Вісник Харківського національного університету імені В.Н. Каразіна
Серія "Математика, прикладна математика і механіка"
Том 94, 2021, с. 24–39
УДК 681.5

Visnyk of V.N.Karazin Kharkiv National University
Ser. “Mathematics, Applied Mathematics and Mechanics”
Vol. 94, 2021, p. 24–39
DOI: 10.26565/2221-5646-2021-94-02

On homogeneous controllability functions

A. Polyakov

Inria Lille-Nord Europe, 40. av. Halley, Villeneuve d’Ascq, 5965, France
andrey.polyakov@inria.fr

The controllability function method, introduced by V. I. Korobov in late 1970s, is known to be an efficient tool for control systems design. This paper bridges the method with the homogeneity theory popular today. In particular, it is shown that the so-called homogeneous norm is a controllability function of the system in some cases. Moreover, the closed-loop control system is homogeneous in a generalized sense. This immediately yields many useful properties of the system such as robustness (Input-to-State Stability) with respect to a rather large class of perturbations.
Keywords: controllability function; generalized homogeneity; robustness.

Поляков А. **Про однорідні функції керованості.** Метод функції керованості, введений В. І. Коробов наприкінці 1970-х років, як відомо, є ефективним інструментом для дослідження систем керування. Ця стаття поєднує цей метод із популярною сьогодні теорією однорідності. Зокрема, показано, що так звана однорідна норма в деяких випадках є функцією керованості системи. Більше того, замкнута система керування є однорідною в узагальненому сенсі. Це відразу дає багато корисних властивостей системи, таких як робастість (стабільність вхідного стану) щодо досить великого класу збурень.
Ключові слова: функція керованості; узагальнена однорідність; робастність.

Поляков А. **Об однородных функциях управляемости.** Метод функции управляемости, введенный В. И. Коробов в конце 1970-х годов, как понятно, является действенным инвентарем для исследования систем управления. Эта статья совмещает этот метод с популярной сегодня теорией однородности. В частности, показано, что так называемая однородная норма в некоторых случаях является функцией управляемости системы. Более того, замкнутая система управления однородна в обобщенном смысле. Это сразу дает много полезных свойств системы, таких как робостность (стабильность входного состояния) относительно достаточно большого класса возмущений.
Ключевые слова: функция управляемости; обобщенная однородность; робастность.

2010 Mathematics Subject Classification: 34H05; 34H15; 93C10; 93C15.

© A. Polyakov, 2021

1. Introduction

Most of physical systems are symmetric in the view of Noether's Theorem [23]. Differential symmetries can discovered in both ODE (Ordinary Differential Equation) and PDE (Partial Differential Equation) models of dynamical systems [9], [8], [27]. One of Lie symmetries, that is rather popular in control theory, is *homogeneity*. The homogeneity is a dilation symmetry known since 18th century, when Leonhard Euler studied functions $x \to f(x)$ which are symmetric with respect to uniform dilation $x \to \lambda x$ of its argument, namely, $f(\lambda x) = \lambda^\nu f(x)$, $\forall \lambda > 0, \forall x$, where ν is a real number. Such functions were called homogeneous and the number ν was referred as the homogeneity degree. It seems that a generalized homogeneity (the symmetry with respect to a non-uniform dilation) was first studied by Vladimir Zubov in [36]. The homogeneity is useful for analysis of nonlinear finite-dimensional dynamical systems (see also [36], [11], [13] [14], [33], [4], [2]) as well as non-linear controllers/observers design (see [13], [7], [10], [1], [19]). Homogeneity degree specifies a convergence rate of any asymptotically stable homogeneous system (see e.g. [22]). Homogeneous approximations of nonlinear models are useful in the case when a linear approximation is not informative or simply impossible [1]. Homogeneous control systems have similar properties to linear ones, e.g., local stability yields global stability, invariance of the compact set is equivalent to stability, etc. However, they may have better control quality, e.g., faster convergence, better robustness and less overshoot. For more details we refer the reader to [27, Chapter 1].

The controllability function method [16] is an efficient tool for control systems design. The monograph [17] presents most detailed study of the method and its applications to different control problems. Formally, a controllability function has no relation with the homogeneity theory. However, in some cases, its design is implicitly inspired by the homogeneity and uses the dilation symmetry of the system. This paper bridges controllability function method with the homogeneity theory showing that the control/controllability function design for linear plants results in a generalized homogeneous control system, that inherits all good properties of homogeneity, e.g., robustness (Input-to-State Stability) with respect to a sufficiently large class of perturbations.

The paper is organized as follows. First, the controllability function method and elements of the homogeneity theory are remained. Next, the main theorem about homogeneous controllability functions for linear plants is proven. Finally, some remarks and conclusions are given.

2. On controllability function method

Following [17] the *controllability function* $\Theta : \mathbb{R}^n \mapsto [0, +\infty)$ is a Lyapunov-like function of a closed-loop system $u = u(x)$ realizing a position control in a finite time. More precisely, the properties of Θ are characterized by the following theorem.

Theorem 1. *[17, Theorem 1.2, page 19] Let us consider the controlled process*

$$\dot{x} = f(x,u), \quad t > 0, \quad f : \mathbb{R}^{n+m} \mapsto \mathbb{R}^m \tag{1}$$

where $x(t) \in \mathbb{R}^n, u \in \Omega \subset \mathbb{R}^m$ and the function f is locally Lipschit continuous on $(\mathbb{R}^n \backslash \{0\}) \times \Omega$, i.e.,

$$\|f(x_1,u_1) - f(x_2,u_2)\| \leq L(\rho_1,\rho_2)(\|x_1 - x_2\| + \|u_1 - u_2\|), \qquad \begin{array}{c} x_i \in K(\rho_1,\rho_2), \\ \|u_i\| \in \Omega, i = 1,2, \end{array}$$

where $K(\rho_1,\rho_2) := \{x : 0 < \rho_1 \leq \|x\| \leq \rho_2\}$.

