was considered (with some drawbacks) in [30]. The corresponding definitions are as follows.

Definition 3. System (2) is said to be ℓ_∞ -finite-time input-to-state stable or ℓ_∞ -FTISS if and only if there exist $\gamma \in \mathcal{K}$ and $\beta \in \mathcal{GKL}$ such that $\beta(r, s) = 0$ for each $s \geq T(r)$ with some $r \mapsto T(r)$ of class $C([0, +\infty[; [0, +\infty[)$ and such that $T(0) = 0$, and, for each $t_0 \in \mathcal{T}$, each $X^0 = \{X_i^0\}_{i=1}^\infty$ in ℓ_∞ , and each $D(\cdot) = \{D_i(\cdot)\}_{i=1}^\infty$ in $L_\infty(\mathbb{R}; \ell_\infty)$, we have $Y(t_0, X^0, D(\cdot), [t_0, +\infty[) \neq \emptyset$ and each solution $X(\cdot) \in Y(t_0, X^0, D(\cdot), [t_0, +\infty[)$ satisfies the following inequality

$$\|X(t)\|_{\ell_\infty} \leq \max\{\beta(\|X^0\|_{\ell_\infty}, t - t_0), \gamma(\|D(\cdot)\|_{L_\infty(\mathbb{R}; \ell_\infty)})\} \quad \text{for all } t \geq t_0. \quad (5)$$

In particular, if $D(\cdot) = 0$, i.e., there is no any external disturbance $D(\cdot)$ in system (2), we obtain the following definition of ℓ_∞ -finite-time stability (ℓ_∞ -FTS)

$$\|X(t)\|_{\ell_\infty} \leq \beta(\|X^0\|_{\ell_\infty}, t - t_0) \quad \text{for all } t \geq t_0, \quad (6)$$

where $\beta \in \mathcal{GKL}$ is the same as above in (5). In both the cases, the above-mentioned function $r \mapsto T(r)$ is called the settling time for system (2).

Remark 1. For comparison, let us quote the original, classical definition of finite-time input-to-state stability (FTISS) of finite-dimensional systems of ordinary differential equations (ODE), which was given in [14]. System of ODE

$$\dot{X}(t) = F(X(t), D(t)), \quad X \in \mathbb{R}^N, \quad D \in \mathbb{R}^M \quad (7)$$

with states $X \in \mathbb{R}^N$, external disturbance input $D(\cdot) \in L_\infty(\mathbb{R}; \mathbb{R}^M)$, and continuous $F(\cdot, \cdot)$ is said to be finite-time input-to-state stable (FTISS), if and only if there exist $\gamma \in \mathcal{K}$ and $\beta \in \mathcal{GKL}$ such that $\beta(r, s) = 0$ for each $s \geq T(r)$ with some $r \mapsto T(r)$ of class $C(\mathbb{R}_+; \mathbb{R}_+)$ such that $T(0) = 0$ and for each $t_0 \in \mathbb{R}$, each $X^0 \in \mathbb{R}^N$ and each $D(\cdot) \in L_\infty(\mathbb{R}; \mathbb{R}^M)$ every solution to (7) with $X(t_0) = X^0$, $D = D(t)$ satisfies the inequality

$$|X(t)| \leq \max\{\beta(\|X^0\|, t - t_0), \gamma(\|D(\cdot)\|_{L_\infty(\mathbb{R}; \mathbb{R}^M)})\} \quad \text{for all } t \geq t_0. \quad (8)$$

(actually, the original Definition 3 from [14] has sum instead of max in the right-hand side of (8), but both these two versions are equivalent, of course). Also following [14] (with some simplification), we say that $V(\cdot)$ of class $C^1(\mathbb{R}^N; \mathbb{R}_+)$ is a finite-time ISS Lyapunov function for system (7), if and only if there

exist $a \in]0, 1[$, $\lambda > 0$ and $\alpha_1(\cdot) \in \mathcal{K}_\infty$, $\alpha_2(\cdot) \in \mathcal{K}_\infty$, $\bar{\gamma}(\cdot) \in \mathcal{K}$ such that $\alpha_1(|X|) \leq V(X) \leq \alpha_2(|X|)$ for all $X \in \mathbb{R}^N$ and such that the following FTI-SS Lyapunov inequality holds

$$V(X) \geq \bar{\gamma}(|D|) \Rightarrow \nabla V(X)F(X, D) \leq -\lambda[V(X)]^a \quad \text{for all } X \in \mathbb{R}^N, D \in \mathbb{R}^M. \quad (9)$$

In comparison with [14], the main simplification here is that $V(\cdot)$ should not be necessarily of class C^1 in [14], but this is an equivalent formulation in our case. As it is noted in [14] (and can be easily shown), if system (7) has a finite-time ISS Lyapunov function, then (7) is FTISS.

Accordingly instead of the small-gain theorem from [7] our current version of finite-time (FT) small gain theorems is as follows.

