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On exact controllability and complete stabilizability for linear systems

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This paper concerns the relation between exact controllability and stabilizability with arbitrary decay rate in infinite dimensional spaces. It appears that in several cases the notions are equivalent, but there are a lot of situations when additional conditions are needed, for example in Banach spaces. This is a short and non exhaustive review of some research on control theory for infinite dimensional spaces initiated by V. I. Korobov during the 70th of the past century in Kharkov State University.
Keywords: Exact controllability; complete stabilizability; infinite dimensional systems; neutral type.

Рабах Рабах. **Про точну керованість та повну стабілізацію для лінійних систем.** У цій статті йдеться про зв'язок між точною керованістю та стабілізацією з довільною швидкістю спаду в нескінченномірних просторах. У кількох випадках поняття є еквівалентними, але є багато ситуацій, коли потрібні додаткові умови, наприклад у просторах Банаха. Це короткий і невичерпний огляд деяких досліджень теорії керування для нескінченновимірних просторів, ініційованих В. І. Коробовим в 70-х роках минулого сторіччя у Харківському державному університеті.
Ключові слова: Точна керованість; повна стабілізація; нескінченновимірні системи; нейтральний тип.

Рабах Рабах. **О точной управляемости и полной стабилизируемости линейных систем.** В статье рассматривается связь между точной управляемостью и стабилизируемостью с произвольной скоростью убывания в бесконечномерных пространствах. В некоторых случаях понятия эквивалентны, но существует множество ситуаций, когда требуются дополнительные условия, например, в банаховых пространствах. Это краткий и неполный обзор некоторых исследований по теории управления для бесконечномерных пространств, начатых В. И. Коробовым в 70-е годы прошлого века в Харьковском государственном университете.
Ключевые слова: Точная управляемость; полная стабилизируемость; бесконечномерные системы; нейтральный тип.

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1. Introduction

We consider linear systems in the general form

$$\dot{x} = \mathcal{A}x + \mathcal{B}u, \tag{1}$$

where the state $x(t)$ and the control $u(t)$ take values in Banach spaces X and U. The linear operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $S(t) = \mathrm{e}^{\mathcal{A}t}$ and $\mathcal{B}$ is a linear bounded operator. We consider also the cases of Hilbert spaces and some situation when $\mathcal{B}$ is an unbounded but admissible operator. The function $u(t)$ is supposed, at least, Bochner integrable.

By exact (null) controllability we mean controllability from any state to any state (or zero state). By complete stabilizability we mean exponential stabilizability with arbitrary decay rate by linear state feedback $u = \mathcal{F}x$. Sometimes the problem is called pole or spectrum assignment.

The well known relation between complete controllability and pole assignment in finite dimensional spaces X and U may be formulated in the following form.

Theorem 1. *Let A and B be (real or complex) matrices of dimension $n \times n$ and $n \times m$ respectively, x and u vector functions of suitable dimension. The system*

$$\dot{x} = Ax + Bu \tag{2}$$

is completely controllable, i.e. $\operatorname{rank}[B \;\; AB \cdots A^{n-1}B] = n$, *if and only if the system is completely stabilizable, i.e. for all $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, there exists a feedback $u = Fx$ such that the spectrum of the closed loop system verifies:*

$$\sigma(A + BF) = \{\lambda_1, \ldots, \lambda_n\},$$

i.e. the problem of pole assignment by linear feedback is solvable.

A nice proof in the general case can be found in [33]. If the system is assumed to be real (matrices are all real), the spectrum is formed by real or complex conjugate numbers.

Another formulation of the same statement is that all the poles are controllable (Hautus criteria):

$$\forall \lambda \in \sigma(A), \quad \operatorname{rank}[\lambda I - A \quad B] = n.$$

Our purpose is to discuss some important results on the same problems in infinite dimensional Hilbert or Banach spaces. Since the spectrum is more complicated and the behavior of infinite dimensional systems is not completely determined by the spectrum, solutions of such problems, when they exist, are not so simple.

We recall some classical results concerning the relation between exact controllability and complete stabilizability.

The first important result in this context was given by Slemrod: if $\mathrm{e}^{\mathcal{A}t}$ is a group, then exact controllability implies complete stabilizability. The converse, for a group, was proved by Zabczyk (see for example [34]). The result was generalized and precised by several authors for the case of a bounded operator $\mathcal{A}$ in Banach spaces, for the case of a semigroup $\mathrm{e}^{\mathcal{A}t}$ and for some classes of systems, governed by partial differential equations or functional-differential equations.