Let there exist a function $\Theta : \mathbb{R}^n \mapsto [0, +\infty)$ such that

1) *$\Theta > 0$ for $x \neq 0$ and $\Theta(0) = 0$;*
2) *there exists $c > 0$ such that the set is $\mathbb{Q} := \{x : \Theta(x) \leq c\}$ is bounded;*
3) *there exists a function $u : \mathbb{Q} \mapsto \Omega$ satisfying the inequality*

$$\frac{\partial \Theta(x)}{\partial x} f(x,u(x)) \leq -q\Theta^{1-\frac{1}{\alpha}}(x), \quad \forall x \in \mathbb{Q}\backslash\{0\} \tag{2}$$

for some $q > 0$ and $\alpha > 0$, such that u satisfies the Lipschitz condition on on any set $K(\rho_1,\rho_2) \cap \mathbb{Q}$.

Then the trajectory $x(t)$ of the closed-loop system (1) with $x(0) = x_0 \in \mathbb{Q}$ reaches the state $x = 0$ at an instant of time $T(x_0) \leq \frac{\alpha}{q}\Theta^{\frac{1}{\alpha}}(x_0)$. Moreover, if $\alpha = \infty$ then $x(t) \to 0$ as $t \to \infty$.

If $\alpha = 1$ and the symbol $\leq$ in (2) is replaced with $=$, then the inequality becomes the Bellman (dynamic programming) equation and the function Θ is the settling time function of the closed-loop system. On the other hand, the controllability function Θ satisfies the classical [20] (resp., a generalized [32], [5]) Lyapunov theorem for $\alpha = +\infty$ (resp. $0 < \alpha < +\infty$).

Example 1. [17, page 21] A controllability function Θ for the chain of integrators

$$\dot{x} = Ax + Bu, \quad x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}, \quad A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{pmatrix}, \tag{3}$$

can be designed implicitly as a solution of the following algebraic equation

$$a_0\Theta = x^\top D(\Theta) F D(\Theta) x, \quad x \neq 0 \tag{4}$$

where $a_0 > 0$, a symmetric matrix $F = F^\top \in \mathbb{R}^{n\times n}$ satisfies[1] the linear matrix inequalities (LMIs)

$$F(A + BK) + (A + BK)^\top F \prec 0, \quad F \succ 0 \tag{5}$$

[1]The sign $\prec$ (resp. $\succ$) denotes the negative (resp. positive) definiteness of a symmetric matrix.

with $K \in \mathbb{R}^{1\times n} : A + BK$ - Hurwitz, and the matrix-valied function D is defined as follows

$$D(\Theta) = \operatorname{diag}\left(\Theta^{-\frac{m+n-2i+1}{2\alpha}}\right) \tag{6}$$

with α and m are natural numbers being selected such that the linear matrix inequality (LMI)

$$F - H_\alpha F - H_\alpha F \succ 0, \quad H_\alpha = \operatorname{diag}\left(-\frac{m+n-2i+1}{2\alpha}\right), \tag{7}$$

is fulfilled. The corresponding controller is given by

$$u(x) = \Theta^{\frac{m-n-1}{2\alpha}} KD(\Theta)x. \tag{8}$$

3. Homogeneity

This section presents a brief summary of the homogeneity theory of dynamical system in $\mathbb{R}^n$.

3.1. Dilations in $\mathbb{R}^n$

The standard (Euler's) dilation in a vector space is the operator $x \to \lambda x$, $\lambda > 0$. Generalized dilations and the generalized homogeneity in $\mathbb{R}^n$ are introduced in [36], [15], [14], [26].

Definition 1. *[14] A family of mappings* $\mathbf{d}(s) : \mathbb{R}^n \mapsto \mathbb{R}^n$, $s \in \mathbb{R}$ *is a said to be a dilation group (or, simply, a* ***dilation****) in* $\mathbb{R}^n$ *if*

- *(**Group property**)* $\mathbf{d}(0)x = x$, $\mathbf{d}(t)\circ\mathbf{d}(s)x = \mathbf{d}(t+s)x$, $\forall t, s \in \mathbb{R}, \forall x \in \mathbb{R}^n$;
- *(**Limit property**)* $\lim\limits_{s\to-\infty} \|\mathbf{d}(s)x\| = 0$ *and* $\lim\limits_{s\to+\infty} \|\mathbf{d}(s)x\| = \infty$ *for all* $x \neq 0$.

Below we deal only with the so-called linear dilation [26] defined as follows:

$$\mathbf{d}(s) := e^{sG_\mathbf{d}} = \sum_{i=0}^{\infty} \frac{s^i G_\mathbf{d}^i}{i!}, \tag{9}$$

where $G_\mathbf{d} \in \mathbb{R}^{n\times n}$ is an anti-Hurwitz matrix being a generator of the dilation. For $G_\mathbf{d} = I_n$ (the identity matrix), we have the standard (Euler's) dilation $\mathbf{d}(s) = e^s I_n$ but the case of diagonal matrix $G_\mathbf{d}$ corresponds to the so-called weighted dilation [36], [31], [12], [18], [1], [28] popular today.

Definition 2. *[26] A linear dilation* $\mathbf{d}$ *is said to be monotone with respect to a norm* $\|\cdot\|$ *if there exists* $\beta > 0$ *such that*

$$\|\mathbf{d}(s)\| \leq e^{\beta s}, \quad \forall s \leq 0, \tag{10}$$

where $\|\mathbf{d}(s)\| = \max\limits_{x\neq 0} \frac{\|\mathbf{d}(s)x\|}{\|x\|}$ *is the matrix norm of* $\mathbf{d}(s)$.

A criterion of the monotonicity of the linear dilation in $\mathbf{R}^n$ is given by the following lemma

Lemma 1. *[26] Any linear dilation* $\mathbf{d}$ *in the finite dimensional space* $\mathbb{R}^n$ *is monotone with respect to the weighted Euclidean norm* $\|x\|_P = \sqrt{x^\top P x}, x \in \mathbb{R}^n$ *provided that the symmetric matrix* $P = P^\top \in \mathbf{R}^{n\times n}$ *satisfies the linear matrix inequality*

$$G_{\mathbf{d}}^\top P + P G_{\mathbf{d}} \succ 0, \quad P \succ 0, \tag{11}$$

where the symbol $\succ 0$ *means that the matrix is positive definite.*

It is known [26] that any vector from $\mathbb{R}^n\backslash\{0\}$ can be uniquely projected to a unit sphere using monotone dilation. More precisely, for any $x \in \mathbb{R}^n\backslash\{0\}$ there exists a unique pair $(z, s) : \|z\| = 1, s \in \mathbb{R}$ such that $\mathbf{d}(s)x = z$. We use this feature of monotone dilation in our further constructions.