Theorem 1. *Suppose that each Φ_i is continuous, inequality (3) holds true and there exist positive definite FTISS Lyapunov functions $V_j(X_j)$ in $C^1(\mathbb{R}^{N_j}; [0, +\infty[)$ such that*

(i) *There exists $\alpha(\cdot) \in \mathcal{K}_\infty$ such that $V_i(X_i) \geq \alpha(|X_i|)$ uniformly for all $X_i \in \mathbb{R}^{N_i}$, $i \in \mathbb{N}$ (i.e., $V_i(\cdot)$ are uniformly radially unbounded)*

(ii) *For each $R > 0$ we have:*

$$\begin{aligned} \sup_{i \in \mathbb{N}} N_i &< +\infty, & \sup_{i \in \mathbb{N}} n_i &< +\infty, \\ \sup_{i \in \mathbb{N}} \max_{|X_i| \leq R} V_i(X_i) &< +\infty, & \sup_{i \in \mathbb{N}} \max_{|X_i| \leq R} \left| \frac{\partial V_i(X_i)}{\partial X_i} \right| &< +\infty \end{aligned} \quad (10)$$

(iii) *There exist $\lambda > 0$, $\mu \in]0, 1[$, $\varepsilon \in]0, 1[$, and $\gamma(\cdot) \in \mathcal{K}$ such that each i -th subsystem of (2) satisfies the following Lyapunov ISS inequality:*

$$\begin{aligned} V_i(X_i) &\geq \max\left\{(1 - \varepsilon) \max_{j \in J(i)} V_j(X_j), \gamma(|D_i|_\infty)\right\} \Rightarrow \\ \nabla V_i(X_i)\Phi_i(X_i, \{X_j\}_{j \in J(i)}, D_i) &\leq -\lambda V_i^{1-\mu}(X_i). \end{aligned} \quad (11)$$

Then the following three statements hold true:

(I) *For each $t_0 \in \mathbb{R}$ each initial $X^0 = \{X_i^0\}_{i=1}^\infty$ in ℓ_∞ , and each $D(\cdot) \in L_\infty(\mathbb{R}; \ell_\infty)$, the set $Y(t_0, X^0, D(\cdot), [t_0, +\infty[)$ is not empty and every trajectory $t \mapsto X(t) = \{X_i(t)\}_{i=1}^\infty \in \ell_\infty$ from $Y(t_0, X^0, D(\cdot), [t_0, +\infty[)$ is well-defined and uniformly bounded on the entire $[t_0, +\infty[$, i.e., $t \mapsto X(t)$ satisfies $X_i(t_0) = X_i^0$ and*

$$\forall i \in \mathbb{N} \quad \dot{X}_i(t) = \Phi_i(X_i(t), \{X_j(t)\}_{j \in J(i)}, D_i(t)) \quad \text{a.e. on } t \in [t_0, +\infty[.$$

For $X^0 = 0 \in \ell_\infty$, and $D_i(\cdot) = 0$, we just have $X(t) = 0 \in \ell_\infty$ for all $t \in [t_0, +\infty[$.

(II) Define the finite-time ISS Lyapunov function for the entire network by

$$V(X) := \sup_{i \in \mathbb{N}} \{V_i(X_i)\} \quad \text{for all } X = \{X_i\}_{i=1}^\infty \in \ell_\infty. \quad (12)$$

Then, for every trajectory $t \mapsto X(t)$ of (2) defined in (I), the function $t \mapsto V(X(t))$ is absolutely continuous on every $[a, b] \subset [t_0, +\infty[$, and

$$V(X(t)) > \gamma(\|D(\cdot)\|_{L_\infty(\mathbb{R}; \ell_\infty)}) \Rightarrow \dot{V}(X(t)) \leq -\lambda V^{1-\mu}(X(t)) \quad \text{a.e. on } [t_0, +\infty[. \quad (13)$$

This immediately implies that (2) is ℓ_∞ -FTISS in the sense of Definition 3.

(III) If $D(\cdot) = 0 \in L_\infty(\mathbb{R}; \ell_\infty)$ then system (2) is ℓ_∞ -FTS in the sense of Definition 3 and the settling time $T(r)$ mentioned in Definition 3 can be obtained from the following estimate:

$$V(X(t)) = 0 \quad \text{for all } t \in [T(V(X^0)), +\infty[,$$

where

$$T(V(X^0)) \leq \frac{1}{\lambda\mu} [V(X^0)]^\mu. \quad (14)$$

Remark 2. Note that Assumptions (i), (ii) imply the existence of $\alpha_{\max}(\cdot) \in \mathcal{K}_\infty$ such that

$$\forall i \in \mathbb{N} \quad \forall X_i \in \mathbb{R}^{N_i} \quad \alpha(|X_i|) \leq V_i(X_i) \leq \alpha_{\max}(|X_i|). \quad (15)$$

In addition, by Assumption (ii), the FT Lyapunov function $V(X)$ from (12) is well-defined by (12) for all $X \in \ell_\infty$, and then, from (15) it follows that

$$\forall X = \{X_i\}_{i=1}^\infty \in \ell_\infty \quad \alpha(|X|) \leq V(X) \leq \alpha_{\max}(|X|). \quad (16)$$

Remark 3. Assumption (iii) with (11) is a finite-time analog of Assumptions (iii), (iv) from [7], the latter being devoted to the problem of uniform asymptotic stabilization for infinite networks (2). For finite networks, they can be formulated in more general form [9], [18], but being motivated by these finite-dimensional and essentially nonlinear results, we note that our version of small gain theorems for infinite networks deals with linear gains similarly to [7]. However, as in [7], we will see that this version does suffice for such important applications as decentralized stabilization of infinite networks composed of nonlinear control systems which are interconnected nonlinearly.

Proof of Theorem 1.