We discuss more precisely the relations between exact (null) controllability and complete stabilizability. In general, for linear systems in Hilbert spaces, exact null controllability implies complete stabilizability, but the converse is not true. We give more recent results on functional-differential systems of neutral type.

2. Preliminaries

In this section we give some definitions and well known basic results on exact controllability and stabilizability for System (1). The mild solution (strong for $x_0 \in D(\mathcal{A})$ and some restrictions on the class of controls u) is given by

$$x(t) = x(t, u(\cdot), x_0) = \mathrm{e}^{\mathcal{A}t}x_0 + \int_0^t \mathrm{e}^{\mathcal{A}(t-\tau)}\mathcal{B}u(\tau)\mathrm{d}\tau. \tag{3}$$

Definition 1. *System (1) is exactly controllable if for all $x_0, x_1 \in X$, there exists a time T and a control $u \in L_p(0,T;U), p \geq 1$ (shortly L_p) such that $x(T, u(\cdot), x_0) = x_1$. If the same holds for $x_1 = 0$, we talk about exact null controllability. If $\mathrm{e}^{\mathcal{A}t}$ is a group, or if it is formed by surjective operators, then both notions are equivalent. If necessary, the class of controls L_p will be specified.*

The notion of stabilizability is based on one concept of asymptotic stability. In infinite dimensional spaces there are several ones. In this paper we consider only exponential stability.

Definition 2. *The semigroup $\mathrm{e}^{\mathcal{A}t}$, or the uncontrollable system, is said to be exponentially stable if for some real $M \geq 1$, $\omega > 0$, and for all $t \geq 0$, we have $\|\mathrm{e}^{\mathcal{A}t}\| \leq M\mathrm{e}^{-\omega t}$.*

In finite dimensional space exponential stability is characterized by the fact that the spectrum of matrix A is in the left half complex plane. The situation in infinite dimensional spaces is much more complicated, see for example [32] and references therein.

Definition 3. *System (1) is said exponentially stabilizable if there exists a linear feedback control $u(t) = \mathcal{F}x(t)$ such that the closed loop system becomes exponentially stable, i.e. for some $\omega > 0$*

$$\|\mathrm{e}^{(\mathcal{A}+\mathcal{B}\mathcal{F})t}\| \leq M_\omega \mathrm{e}^{-\omega t}, \quad M_\omega \geq 1.$$

The system is said to be completely stabilizable if it is exponentially stabilizable for all $\omega > 0$.

In some cases the feedback F is allowed to be not bounded but admissible in some sense.

Complete stabilizability replaces the notion of spectrum assignment which is difficult to be expressed in infinite dimensional spaces. The first reason is that the structure of the spectrum may be complicated and the second one is that, what is noted before, the behavior of the solutions of the systems is not determined by the spectrum only. However, in some cases it is possible to consider this concept also, as we will see later.

Let us now give some general characterizations of the given concepts. We denote by $\mathcal{R}_T$ the "reachability"operator

$$\mathcal{R}_T u(\cdot) = \int_0^T \mathrm{e}^{\mathcal{A}(T-\tau)}\mathcal{B}u(\tau)\mathrm{d}\tau.$$

It is easy to see that this is a linear bounded operator from L_p to X. System (1) is exactly controllable if and only if $\operatorname{Im}\mathcal{R}_T = X$, it is exactly null controllable if and only if $\operatorname{Im}\mathcal{R}_T \subset \operatorname{Im}\mathrm{e}^{\mathcal{A}T}$, for some $T > 0$.

The surjectivity of the operator $\mathcal{R}_T$ in Banach spaces can be characterized by the following property of the adjoint operator $\mathcal{R}_T^*$ (see, for instance, [27])

$$\exists\delta > 0: \quad \forall x^* \in X^*, \quad \|\mathcal{R}_T^* x^*\| \geq \delta\|x^*\|, \tag{4}$$

where norms are operator or vector norms in the adjoint space. To obtain a more or less explicit condition of exact controllability we need the expression of adjoint operator and the norm $\|\mathcal{R}_T^* x^*\|$. We have the following expressions [19]:

$$\begin{cases} \|\mathcal{R}_T^* x^*\| = \left(\int_0^T \|\mathcal{B}^* S^*(t)x^*\|^q \mathrm{d}t\right)^{\frac{1}{q}}, & \text{if } u \in L_p,\ p > 1, \\ \|\mathcal{R}_T^* x^*\| = \operatorname{ess\,sup}\{\|\mathcal{B}^* S^*(t)x^*\|,\ t \in [0,T]\}, & \text{if } u \in L_1, \end{cases} \tag{5}$$

where ess sup denotes the essential supremum of a set. In fact, it is possible to show that the conditions of exact controllability in the class of controls L_p for $p > 1$ are equivalent and exact controllability in the class L_1 is essentially different. This can be summarized by the following formulation.