3.2. Canonical homogeneous norm. Any compact and convex neighborhood of the origin induces a norm in $\mathbb{R}^n$ by means of the standard dilation $x \to \lambda x, \lambda > 0$ (see, e.g.,[30]). In the general case, the similar construction for monotone dilation leads to the so-called homogeneous norm [10].

Definition 3. *The function* $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \mapsto [0, +\infty)$ *defined as follows* $\|0\|_{\mathbf{d}} = 0$ *and*

$$\|x\|_{\mathbf{d}} = e^{s_x} \quad : \|\mathbf{d}(-s_x)x\| = 1, \quad x \neq 0,$$

is called the canonical homogeneous norm, where $\mathbf{d}$ *is a monotone dilation with respect to the norm* $\|\cdot\|$ *in* $\mathbb{R}^n$.

The canonical homogeneous norm is continuous on $\mathbb{R}^n$ and locally Lipschitz continuous on $\mathbb{R}^n\backslash\{0\}$ (see, [26]). Moreover, in [27, Theorem 7.1., page 188] it is proven that $\|\cdot\|_{\mathbf{d}}$ is, in fact, a norm (in the classical sense) in a finite-dimensional space $\tilde{\mathbb{R}}^n$ homeomorphic[2] to $\mathbb{R}^n$.

Below we show that the canonical homogeneous norm is the controllability function for linear control systems considered in [17, Chapter 1]. The related analysis would require differentiability of $\|\cdot\|_{\mathbf{d}}$ on $\mathbb{R}^n\backslash\{0\}$. The latter can always be guaranteed for a canonical homogeneous norm induced by a weighted Euclidean norm.

Lemma 2. *[26] If the canonical homogeneous norm* $\|\cdot\|_{\mathbf{d}}$ *is induced by the norm* $\|x\| = \sqrt{x^\top P x}$ *with a symmetric matrix* $P = P^\top \in \mathbb{R}^{n\times n}$ *satisfying* (11) *then* $\|\cdot\|_{\mathbf{d}} \in C(\mathbb{R}^n) \cap C^1(\mathbb{R}^n\backslash\{0\})$ *and*

$$\frac{\partial \|x\|_{\mathbf{d}}}{\partial x} = \|x\|_{\mathbf{d}} \frac{x^\top \mathbf{d}^\top(-\ln\|x\|_{\mathbf{d}}) P \mathbf{d}(-\ln\|x\|_{\mathbf{d}})}{x^\top \mathbf{d}^\top(-\ln\|x\|_{\mathbf{d}}) P G_{\mathbf{d}} \mathbf{d}(-\ln\|x\|_{\mathbf{d}})x}, \quad x \neq 0. \tag{12}$$

[2]The space $\tilde{\mathbb{R}}^n$ consists of vectors from $\mathbb{R}^n$, but a sum of vectors and a multiplication of a vector by a scalar are defined differently.

Below we show that a properly designed canonical homogeneous norm is a controllability function for linear control plant.

3.3. Homogeneous systems. By definition, we say that an ODE is homogeneous if its right-hand side is a homogeneous vector field.

Definition 4. *[14] A vector field (resp. a function)* $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ *(resp.* $h : \mathbb{R}^n \to \mathbb{R}$*) is said to be* $\mathbf{d}$*-homogeneous of the degree* $\nu \in \mathbb{R}$ *is*

$$\begin{array}{c} f(\mathbf{d}(s)x) = e^{\nu s}\mathbf{d}(s)f(x), \quad \forall s \in \mathbb{R}, \quad \forall x \in \mathbb{R}^n \\ (resp.\ h(\mathbf{d}(s)x) = e^{\nu s}h(x), \quad \forall s \in \mathbb{R}, \quad \forall x \in \mathbb{R}^n), \end{array} \tag{13}$$

where $\mathbf{d}$ *is a dilation in* $\mathbb{R}^n$.

The canonical homogeneous norm is the simplest example of the homogeneous function of the degree 1.

Example 2: The system

$$\dot{x} = f(x) := \begin{pmatrix} x_2 \\ -k_1 x_1^{\frac{1}{5}} - k_2 x_2^{\frac{1}{3}} \end{pmatrix}$$

is $\mathbf{d}$-homogeneous of the degree $\nu = -2$ with respect to the dilation

$$\mathbf{d}(s) = \begin{pmatrix} e^{5s} & 0 \\ 0 & e^{3s} \end{pmatrix}.$$

Indeed, simple computations show

$$f(\mathbf{d}(s)x) = e^{-2s}\mathbf{d}(s)f(x), \quad \forall x \in \mathbb{R}^n, \quad \forall s \in \mathbb{R}.$$

Moreover, one can be shown [29] that

$$V(x) = c\left(\frac{5k_1|x_1|^{\frac{6}{5}}}{6} + \frac{x_2^2}{2}\right)^{\frac{4}{3}} + k_1 x_1 x_2, \quad c > \max\left\{\left(\frac{3}{4}\right)^{\frac{4}{3}}, \frac{k_1^{\frac{5}{6}} 2^{\frac{2}{3}}(1 + k_2/6)}{4k_2/3}\right\}$$

is the controllability function (in the sense of Theorem 1)

$$\exists q > 0 : \dot{V}(x) = \frac{\partial V}{\partial x} f(x) \le -qV^{\frac{3}{4}}(x), \quad \forall x \neq 0,$$

of the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u$$

with the feedback control

$$u(x) = -k_1 x_1^{\frac{1}{5}} - k_2 x_2^{\frac{1}{3}}, \quad k_1 > \frac{5}{6}k_2.$$

To highlight some features the homogeneous systems we present the series of known results.