Step 1. As in [7] we first prove the existence of the corresponding trajectories of (2). Take and fix any initial $t_0 \in \mathbb{R}$ and $X^0 = \{X_i^0\}_{i=1}^\infty \in \ell_\infty$, and any $D(\cdot) \in L_\infty(\mathbb{R}; \ell_\infty)$. Without loss of generality, we first assume that

$$\|X^0\|_{\ell_\infty} + \|D(\cdot)\|_{L_\infty(\mathbb{R}; \ell_\infty)} > 0$$

(the trivial case $X^0 = 0 \in \ell_\infty$, $D(\cdot) = 0 \in L_\infty(\mathbb{R}; \ell_\infty)$ will be discussed in the very end of the proof). This Step 1 is similar to the corresponding Step 1 from [7]. Define

$$V^0 := V(X(t_0)) = \sup_{i \in \mathbb{N}} V_i(X_i(t_0)); \quad \hat{D} := \gamma(\|D(\cdot)\|_{L_\infty(\mathbb{R}; \ell_\infty)}); \quad (17)$$

and taking any $\sigma \in]0, \max\{\frac{V^0}{4}, \frac{\hat{D}}{4}\}[$, define

$$\begin{aligned} R_0 &:= 2 \|X^0\|_{\ell_\infty} + \alpha^{-1}(2V^0 + 2\hat{D} + 1) + 2V^0 + 2\hat{D} + 1; \\ M_0 &:= 1 + \sup_{i \in \mathbb{N}} \max_{|X_i| \leq 2R_0 + 1, |D_i|_\infty \leq \gamma^{-1}(2\hat{D})} \max_{|X_j| \leq 2R_0 + 1, j \in J(i)} |\Phi_i(X_i, \{X_j\}_{j \in J(i)}, D_i)| \\ &+ \sup_{i \in \mathbb{N}} \max_{|X_i| \leq 2R_0 + 1, |D_i|_\infty \leq \gamma^{-1}(2\hat{D})} \max_{|X_j| \leq 2R_0 + 1, j \in J(i)} \left| \frac{\partial V_i(X_i)}{\partial X_i} \right| |\Phi_i(X_i, \{X_j\}_{j \in J(i)}, D_i)| \\ \theta &:= \frac{\sigma}{4M_0 + 1}. \end{aligned} \quad (18)$$

Then we define the following standard iterations $X^{(m)}(\cdot) = \{X_i^{(m)}(\cdot)\}_{i=1}^\infty$ on $[t_0 - \theta, t_0 + \theta]$ for $m = 0, 1, 2, \dots$:

$$\begin{aligned} X_i^{(0)}(t) &= X_i^0, \quad t \in [t_0 - \theta, t_0 + \theta], \quad i \in \mathbb{N}, \\ X_i^{(m)}(t) &= X_i^0 + \int_{t_0}^t \Phi_i(X_i^{(m-1)}(s), \{X_j^{(m-1)}(s)\}_{j \in J(i)}, D_i(s)) ds, \\ t_0 - \theta &\leq t \leq t_0 + \theta, \quad i \in \mathbb{N}, \quad m \in \mathbb{N}. \end{aligned} \quad (19)$$

It is straightforward that each $X_i^{(m)}(\cdot)$ is of class $AC([t_0 - \theta, t_0 + \theta]; \mathbb{R}^{N_i})$, and, using (i) and (18), we obtain

$$\forall i \in \mathbb{N} \quad \forall m \in \mathbb{Z}_{\geq 0} \quad \forall t \in [t_0 - \theta, t_0 + \theta] \quad |X_i^{(m)}(t)| \leq R_0. \quad (20)$$

As in [7], we apply the Arzela-Ascoli lemma and Cantor's diagonal argument and prove the existence of a subsequence $X^{(m_q)}(\cdot) = \{X_i^{(m_q)}(\cdot)\}_{i=1}^\infty \in C^1([t_0 - \theta, t_0 + \theta]; \mathbb{R}^{N_i})_{i=1}^\infty$, $q \in \mathbb{N}$ such that for every fixed $i \in \mathbb{N}$ there is $X_i(\cdot)$ of class $C([t_0 - \theta, t_0 + \theta]; \mathbb{R}^{N_i})$ such that

$$\|X_i^{(m_q)}(\cdot) - X_i(\cdot)\|_{C([t_0 - \theta, t_0 + \theta]; \mathbb{R}^{N_i})} \rightarrow 0 \quad \text{as } q \rightarrow \infty.$$

Combining this with (19), (20), we obtain:

$$\begin{aligned} X_i(t) &= X_i^0 + \int_{t_0}^t \Phi_i(X_i(s), \{X_j(s)\}_{j \in J(i)}, D_i(s)) ds, \\ t &\in [t_0 - \theta, t_0 + \theta] \quad i \in \mathbb{N}. \end{aligned} \quad (21)$$

Hence $X(\cdot) = \{X_i(\cdot)\}_{i=1}^{+\infty}$ belongs to $Y(t_0, X^0, D(\cdot), [t_0 - \theta, t_0 + \theta])$ with $X_i(\cdot) \in AC([t_0 - \theta, t_0 + \theta]; \mathbb{R}^{N_i})$ and

$$\forall i \in \mathbb{N} \quad \forall t \in [t_0 - \theta, t_0 + \theta] \quad |X_i(t)| \leq R_0. \quad (22)$$

Step 2. First we assume that $V^0 > \hat{D}$; the other cases will be discussed in the end of the proof. Define

$$\varepsilon^* := \frac{1}{2} \min\{\varepsilon, V^0 - \hat{D}, \varepsilon \hat{D}, \frac{\varepsilon}{4} V^0\}. \tag{23}$$

Define $\theta > 0$ as in Step 1 (see (18)), and let $t \mapsto X(t)$ be any trajectory of (2) with $X(t_0) = X^0$ defined on some $[t_0 - \theta^*, t_0 + \theta^*]$ with some $\theta^* \in]0, \theta]$, i.e., $t \mapsto X(t)$ satisfies (21) with $\theta^* \in]0, \theta]$ instead of θ .

Using (3),(10),(15),(21), find $L = L(R_0, V_0) > 0$ and $\tau^* \in]0, \frac{\theta^*}{2}]$ such that

$$\forall i \in \mathbb{N} \quad \forall t' \in [t_0, t_0 + \tau^*] \quad \forall t'' \in [t_0, t_0 + \tau^*] \quad |V_i(X_i(t')) - V_i(X_i(t''))| \leq L|t' - t''|. \tag{24}$$

(Indeed, as we noted above, $t \mapsto X(t)$ should be of class $AC([t_0 - \tau^*, t_0 + \tau^*]; \ell_\infty)$, and therefore $\sup_{i \in \mathbb{N}} \|X_i(\cdot)\|_{C([t_0 - \tau^*, t_0 + \tau^*]; \mathbb{R}^{N_i})}$ should be uniformly bounded; then we apply (10),(15),(21)). Then, fix any $\tau \in]0, \tau^*]$ such that

$$\forall t \in [t_0, t_0 + \theta^* - \tau] \quad \forall s \in [0, \tau] \quad \forall i \in \mathbb{N} \quad |V_i(X_i(t+s)) - V_i(X_i(t))| \leq \frac{\varepsilon^*}{4} \leq \frac{\varepsilon}{32} V^0. \tag{25}$$

Then, in particular,

$$\forall i \in \mathbb{N} \quad \forall t \in [t_0, t_0 + \tau] \quad |V_i(X_i(t)) - V_i(X_i(t_0))| \leq \frac{\varepsilon^*}{4} \leq \frac{\varepsilon}{32} V^0, \tag{26}$$

and

$$\forall t \in [t_0, t_0 + \tau] \quad V(X(t)) \leq V^0 + \frac{\varepsilon^*}{4}. \tag{27}$$

For every $\delta \in]0, \varepsilon^*]$, by $I(\delta) \subset \mathbb{N}$ denote the following set of indices

$$I(\delta) := \{j \in \mathbb{N} \mid V_j(X_j(t_0)) \geq V^0 - \delta\}. \tag{28}$$

As in [7] we obtain the following lemma.

Lemma 1. *The following statements hold true.*

(S1) *For each $i \in \mathbb{N}$ and each $t \in [t_0, t_0 + \tau]$ we have:*

$$V_i(X_i(t)) \geq V^0 - \frac{3\varepsilon^*}{4} \Rightarrow \dot{V}_i(X_i(t)) \leq -\lambda[V_i(X_i(t))]^{1-\mu}$$

(S2) *For each $i \in \mathbb{N}$ and each $t \in [t_0, t_0 + \tau]$ we have:*

$$V_i(X_i(t)) \leq \max\{V_i(X_i(t_0)), V^0 - \frac{3\varepsilon^*}{4}\}.$$

(S3) *For each $i \in I(\frac{\varepsilon^*}{2})$ we have:*

$$\forall t \in [t_0, t_0 + \tau] \quad \dot{V}_i(X_i(t)) \leq -\lambda V_i^{1-\mu}(X_i(t)) \tag{29}$$

(S4) For each $j \in \mathbb{N} \setminus I(\frac{\varepsilon^*}{2})$ and each $i \in I(\frac{\varepsilon^*}{4})$ we have: $V_j(X_j(t)) \leq V_i(X_i(t))$ for all $t \in [t_0, t_0 + \tau]$.

Proof of Statement (S1) follows from (23),(26),(27), from (iii), (11) and from (12),(17). Proof of Statement (S2) follows from Statement (S1). Proof of Statement (S3) follows from (26)-(28) and from Statement (S1). Proof of Statement (S4) follows from (25),(26),(28) and from Statements (S1),(S2).

Since $I(\frac{\varepsilon^*}{4}) \subset I(\frac{\varepsilon^*}{2})$, Statement (S4) of Lemma 1 yields:

$$\forall t \in [t_0, t_0 + \tau] \quad V(X(t)) = \sup_{i \in \mathbb{N}} V_i(X_i(t)) = \sup_{i \in I(\frac{\varepsilon^*}{2})} V_i(X_i(t)). \quad (30)$$

Finally, from Statement (S4) of Lemma 1 we obtain integrating the inequality (29):

$$\begin{aligned} \forall i \in I(\frac{\varepsilon^*}{2}) \quad [V_i(X_i(t+h))]^\mu &\leq [V_i(X_i(t))]^\mu - \lambda\mu h \quad \text{for all } t \geq t_0, \quad h > 0 \\ \text{such that } t_0 \leq t \leq t+h &\leq t+\tau, \end{aligned}$$

Taking $\sup_{i \in \mathbb{N}}$, and using (12),(30), we obtain

$$\begin{aligned} [V(X(t+h))]^\mu &\leq [V(X(t))]^\mu - \lambda\mu h \quad \text{for all } t \geq t_0, \quad h > 0 \\ \text{such that } t_0 \leq t \leq t+h &\leq t+\tau, \end{aligned}$$

which yields for any $h > 0$

$$[V(X(t+h))] - V(X(t)) \leq [[V(X(t))]^\mu - \lambda\mu h]^{\frac{1}{\mu}} - V(X(t)),$$

i.e.,

$$\begin{aligned} \frac{V(X(t+h)) - V(X(t))}{h} &\leq V(X(t)) \frac{\left[1 - \frac{\lambda\mu h}{[V(X(t))]^\mu}\right]^{\frac{1}{\mu}} - 1}{h} \quad \text{for all } t \geq t_0, \quad h > 0 \quad (31) \\ \text{such that } t_0 \leq t \leq t+h &\leq t+\tau, \end{aligned}$$

From (24), (30), it follows that that $t \mapsto V(X(t))$ satisfies (24) on $[t_0, t_0 + \tau]$; hence $t \mapsto V(X(t))$ is absolutely continuous and differentiable almost everywhere on $[t_0, t_0 + \tau]$. If $\hat{D} = 0$ then, taking $\lim_{h \rightarrow +0}$ in(31) we obtain (III).