Theorem 2. *[36] If $x(t, x_0)$ is a solution of a* **d***-homogeneous system*

$$\dot{x} = f(x), \quad t > 0, \quad f : \mathbb{R}^n \mapsto \mathbb{R}^n, \quad f \in C(\mathbb{R}^n) \tag{14}$$

with the initial condition $x(0) = x_0$ then

$$x(t, \mathbf{d}(s)) = \mathbf{d}x(e^{\mu s}t, x_0)$$

is a solution of the same ODE with the scaled initial condition $x(0) = \mathbf{d}(s)x_0$.

The latter theorem implies that any local property of the system (14) can be extended globally. For instance, local stability implies global stability, global-in-time existence of solutions for small initial data yields global-in-time existence of solutions for large initial data, etc.

The Zubov-Rosier Theorem [36], [31] given below is the converse Lyapunov theorem for homogeneous systems. For shortness, we say that a "system is stable" if its zero solution is stable.

Theorem 3. *[31] A* **d***-homogeneous system* (14) *is asymptotically stable if and only if there exists a positive definite* **d***-homogeneous function $V : \mathbb{R}^n \mapsto [0, +\infty)$ of degree* 1 *such that $V \in C(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \backslash \{0\})$ and*

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq -qV^{1+\mu}(x), \quad \forall x \neq 0, \tag{15}$$

where μ is the homogeneity degree of the vector field f and $q > 0$ is a positive parameter.

In other words, any stable homogeneous system admits a homogeneous Lyapunov function. Notice that, for $\mu < 0$ the inequality (15) repeats the inequality (2). The estimate (15) yields the following corollary which characterize the convergence rates of homogeneous systems.

Corollary 1. *[22] If the system is asymptotically stable then it is*

- *globally uniformly finite-time stable*[3] *for $\mu < 0$;*
- *globally uniformly exponentially stable for $\mu = 0$;*
- *globally uniformly nearly fixed-time stable*[4] *for $\mu > 0$,*

where $\mu \in \mathbb{R}$ is a homogeneity degree of the vector field f.

[3] A system is globally uniformly finite-time stable if it is Lyapunov stable and there exists a locally bounded function $T : \mathbb{R}^n \to [0, +\infty)$ such that $\|x(t)\| = 0, \forall t \geq T(x_0), \forall x_0 \in \mathbb{R}^n$. For more details about finite-time stability see [5], [32].

[4] A system is globally uniformly nearly fixed-time stable if it Lyapunov stable and $\forall r > 0, \exists T_r > 0 : \|x(t)\| < r, \forall t \geq T_r$ independently of $x_0 \in \mathbb{R}^n$. For more details about fixed-time stability see [24].

One of useful properties of homogeneous control systems is robustness (Input-to-State Stability) in the sense of the following definition.

Definition 5. *[34] The system*

$$\dot{x} = f(x, \delta), \quad f : \mathbb{R}^{n+k} \mapsto \mathbb{R}^n, \quad f \in C(\mathbb{R}^{n+k}) \tag{16}$$

is said to be Input-to-State Stable (ISS) if there exist[5] $\beta \in \mathcal{KL}$ *and* $\gamma \in \mathcal{K}$ *:*

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma\left(\sup_{\tau\in[0,t]} \|\delta(\tau)\|\right), \quad \forall t > 0.$$

The ISS analysis is a non-trivial problem in the general case [35], however, ISS of a perturbed homogeneous systems follows from asymptotic stability of a non-perturbed one provided that the disturbances are involved to the system in a homogeneous way [33], [12], [1].

Theorem 4. *[3] Let* $\mathbf{d}$ *be a dilation in* $\mathbb{R}^n$*,* $\mathbf{d}$ *is a dilation in* $\mathbb{R}^k$*, and* $t\mu \in \mathbb{R}$*.If*

$$f(\mathbf{d}(s)x, \tilde{\mathbf{d}}(s)\delta) = e^{\mu s}\mathbf{d}(s)f(x, \delta), \quad \forall s \in \mathbb{R}, \forall x \in \mathbb{R}^n, \forall \delta \in \mathbb{R}^k.$$

and the system (16) *with* $\delta = 0$ *is asymptotically stable then this system is ISS.*

Example 3. The control system considered in Example 2 is robust (Input-to-State Stable) with respect to bounded additive exogenous perturbations and additive measurement noises. Indeed, let us denote

$$f(x, \delta) = \begin{pmatrix} x_2 \\ u(x+\delta) \end{pmatrix} + \delta_1, \quad x = (x_1, x_2)^\top \in \mathbb{R}^2, \quad \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} \in \mathbb{R}^4,$$

where the feedback control u is defined in Example 2, $\delta_1 \in \mathbb{R}^2$ is a measurement noise and δ_2 is an exogenous perturbation. Let the dilation $\mathbf{d}$ be defined as in Example 2. Then

$$\tilde{\mathbf{d}}(s) = \begin{pmatrix} \mathbf{d}(s) & 0 \\ 0 & e^{-2s}\mathbf{d}(s) \end{pmatrix},$$

is a dilation in $\mathbb{R}^4$ with the generator

$$G_{\tilde{\mathbf{d}}} = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

[5] A function $\gamma : [0, +\infty) \mapsto [0, +\infty)$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and $\gamma(0) = 0$.
A function $\beta : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is of class $\mathcal{KL}$ if the function $r \mapsto \beta(r, s)$ is of class $\mathcal{K}$ for any fixed $s \in \mathbb{R}$ and for any fixed $r \geq 0$ the function $s \mapsto \beta(r, s)$ is decreasing to zero as $s \to +\infty$.

being an anti-Hurwitz matrix. In Example 2, it was shown that the system

$$\dot{x} = f(x, 0)$$

is globally uniformly asymptotically stable. Since

$$f(\mathbf{d}(s)x, \tilde{\mathbf{d}}(s)\delta) = e^{-2s}\mathbf{d}(s)f(x, \delta), \quad \forall x \in \mathbb{R}^2, \quad \forall \delta \in \mathbb{R}^4, \forall s \in \mathbb{R}$$

then, by Theorem 4, this system is ISS.

4. Homogeneous controllability functions for linear plants

Let us consider the linear control system

$$\dot{x} = Ax + Bu, \quad t > 0, \quad x(0) = x_0 \tag{17}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $A \in \mathbb{R}^{n\times n}$ is the system matrix, $u(t) \in \mathbb{R}^m$ is a control input and $B \in \mathbb{R}^{n\times m}$.