If $V^0 > \hat{D} > 0$, then we repeat the argument from [7], Proof of Theorem 1, Steps 2,3 beginning with eq.(31) of [7] and until the very end of the Proof of Theorem 1 from [7] and obtain (I),(II) (more specifically, using Lemma 1 we note that $Y(t_0, X^0, D(\cdot), [t_0, +\infty])$ is not empty, since (18) implies that the solution constructed on $[t_0, t_0 + \theta]$ in Step 1 can be extended inductively to $[t_0 + \theta, t_0 + 2\theta]$, with the new initial condition at $t = t_0 + \theta$, then to $[t_0 + 2\theta, t_0 + 3\theta]$, etc. and the length of each new interval will be not less than $\theta > 0$ defined in (18), because the inequality $\frac{dV(X(t))}{dt} > 0$ is not possible whenever $V(X(t)) > \hat{D}$ according to

Lemma 1 and the assumption $V^0 > \hat{D}$, hence, after each extension of the time interval, the inequality (22) will hold on each extended time interval. Then, as in [7], for any solution $t \mapsto X(t)$ from $Y(t_0, X^0, D(\cdot), [t_0, +\infty[)$, we find $\tilde{\theta} := \sup\{\bar{\theta} > 0 \mid V(X(t_0 + \bar{\theta})) > \hat{D}\}$ and obtain that $V(X(t_0 + s)) \leq \hat{D}$ for all $s \geq \tilde{\theta}$, because, otherwise, we again obtain contradiction with Lemma 1. Finally, if $V^0 \leq \hat{D}$, then again from Lemma 1 it follows that $V(X(t_0 + s)) \leq \hat{D}$ for all $s \geq 0$, since $\exists s > 0 \mid V(X(t_0 + s)) > \hat{D}$ implies $\exists s' > 0$ s.t. $V(X(t_0 + s')) > \hat{D}$ and $\frac{d}{ds}V(X(t_0 + s')) > 0$, which is again impossible due to Lemma 1 and for every solution $t \mapsto X(t)$ from $Y(t_0, X^0, D(\cdot), [t_0, +\infty[)$, whereas $Y(t_0, X^0, D(\cdot), [t_0, +\infty[)$ is again not empty, which again follows from the inductive extension of construction in Step 1 to $[t_0 + \theta, t_0 + 2\theta], [t_0 + 2\theta, t_0 + 3\theta], \dots$. The same can be obtained for the trivial case $V^0 = \hat{D} = 0$: on the one hand, $X(t) = 0$ belongs to $Y(t_0, X^0, D(\cdot), [t_0, +\infty[)$, and, on the other hand there no any other solutions from $Y(t_0, X^0, D(\cdot), [t_0, +\infty[)$, because $\frac{dV(X(t))}{dt} > 0$ and $V(X(t)) > 0$ is not possible for any $t \geq t_0$ for the same reasons as above).

The proof of Theorem 1 is complete.

4. Applications: decentralized finite-time stabilization of infinite networks

Motivated by [20, 25, 26, 29, 31] consider the following infinite network of interconnected strict-feedback form control systems

$$\begin{cases} \dot{x}_{i,k} = x_{i,k+1} + \Delta_{i,k}(X_{i,k}, \mathcal{X}_{i,k}, D_i), & k = 1, \dots, \nu_i - 1, \\ \dot{x}_{i,\nu_i} = u_i + \Delta_{i,\nu_i}(X_{i,\nu_i}, \mathcal{X}_{i,\nu_i}, D_i), \end{cases} \quad i \in \mathbb{N}, \quad (32)$$

with controls $u_i \in \mathbb{R}^1$, $i \in \mathbb{N}$, with the state vector $X = \{X_{i,\nu_i}\}_{i=1}^\infty \in \ell_\infty$, where $X_{i,k}$, $\mathcal{X}_{i,k}$, are given by

$$X_{i,k} := [x_{i,1}, \dots, x_{i,k}]^\top, \quad \mathcal{X}_{i,k} := \{X_{\kappa, \min\{k, \nu_\kappa\}}\}_{\kappa \in J(i)}, \quad k = 1, \dots, \nu_i \quad (33)$$

for all $i \in \mathbb{N}$, and with external disturbance inputs $\mathbb{R} \ni t \mapsto D(t) = \{D_i(t)\}_{i=1}^\infty$ such that for each $i \in \mathbb{N}$ we have $D_i(\cdot) \in L_\infty(\mathbb{R}; \mathbb{R}^{\nu_i})$ and $D(t) = \{D_i(t)\}_{i=1}^\infty \in \ell_\infty$ a.e. on $t \in \mathbb{R}$, and such that $\sup_{i \in \mathbb{N}} \max_{j=1, \dots, \nu_i} \|D_{i,j}(\cdot)\|_{L_\infty(\mathbb{R}; \mathbb{R})} < +\infty$. As in Section 3, $J(i) \subset \mathbb{N}$ can be considered as the set of “neighbors” affecting i -th agent (node) of (32). As in Section 3, we assume without loss of generality that $i \notin J(i)$ for all $i \in \mathbb{N}$. Let us remark that *finite but large-scale* networks of form (32) with hierarchical structure of interconnections (32), (33) have engineering and physical motivation, see, for instance, [26, 25]. The case of infinite networks can be interpreted, for instance, as “open multi-agent systems”, when some agents (nodes) may unexpectedly arrive, some agents (nodes) may unexpectedly depart, and the maximal number of nodes is unknown [12].

We suppose that (32) satisfies the following assumptions:

- (A1) Every $J(i) \subset \mathbb{N}$ is *finite* for each $i \in \mathbb{N}$, and $\sup_{i \in \mathbb{N}} |J(i)| < +\infty$;

(A2) $\Delta_{i,k}(0,0,0) = 0$ and $\Delta_{i,k}(\cdot, \cdot, \cdot)$ are functions of class C^1 ;

(A3) The dimensions of the state spaces and disturbances of all the agents are uniformly bounded, i.e., $\nu := \sup_{i \in \mathbb{N}} \nu_i < +\infty$, and $n := \sup_{i \in \mathbb{N}} n_i < +\infty$;

(A4) For every $R > 0$ we have:

$$\sup_{i \in \mathbb{N}} \max_{1 \leq k \leq \nu_i} \max_{|X_{i,k}| \leq R, |\mathcal{X}_{i,k}| \leq R, |D_i| \leq R} |\Delta_{i,k}(X_{i,k}, \mathcal{X}_{i,k}, D_i)| < \infty; \quad (34)$$

$$\sup_{i \in \mathbb{N}} \max_{z \in J(i) \cup \{i\}} \max_{1 \leq k \leq \nu_i} \max_{\substack{|X_{i,k}| \leq R, \\ |\mathcal{X}_{i,k}| \leq R, \\ |D_i| \leq R}} \left| \frac{\partial \Delta_{i,k}(X_{i,k}, \mathcal{X}_{i,k}, D_i)}{\partial X_{z,k}} \right| < \infty; \quad (35)$$

$$\sup_{i \in \mathbb{N}} \max_{z \in J(i) \cup \{i\}} \max_{1 \leq k \leq \nu_i} \max_{\substack{|X_{i,k}| \leq R, \\ |\mathcal{X}_{i,k}| \leq R, \\ |D_i| \leq R}} \left| \frac{\partial \Delta_{i,k}(X_{i,k}, \mathcal{X}_{i,k}, D_i)}{\partial D_i} \right| < \infty; \quad (36)$$

As a corollary of our main Theorem 1, we obtain the following result.

Theorem 2. *Under the above Assumptions (A1)-(A4), there is a decentralized continuous feedback $u_i = u_i(X_{i,\nu_i})$ with $u_i(0) = 0$, $i \in \mathbb{N}$, which renders (32) ℓ_∞ -FTISS in the sense of Definition 3, i.e., there exist $\gamma \in \mathcal{K}$ and $\beta \in \mathcal{GKL}$ such that $\beta(r, s) = 0$ for each $s \geq T(r)$ with some $r \mapsto T(r)$ of class $C([0, +\infty[; [0, +\infty[)$ such that $T(0) = 0$, and such that, first, there exists at least one solution to the closed-loop system (32) with this decentralized feedback $u_i = u_i(x_{i,1}, \dots, x_{i,\nu_i})$, $i \in \mathbb{N}$ with every initial condition $X_{i,\nu_i}(t_0) = X_{i,\nu_i}^0 \in \mathbb{R}^{\nu_i}$, with every $X^0 = \{X_{i,\nu_i}^0\}_{i=1}^\infty \in \ell_\infty$, and with every disturbance input $\mathbb{R} \ni t \mapsto D(t) = \{D_i(t)\}_{i=1}^\infty$ such that $D_i(\cdot) \in L_\infty(\mathbb{R}; \mathbb{R}^{n_i})$ for all $i \in \mathbb{N}$ and $D(t) = \{D_i(t)\}_{i=1}^\infty \in \ell_\infty$ a.e. on $t \in \mathbb{R}$, and $\sup_{i \in \mathbb{N}} \max_{j=1, n_i} \|D_{i,j}(\cdot)\|_{L_\infty(\mathbb{R}; \mathbb{R})} < +\infty$, and, second, every such a solution $t \mapsto X(t) \in \ell_\infty$ can be extended to the entire $[t_0, +\infty[$ and it always satisfies the following inequality*

$$\|X(t)\|_{\ell_\infty} \leq \max\{\beta(\|X^0\|_{\ell_\infty}, t - t_0), \gamma(\|D(\cdot)\|_{L_\infty(\mathbb{R}; \ell_\infty)})\} \quad \text{for all } t \geq t_0.$$

The design of the above-mentioned decentralized feedback $u_i = u_i(x_{i,1}, \dots, x_{i,\nu_i})$, $i \in \mathbb{N}$ is constructive, the settling time is finite and estimated by our main Theorem 1, and the controllers $u_i(\cdot)$ along with the ℓ_∞ -FTISS Lyapunov function and with the settling time are derived explicitly.

In the special case, when $D_i(\cdot) = 0$, or D_i are absent from (32), the same decentralized feedback $u_i = u_i(x_{i,1}, \dots, x_{i,\nu_i})$, $i \in \mathbb{N}$ renders (32) ℓ_∞ -FTS in the sense of Definition 3.