Inspired by Theorem 1 and [26] we present the following result.

Theorem 5. *Let the pair $\{A, B\}$ be controllable and the pair $Y_0 \in \mathbf{R}^{m\times n}$, $G_0 \in \mathbb{R}^{n\times n}$ be a solution of the linear algebraic equation*

$$AG_0 - G_0A + BY_0 = A, \quad G_0B = 0 \tag{18}$$

with respect to G_0, Y_0, then

1) *$G_{\mathbf{d}} = I_n + \mu G_0$ is anti-Hurwitz for any $\mu \in [-1, 0)$;*

2) *$G_0 - I_n$ is invertible and the matrix $A_0 = A + BY_0(G_0 - I_n)^{-1}$ satisfies*

$$A_0G_{\mathbf{d}} = (G_{\mathbf{d}} + \mu I_n)A_0, \quad G_{\mathbf{d}}B = B. \tag{19}$$

If a solution $X \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{m\times n}$ satisfies the system of linear matrix inequalities (LMIs)

$$A_0X + XA_0^\top + BY + Y^\top B^\top \prec 0, \quad G_{\mathbf{d}}X + XG_{\mathbf{d}}^\top \succ 0, \quad X = X^\top \succ 0 \tag{20}$$

then

3) *the canonical homogeneous $\|\cdot\|_{\mathbf{d}}$ induced by the weighted Euclidean norm $\|x\| = \sqrt{x^\top X^{-1}x}$ is a **controllability function** of the system (17) with the control*

$$u(x) = Y_0(G_0 - I_n)^{-1}x + \|x\|_{\mathbf{d}}^{\mu+1}YX^{-1}\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x \tag{21}$$

and

$$\frac{d}{dt}\|x\|_{\mathbf{d}} \le \frac{\lambda_{\max}(A_0X + XA_0^\top + BY + Y^\top B^\top)}{\lambda_{\max}(G_{\mathbf{d}}X + XG_{\mathbf{d}}^\top)}\|x\|_{\mathbf{d}}^{1+\mu}, \quad x \neq 0, \tag{22}$$

where $\lambda_{\max}(Q)$ denotes a maximal eigenvalue of a symmetric matrix Q and $\mathbf{d}$ is a dilation generated by $G_{\mathbf{d}}$;

4) $u \in C^1(\mathbf{R}^n \backslash \{0\})$ for any $\mu \in [-1, 0)$, u is continuous at zero if $\mu \in (-1, 0)$ and u is discontinuous at zero but locally bounded if $\mu = -1$;

5) the closed-loop system (17), (21) *is* **d**-*homogeneous of the degree* μ.

Proof. 1) Since the pair $\{A, B\}$ then there exists [21], [28] a coordinate transformation $M \in \mathbb{R}^{n\times n}$ such that

$$M^{-1}B = \begin{pmatrix} 0 \\ ... \\ 0 \\ A_{k\,k+1} \end{pmatrix}, \quad M^{-1}A_0M = \begin{pmatrix} 0 & A_{12} & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & A_{k-1\,k} \\ A_{k1} & A_{k2} & A_{k3} & ... & A_{kk} \end{pmatrix}, \quad (23)$$

where $A_{i\,i+1} \in \mathbb{R}^{n_i\times n_{i+1}}$, $\operatorname{rank}(A_{i\,i+1}) = n_i$, $n_i \le n_{i+1}$, $i = 1, 2, ..., k$, $n_1 + ... + n_k = n$, $n_k \le m$, $n_{k+1} = m$ and k is a minimal natural number such that $\operatorname{rank}(B, AB, ..., A^{k-1}B) = n$.

In this case, the equation (18) can be equivalently rewritten as follows

$$\tilde{A}\tilde{G}_0 - \tilde{G}_0\tilde{A} + \tilde{B}\tilde{Y}_0 = \tilde{A}, \quad \tilde{G}_0\tilde{B} = 0 \quad (24)$$

where $\tilde{B} = M^{-1}B$, $\tilde{A} = M^{-1}A_0M, \tilde{G}_0 = M^{-1}G_0M$ and $\tilde{Y}_0 = Y_0M$.

Taking into account, the structure of $\tilde{A}$ and $\tilde{B}$ we conclude that the linear equation (24) has a solution with respect to $\tilde{Y}_0, \tilde{G}_0$ and the matrix $\tilde{G}_0$ has the block lower triangular form

$$\tilde{G}_0 = \begin{pmatrix} -(k-1)I_{n_1} & 0 & 0 & ... & 0 & 0 \\ * & -(k-2)I_{n_2} & 0 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & \\ * & * & * & ... & -I_{n_{k-1}} & 0 \\ * & * & * & ... & * & 0 \end{pmatrix},$$

where $*$ denotes a possibly non-zero block. The latter means $G_0 - I$ is invertible and $G_{\mathbf{d}} = I_n + \mu G_0$ is anty-Hurwitz for $\mu < 0$.

2) Since $G_0 - I_n$ is invertible and the pair G_0, Y_0 satisfies (18) then

$$A - A + \mu AG_0 - \mu G_0A + \mu BY_0 = \mu A$$

or, equivalently,

$$AG_{\mathbf{d}} + \mu BY_0 = (G_{\mathbf{d}} + \mu I_n)A.$$

On the one hand, obviously, $G_{\mathbf{d}}B = B$ and

$$(G_{\mathbf{d}} + \mu I_n)A = (G_{\mathbf{d}} + \mu I_n)\left(A + BY_0(G_0 - I_n)^{-1} - BY_0(G_0 - I_n)^{-1}\right) =$$

$$(G_{\mathbf{d}} + \mu I_n)A_0 - (1 + \mu)BY_0(G_0 - I_n)^{-1}.$$

On the other hand, one has

$$AG_{\mathbf{d}} + \mu BY_0 = \left(A + BY_0(G_0 - I_n)^{-1} - BY_0(G_0 - I_n)^{-1}\right)G_{\mathbf{d}} + \mu BY_0 =$$

$$A_0G_{\mathbf{d}}-BY_0(G_0-I_n)^{-1}G_{\mathbf{d}}+\mu BY_0 = A_0G_{\mathbf{d}}-BY_0(G_0-I_n)^{-1}(G_{\mathbf{d}}-\mu(G_0-I_n)) = A_0G_{\mathbf{d}}-(\mu+1)BY_0(G_0-I_n)^{-1}.$$

Hence, we derive the identity (19).

3) First of all, notice that, the canonical homogeneous is well defined since X^{-1} is satisfies the second and the third inequalities from (20). It is continuous on $\mathbb{R}^n$ and continuously differentiable on $\mathbb{R}^n\backslash\{0\}$ in the view of Lemma 2.

On the other hand, the formula $\mathbf{d}(s)=e^{sG_{\mathbf{d}}}$ and the identity (19) yield

$$A_0\mathbf{d}(s)=\sum_{i=0}^{\infty}\frac{s^iA_0G_{\mathbf{d}}^i}{i!}=\sum_{i=0}^{\infty}\frac{s^i(G_{\mathbf{d}}-\mu I_n)^iA_0}{i!}=e^{sG_{\mathbf{d}}}A_0=e^{\mu s}\mathbf{d}(s)A_0. \tag{25}$$

and

$$\mathbf{d}(s)B=e^sB,\quad \forall s\in\mathbb{R}. \tag{26}$$

In this case, for the closed loop system, using the formula (12) we derive

$$\frac{d}{dt}\|x\|_{\mathbf{d}}=\|x\|_{\mathbf{d}}\frac{x^\top\mathbf{d}^\top(-\ln\|x\|_{\mathbf{d}})X^{-1}\mathbf{d}(-\ln\|x\|_{\mathbf{d}})(A_0x+\|x\|_{\mathbf{d}}^{\mu+1}BYX^{-1}\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x)}{x^\top\mathbf{d}^\top(-\ln\|x\|_{\mathbf{d}})X^{-1}G_{\mathbf{d}}\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x}=$$

$$\|x\|_{\mathbf{d}}^{1+\mu}\frac{x^\top\mathbf{d}^\top(-\ln\|x\|_{\mathbf{d}})X^{-1}(A_0+BYX^{-1})\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x}{x^\top\mathbf{d}^\top(-\ln\|x\|_{\mathbf{d}})X^{-1}G_{\mathbf{d}}\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x}=$$

$$\|x\|_{\mathbf{d}}^{1+\mu}\frac{x^\top\mathbf{d}^\top(-\ln\|x\|_{\mathbf{d}})X^{-1}(A_0X+BY+XA_0^\top+Y^\top B^\top)X^{-1}\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x}{x^\top\mathbf{d}^\top(-\ln\|x\|_{\mathbf{d}})X^{-1}(G_{\mathbf{d}}X+XG_{\mathbf{d}}^\top)X^{-1}\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x}.$$

Hence, taking into account that X and Y satisfy (20) we derive (22).

4) Since the canonical homogeneous is continuous on $\mathbb{R}^n$ and continuously differentiable on $\mathbb{R}^n\backslash\{0\}$ then, by construction, u is continuously differentiable on $\mathbb{R}^n\backslash\{0\}$. Moreover, from the definition of the canonical homogeneous norm we have

$$x^\top\mathbf{d}^\top(-\ln\|x\|_{\mathbf{d}})X^{-1}\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x=1,$$

so

$$\|u\|\le r_1\|x\|+r_2\|x\|_{\mathbf{d}}^{\mu+1},\quad \forall x\in\mathbb{R}^n$$

with $r_1\ge 0$ and $r_2>0$ dependent of Y_0, G_0, X, and Y. The latter means that u is locally bounded for any $\mu\in[-1,0)$ and continuous at zero for $\mu\in(-1,0)$.

5) Let us denote the right-hand side of the closed-loop system by

$$f(x)=A_0x+\|x\|_{\mathbf{d}}^{\mu+1}BYX^{-1}\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x$$

Using the identities (25), (26) and $\|\mathbf{d}(s)x\|_{\mathbf{d}}=e^s\|x\|_{\mathbf{d}}$ we derive

$$f(\mathbf{d}(s)x)=A_0\mathbf{d}(s)x+\|\mathbf{d}(s)x\|_{\mathbf{d}}^{\mu+1}BYX^{-1}\mathbf{d}(-\ln\|\mathbf{d}(s)x\|_{\mathbf{d}})\mathbf{d}(s)x=$$

$$e^{\mu s}\mathbf{d}(s)A_0x+e^{s(1+\mu)}\|x\|_{\mathbf{d}}^{\mu+1}BYX^{-1}\mathbf{d}(-\ln\|x\|_{\mathbf{d}}-s)\mathbf{d}(s)x=$$

$$e^{\mu s}\mathbf{d}(s)\left(A_0x+\|x\|_{\mathbf{d}}^{\mu+1}BYX^{-1}\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x\right)=e^{\mu s}\mathbf{d}(s)f(x),$$

for all $x\in\mathbb{R}^n$ and all $s\in\mathbb{R}$. The proof is complete.

Remark 1. *A solution of the linear matrix equation* (18) *is not unique in the general case of the multiply-input linear control system. In the single input case* ($m = 1$)*, the solution if always unique and* $\tilde{G}_0$ *is a diagonal matrix.*

Remark 2. *The system of linear matrix inequalities* (20) *is always feasible at least for* μ *close to 0. Indeed, the feasibility of the first LMI together with the third one follows from controllability of the pair* $\{A, B\}$ *(see, [6]), while the second LMI becomes the third one for* μ *tending to 0 (reps.* $G_{\mathbf{d}} \to I_n$ *as* $\mu \to 0$*). Notice that, the system of LMIs* (20) *is always feasible for* $\mu = [-1, 0)$*, at least, in the single input case [25].*

As a straightforward corollary of Theorems 4 and 5 (see also Example 3 as the hint of the proof) we provide the following result.