Proof of Theorem 2. The proof of Theorem 2 is a combination of our main Theorem 1 with the gain assignment obtained in the decentralized backstepping design proposed in [29]. The only remark in comparison with [29] is that our network is now composed on *infinite (countable)* set of nodes, whereas the network in [29] was finite and without external disturbance inputs. However our conditions (A1)-(A4) will eventually provide conditions (i)-(iii) of our main Theorem 1 inside the design borrowed from [29] (and our external disturbance inputs can be included into common inputs in the gain assignment borrowed from [29]).

More specifically, to reduce our proof to Theorem 1, we first fix any $\varepsilon \in]0, \frac{1}{4}[$, and fix any finite sequences of positive real numbers $\varepsilon^{(k)} \in]0, \varepsilon[$ and $\gamma^{(k)} = 1 - \varepsilon^{(k)} > 0$, $k = 1, 2, \dots, \nu$, such that

$$0 < \gamma^{(k-1)} < \gamma^{(k)} < 1 - \varepsilon \quad \text{for all } k = 2, \dots, \nu, \tag{37}$$

where $\nu \in \mathbb{N}$ is defined in Assumption (A3).

Second, we take any $n \in \mathbb{N}$ such that $n \geq \nu := \sup_{i \in \mathbb{N}} \nu_i = \max_{i \in \mathbb{N}} \nu_i$. As in [40], define $d = \frac{4n}{2n+1}$ and

$$V_{i,1}(x_{i,1}) := \frac{x_{i,1}^2}{2}; \quad \xi_{i,1} := x_{i,1}; \quad q_k := \frac{2n-2k+3}{2n+1}, \quad k = \overline{1, n}. \tag{38}$$

and denote

$$\bar{\chi}_{i,k} := (\mathcal{X}_{i,k}, D_i) \quad \text{for all } k = 1, \dots, \nu_i - 1, \quad i \in \mathbb{N}, \tag{39}$$

and $\bar{X}_{i,k} := (X_{i,k}, \bar{\chi}_{i,k})$. Then we rewrite our system (32) as

$$\begin{cases} \dot{x}_{i,j} = x_{i,j+1} + \Delta_{i,j}(X_{i,j}, \bar{\chi}_{i,j}), & j = 1, \dots, \nu_i - 1, \\ \dot{x}_{i,k} = u_i + \Delta_{i,\nu_i}(X_{i,\nu_i}, \bar{\chi}_{i,\nu_i}), & i \in \mathbb{N}, \end{cases} \tag{40}$$

and thus we unify our notation with [29]. Then, with this new notation, we repeat (almost copy and paste) the passage from Section 5 of [29] beginning with (11) from [29] until the very end of Section 5. The only updates will be as follows: $\pi_{i,k,\sigma}(\theta, \bar{X}_k)$ should be everywhere replaced with $\Delta_{i,k}(\bar{X}_k) = \Delta_{i,k}(X_{i,k}, \bar{\chi}_{i,k})$ and $\varkappa \in \{1, \dots, N\} \setminus \{i\}$ should be replaced everywhere with $j \in J(i)$.

Using this backstepping algorithm from [29], i.e., recursive design of controllers, FT Lyapunov functions, and gain assignment by induction on $k = 1, 2, \dots, \nu_i$ for each fixed $i \in \mathbb{N}$ and for each reduced order system

$$\begin{cases} \dot{x}_{i,j} = x_{i,j+1} + \Delta_{i,j}(X_{i,j}, \bar{\chi}_{i,j}), & j = 1, \dots, k - 1, \\ \dot{x}_{i,k} = x_{i,k+1} + \Delta_{i,k}(X_{i,k}, \bar{\chi}_{i,k}), & i \in \mathbb{N}, \end{cases} \tag{41}$$

and having designed inductively the feedbacks and FTISS Lyapunov functions which satisfy Assumption (iii) of our main Theorem 1 with $\varepsilon^{(k)} > 0$, $k = 1, 2, \dots, \nu$ from (37) instead of ε , we finally need to explain why Assumptions (i),(ii) hold true as well.

To obtain the uniform estimates from (i)-(ii), we note that, by (34)-(36) from Assumption (A4), functions $\lambda_{i,1}(X_{i,1}) > 0$ from eq. (13) in [29] can be designed such that all $\lambda_{i,1}(X_{i,1}) > 0$ and all their partial derivatives are uniformly bounded w.r.t. $i \in \mathbb{N}$, on every closed ball of every fixed radius. From this, we will obtain (ii) for the closed-loop system (41) with $x_{i,2} = x_{i,2}^*$ in the Base Case $k = 1$. Condition (i) is straightforward in the Base Case $k = 1$ due to (38).

To obtain (i),(ii) at every Inductive Step $(k - 1) \rightarrow k$, one uses the formulas

$$\frac{\partial V_{i,1}}{\partial x_{i,1}} = x_{i,1}, \quad \text{and} \quad \frac{\partial W_{i,k}}{\partial x_{i,k}} = \xi_{i,k}^{2-q_k}, \quad \text{and}$$


$$\frac{\partial W_{i,k}}{\partial x_{i,l}} = -(2 - q_k) \frac{\partial(x_{i,k}^*)^{\frac{1}{q_k}}}{\partial x_{i,l}} \int_{x_{i,k}^*}^{x_{i,k}} (s^{\frac{1}{q_k}} - x_{i,k}^*)^{1-q_k} ds \quad (l < k). \quad (42)$$