Corollary 2. *The control system* (17), (21) *designed by Theorem 5 is ISS with respect bounded additive measurement noises if* $\mu \in [-1, 0)$ *and ISS with respect to bounded additive exogenous perturbations if* $\mu \in (-1, 0)$*.*

Remark 3. *If* $m = 1$ *and* A *as in Example 1 then* $Y_0 = 0$*,*

$$G_0 = \begin{pmatrix} -n+1 & 0 & \dots & 0 & 0 \\ 0 & -n+2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -1 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \quad \mathbf{d}(s) = \begin{pmatrix} e^{s(1-\mu(n-1))} & 0 & \dots & 0 & 0 \\ 0 & e^{1-\mu(n-2))} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e^{s(1-\mu)} & 0 \\ 0 & 0 & \dots & 0 & e^{s} \end{pmatrix}.$$

Obviously, for $F = a_0 X^{-1}$*, an integer* $\alpha = -\frac{1}{\mu}$ *and* $m = \alpha + n - 1$ *the canonical homogeneous norm defined in Theorem 5 simply coincides with the controllability function given by* (4), (6) *and the system of LMIs* (20) *implies* (5), (7) *with* $K = YX^{-1}$*. Therefore, the controller* (21) *coincides with* (8)*. The latter means that the control system* (3), (8) *is* **d***-homogeneous.*

Similar conclusions can be made for controllability functions designed in [17, Chapter 1, §5] and [17, Chapter 1, §6].

5. Conclusions

The paper shown that linear autonomous control system always admits a generalized homogeneous controllability function. Moreover, the well-known controllability function studied in [17, Chapter 1] are homogeneous as well! This immediately results in robustness (Input-to-State Stability) of the corresponding control systems with respect to a rather large class of disturbances. An interesting open problem in this context is the homogeneity analysis of controllability functions designed by means of integral operators (see [17, Chapter 5]).

ORCID ID

A.Polyakov, https://orcid.org/0000-0002-5876-3495

REFERENCES

1. V. Andrieu, L. Praly, and A. Astolfi. Homogeneous Approximation, Recursive Observer Design, and Output Feedback, SIAM Journal of Control and Optimization, – 2008. – **47(4).** – P. 1814–1850. DOI: 10.1137/060675861.

2. E. Bernuau, D. Efimov, W. Perruquetti, and A. Polyakov. On homogeneity and its application in sliding mode control, Journal of The Franklin Institute, – 2014. – **351(4).** – P. 1866–1901. DOI: 10.1016/j.jfranklin.2014.01.007.

3. E. Bernuau, A. Polyakov, D. Efimov, and W. Perruquetti. Verification of ISS, iISS and IOSS properties applying weighted homogeneity, System & Control Letters, – 2013. – **62(12).** – P. 1159–1167. DOI: 10.1016/j.sysconle.2013.09.004.

4. S. P. Bhat and D. S. Bernstein. Geometric homogeneity with applications to finite-time stability, Mathematics of Control, Signals and Systems, – 2005. – **17.** – P. 101–127. DOI: 10.1007/s00498-005-0151-x.

5. S. P. Bhat and D. S. Bernstein. Finite time stability of continuous autonomous systems, SIAM J. Control Optim., – 2000. – **38(3).** – P. 751–766. DOI: 10.1137/S0363012997321358.

6. S. Boyd, E. Ghaoui, E. Feron, and V. Balakrishnan. Linear Matrix Inequalities in System and Control Theory. 1994. SIAM, Philadelphia, ix + 185 p. DOI: 10.1137/1.9781611970777.

7. J.-M. Coron and L. Praly. Adding an integrator for the stabilization problem, Systems & Control Letters, – 1991. – **17(2).** – P. 89–104. DOI: 10.1016/0167-6911(91)90034-C.

8. V. Fischer and M. Ruzhansky. Quantization on Nilpotent Lie Groups. 2016. Springer, XIII + 557 p. DOI: 10.1007/978-3-319-29558-9.

9. G. Folland. Subelliptic estimates and function spaces on nilpotent Lie groups, Arkiv for Matematik, – 1975. – **13(1-2).** – P. 161–207.

10. L. Grüne. Homogeneous state feedback stabilization of homogeneous systems, SIAM Journal of Control and Optimization, – 2000. – **38(4).** – P. 1288–1308. DOI: 10.1137/S0363012998349303.

11. H. Hermes. Nilpotent approximations of control systems and distributions, SIAM Journal of Control and Optimization, – 1986. – **24(4).** – P. 731–736. DOI: 10.1137/0324045

12. Y. Hong. H_∞ control, stabilization, and input-output stability of nonlinear systems with homogeneous properties, Automatica, – 2001. – **37(6).** – P. 819–829. DOI: 10.1016/S0005-1098(01)00027-9.

13. M. Kawski. Homogeneous stabilizing feedback laws, Control Theory and Advanced Technology, – 1990. – **6(4).** – P. 497–516.

14. M. Kawski. Families of dilations and asymptotic stability, Analysis of Controlled Dynamical Systems, – 1991. – **8.** – P. 285–294. DOI: 10.1007/978-1-4612-3214-8_25.

15. V. V. Khomenuk. On systems of ordinary differential equations with generalized homogenous right-hand sides, Izvestia vuzov. Mathematica, – 1961. – **3(22).** – P. 157–164 (in Russian).

16. V. I. Korobov. A solution of the synthesis problem using controlability function, Doklady Academii Nauk SSSR, – 1979. – **248.** – P. 1051–1063.

17. V. I. Korobov. Controllability Function Method. 2007, Institute of Computer Research "Regular and chaotic dynamics", M.–Izhevsk, 576 p. (in Russian). ISBN 978-5-93972-610-8

18. A. Levant. Higher-order sliding modes, differentiation and output-feedback control, International Journal of Control, – 2003. – **76(9-10).** – P. 924–941. DOI: 10.1080/0020717031000099029.

19. F. Lopez-Ramirez, A. Polyakov, D. Efimov, and W. Perruquetti. Finite-time and fixed-time observer design: Implicit Lyapunov function approach, Automatica, – 2018. – **87.** – P. 52–60. DOI: 10.1016/j.automatica.2017.09.007.

20. A. M. Lyapunov. The general problem of the stability of motion. 1992. Taylor & Francis.

21. M. Misrikhanov and V. Ryabchenko. Pole placement for controlling a large scale power system, Automation and Remote Control, – 2011. – **72.** – P. 2123–2146. DOI: 10.1134/S0005117911100110.

22. H. Nakamura, Y. Yamashita, and H. Nishitani. Smooth Lyapunov functions for homogeneous differential inclusions, In Proceedings of the 41st SICE Annual Conference, – 2002. – **3.** – P. 1974–1979. DOI: 10.1109/SICE.2002.1196633.

23. E. Noether. Invariante variationsprobleme, Kgl. Ges. d. Wiss. Nachrichten, Math.-phys. Klasse, – 1918. – P. 235–257 (In German).

24. A. Polyakov. Nonlinear feedback design for fixed-time stabilization of linear control systems, IEEE Transactions on Automatic Control, – 2012. – **57(8).** – P. 2106–2110. DOI: 10.1109/TAC.2011.2179869.

25. A. Polyakov. Time-suboptimal feedback design via linear matrix inequalities, Automation and Remote Control, – 2015. – **76.** – P. 847–862. DOI: 10.1134/S0005117915050100.

26. A. Polyakov. Sliding mode control design using canonical homogeneous norm, International Journal of Robust and Nonlinear Control, – 2018. – **29(3)** – P. 682–701. DOI: 10.1002/rnc.4058.

27. A. Polyakov. Generalized Homogeneity in Systems and Control. 2020. Springer, XVIII + 447 p. DOI: 10.1007/978-3-030-38449-4.

28. A. Polyakov, D. Efimov, and W. Perruquetti. Robust stabilization of MIMO systems in finite/fixed time, International Journal of Robust and Nonlinear Control, – 2016. – **26(1)** – P. 69–90. DOI: 10.1002/rnc.3297.

29. A. Polyakov, Y. Orlov, H. Oza, and S. Spurgeon. Robust finite-time stabilization and observation of a planar system revisited, In Conference on Decision and Control, – 2015. – P. 5689-5694. DOI: 10.1109/CDC.2015.7403112.

30. R. T. Rockafellar. Convex Analysis. 1970. Princeton University Press, 470 p.

31. L. Rosier. Homogeneous Lyapunov function for homogeneous continuous vector field, Systems & Control Letters, – 1992. – **19(6).** – P. 467–473. DOI: 10.1016/0167-6911(92)90078-7.

32. E. Roxin. On finite stability in control systems, Rendiconti del Circolo Matematico di Palermo, – 1966. – **15.** – P. 273–283.

33. E. P. Ryan. Universal stabilization of a class of nonlinear systems with homogeneous vector fields, Systems & Control Letters, – 1995. – **26(3).** – P. 177–184. DOI: 10.1016/0167-6911(95)00013-Y.

34. E. D. Sontag. Smooth stabilization implies coprime factorization, IEEE Transactions on Automatic Control, – 1989. – **34(4).** – P. 435–443. DOI: 10.1109/9.28018.

35. E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability property, Systems & Control Letters, – 1996. – **24(5).** – P. 351–359. DOI: 10.1016/0167-6911(94)00050-6.

36. V. I. Zubov. On systems of ordinary differential equations with generalized homogenous right-hand sides, Izvestia vuzov. Mathematica, – 1958. – **1.** – P. 80–88 (in Russian).

Про однорідні функції керованості

Поляков. А.
Інрія Лілль-Норд Європа, пр-т. Алле, 40, Вільньов д'Аск, Франція, 5965.

Метод функції керованості, введений В. І. Коробовим наприкінці 1970-х років, як відомо, є ефективним інструментом для проектування систем керування. Він розроблений як для лінійних/нелінійних, так і для скінченно/нескінченновимірних систем. Ця стаття поєднує цей метод із теорією

однорідності, що корінням сягає початку 18 століття та являє собою симетрією функції щодо рівномірного масштабування її аргументу. Узагальнення такого ефективного підходу були введені в 20 столітті. У цій роботі показано, що так звана однорідна норма є функцією керованості лінійної автономної системи керування, а відповідна замкнута система є однорідною в узагальненому сенсі. Це відразу дає багато корисних властивостей, відомих для однорідних систем, таких як робастість (стабільність вхідних даних) щодо досить великого класу збурень, зокрема щодо обмежених адитивних шумів вимірювання та обмежених адитивних екзогенних збурень. Основна теорема, представлена в цій роботі, дещо уточнює побудову функції керованості для лінійних автономних систем керування з кількома входами. Процедура полягає в розв'язанні лінійного алгебраїчного рівняння та систему лінійних матричних нерівностей. Сама однорідність і використання канонічної однорідної норми істотно спрощують знаходження функції керованості та аналіз замкнутої системи. Теоретичні результати підкріплені прикладами. Перспективним напрямком для майбутніх досліджень є подальше вивчення побудови функцій керованості на основі однорідності.
Ключові слова: функція керованості; узагальнена однорідність; робастність.

On homogeneous controllability functions

A. Polyakov
Inria Lille-Nord Europe, 40. av. Halley, Villeneuve d'Ascq, 5965, France

The controllability function method, introduced by V. I. Korobov in late 1970s, is known to be an efficient tool for control systems design. It is developed for both linear/nonlinear and finite/infinite dimensional systems. This paper bridges the method with the homogeneity theory popular today. The standard homogeneity known since 18th century is a symmetry of function with respect to uniform scaling of its argument. Some generalizations of the standard homogeneity were introduced in 20th century. This paper shows that the so-called homogeneous norm is a controllability function of the linear autonomous control system and the corresponding closed-loop system is homogeneous in the generalized sense. This immediately yields many useful properties known for homogeneous systems such as robustness (Input-to-State Stability) with respect to a rather large class of perturbations, in particular, with respect to bounded additive measurement noises and bounded additive exogenous disturbances. The main theorem presented in this paper slightly refines the design of the controllability function for a multiply-input linear autonomous control systems. The design procedure consists in solving subsequently a linear algebraic equation and a system of linear matrix inequalities. The homogeneity itself and the use of the canonical homogeneous norm essentially simplify the design of a controllability function and the analysis of the closed-loop system. Theoretical results are supported with examples. The further study of homogeneity-based design of controllability functions seems to be a promising direction for future research.
Keywords: controllability function; generalized homogeneity; robustness.

Article history: Received: 28 July 2021; Final form: 11 November 2021;
Accepted: 15 November 2021.