from [29] and notes that our Assumption (A4) implies that the functions $\phi_{i,k,j}(\cdot)$, $\psi_{z,k,j}(\cdot)$ from eq. (12) in [29] are bounded on every compact subset of their domain *uniformly w.r.t.* $i \in \mathbb{N}$. Similarly, Assumption (A4) implies that the functions $\tilde{c}_{i,k,l,j}(\cdot)$, from Lemma 1 of [29], $\bar{\rho}_{i,k}(\cdot)$, from (23) from Lemma 3 of [29] are bounded on every compact subset of their domain *uniformly w.r.t.* $i \in \mathbb{N}$. Since they are involved in (24), (25) of [29] we obtain that the coefficients A, B, Q from (26)-(28) in [29] are also bounded on every compact subset of the corresponding domain uniformly w.r.t. $i \in \mathbb{N}$. Finally, $c_{i,k}(\cdot)$, from Lemma 4 of [29], are also bounded on every compact subset of the corresponding domains uniformly w.r.t. $i \in \mathbb{N}$ similarly by (A4).

All this proves that $\lambda_{i,1}(X_{i,1}) > 0, \dots, \lambda_{i,k-1}(X_{i,k}) > 0$ and all their partial derivatives are uniformly bounded w.r.t. $i \in \mathbb{N}$, on every closed ball of every fixed radius. From this, we obtain (i)-(ii) for the closed-loop system (41) with $x_{i,k} = x_{i,k}^*$ at every Inductive Step $(k - 1) \rightarrow k$, and finally for $k = \nu_i$, which ends the proof of Theorem 2.

Acknowledgement. This result was partially obtained when the author was with the University of Groningen, and its special case devoted to finite-time stability and stabilization of infinite networks without any external disturbance inputs and even without the precise definition of ℓ_∞ -FTS and its analysis was published in the conference predecessor [30] of the current work.

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**Теорема про мале посилення для стійкості вхід-стан за
скінченний час нескінченних мережевих систем і її застосування**

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Ми доводимо достатню умову стійкості вхід-стан за скінченний час нескінченних мережевих систем в термінах малого посилення (small gain condition). Мережева система, що розглядається, складається зі зліченної множини скінченновимірних систем звичайних диференціальних рівнянь, кожна з яких з'єднана тільки зі скінченною множиною сусідніх підсистем, а також містить зовнішнє збурення. Передбачається, що кожен вузол мережі (кожна підсистема) є стійкою вхід-стан за скінченний час відносно його скінченновимірних входів утворених фазовими змінними сусідніх підсистем і зовнішнім збуренням. Як застосування цього результату (наслідок) ми отримуємо нову теорему про децентралізовану стабілізацію вхід-стан за скінченний час для нескінченних мережевих систем, які представляють собою зліченний набір з'єднаних трикутних систем звичайних диференціальних рівнянь. Для цього ми комбінуємо доведену в даній роботі теорему малого посилення (small

gain theorem) з методом побудови децентралізованих стабілізуючих керувань, який отримано в роботі S. Pavlichkov and C. K. Pang (NOLCOS-2016) для кінцевих мережевих систем. Дана робота переносить результати недавньої роботи S. Dashkovskiy and S. Pavlichkov, Stability conditions for infinite networks of nonlinear systems and their application for stabilization, Automatica. – 2020. – 112. – 108643 на випадок стабілізації за скінченний час. Ця стаття поширює та узагальнює свого попередника – конференційну статтю на випадок стійкості вхід-стан за скінченний час та децентралізованої стабілізації за наявності зовнішніх входів-збурень. В окремому випадку, коли всі зовнішні збурення є нулями, ми просто отримуємо стійкість за скінченний час та відповідно децентралізовану стабілізацію нескінченних мережевих систем за скінченний час.

Ключові слова: нелінійні системи; стійкість вхід-стан; умови малого посилення.

A small gain theorem for finite-time input-to-state stability of infinite networks and its applications

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We prove a small-gain sufficient condition for (global) finite-time input-to-state stability (FTISS) of infinite networks. The network under consideration is composed of a countable set of finite-dimensional subsystems of ordinary differential equations, each of which is interconnected with a finite number of its “neighbors” only and is affected by some external disturbances. We assume that each node (subsystem) of our network is finite-time input-to-state stable (FTISS) with respect to its finite-dimensional inputs produced by this finite set of the neighbors and with respect to the corresponding external disturbance. As an application we obtain a new theorem on decentralized *finite-time* input-to-state stabilization with respect to external disturbances for infinite networks composed of a countable set of strict-feedback form systems of ordinary differential equations. For this we combine our small-gain theorem proposed in the current work with the controllers design developed by S. Pavlichkov and C. K. Pang (NOLCOS-2016) for the gain assignment of the strict-feedback form systems in the case of finite networks. The current results address the finite-time input-to-state stability and decentralized finite-time input-to-state stabilization and redesign the technique proposed in recent work S. Dashkovskiy and S. Pavlichkov, Stability conditions for infinite networks of nonlinear systems and their application for stabilization, Automatica. – 2020. – 112. – 108643, in which the case of ℓ_∞ -ISS of infinite networks was investigated. The current paper extends and generalizes its conference predecessor to the case of finite-time *ISS* stability and decentralized stabilization in presence of external disturbance inputs and with respect to these disturbance inputs. In the special case when all these external disturbances are zeroes (i.e. are absent), we just obtain finite-time stability and finite-time decentralized stabilization of infinite networks accordingly.

Keywords: nonlinear systems; input-to-state stability; small gain conditions.

Article history: Received: 2 September 2021; Final form: 22 October 2021;

Accepted: 25 October 2021.