Theorem 2 ([19])**.** *System* (1) *is exactly controllable in the class* L_p *if and only if*

1. *For* $p = 1$*, if* $\lim\limits_{n\to\infty} \mathcal{B}^* S^*(t)x_n^* = 0$ **uniformly** *almost everywhere on* $[0,T]$ *implies* $\lim\limits_{n\to\infty} x_n^* = 0$.
2. *For* $p > 1$*, if* $\lim\limits_{n\to\infty} \mathcal{B}^* S^*(t)x_n^* = 0$ *almost everywhere on* $[0,T]$ *implies* $\lim\limits_{n\to\infty} x_n^* = 0$.

This implies a necessary condition of exact controllability in the class L_p for $p > 1$: operators $S(t) = \mathrm{e}^{\mathcal{A}t}$ are surjective for all $t \geq 0$.

This gives, in particular, the lack of exact controllability in the class L_p for $p > 1$ for compact or analytic semigroups (see [30, 31, 11]). The following example shows that there exist systems which are exactly controllable in the class L_1, but not exactly controllable in classes $L_p,\ p > 1$.

Example 1 ([19])**.** Let $X = U = \ell_2$, $\{e_i, i = 1, 2, \dots\}$ be the canonical basis in ℓ_2. Let $\mathcal{B}$ be defined as the identity operator $\mathcal{B} = \mathcal{I}$, and

$$\mathcal{A}x = \sum_{i=1}^{\infty} -i\langle x, e_i\rangle e_i, \quad x = \sum_{i=1}^{\infty}\langle x, e_i\rangle e_i,\ \sum_{i=1}^{\infty} i^2|\langle x, e_i\rangle|^2 < \infty.$$

It is easy to see that $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup

$$S(t)x = \sum_{i=1}^{\infty} \mathrm{e}^{-it}\langle x, e_i\rangle e_i, \quad t \geq 0$$

of compact operators for $t > 0$. The system is exactly controllable in the class L_1, because $\|R_T^* x^*\|_{L_\infty} = \|x^*\|$, but not in the class $L_p,\ p > 1$, the operators $S(t)$ being not surjective for $t > 0$. □

Let us now consider the problem of null exact controllability. The characterization of this concept is not so clear in general Banach spaces. It is more simple when X and U are Hilbert spaces. This is due to the following classical result.

Theorem 3 (Douglas (1966), [4])**.** *Let C and D be linear bounded operators in Hilbert spaces: $C \in \mathscr{L}(H_1, H_2), D \in \mathscr{L}(H_3, H_2)$. The following conditions are equivalent:*

1. $\operatorname{Im} C \subset \operatorname{Im} D$.
2. *There is a linear bounded operator E such that $C = DE$.*
3. $\exists \mu \geq 0 :\ \forall x \in H_2$ *we have* $\|C^* x\|^2 \leq \mu^2\|D^* x\|^2$.

This means that in Hilbert spaces the condition of exact null controllability $\operatorname{Im} \mathcal{R}_T \subset \mathrm{e}^{\mathcal{A}T}$ may be characterized as follows.

Theorem 4. *Let us suppose that X and U are Hilbert spaces. Then the system (1) is exactly null controllable in the class L_2 if and only if*

$$\exists \delta > 0 :\ \forall x \in X,\ \int_0^T \|\mathcal{B}^* S^*(t)x\|^2 \mathrm{d}t \geq \delta^2\|S^*(T)x\|^2.$$

As it was pointed out by several authors (including Douglas, see for example [6]), Theorem 3 is not true in this form in general Banach spaces. Some other formulations are possible (see, for instance, [6, 1]). We cannot develop this question here, but we will return to the application of Douglas theorem in Section 4.1.

3. Exact controllability implies complete stabilizability

In the section we will see how exact controllability can imply complete stabilizability. The main framework here is the Hilbert spaces setting.

Consider System (1) in the case of Hilbert spaces X and U. Under the condition that the operator $\mathcal{A}$ is the infinitesimal generator of a group $\mathrm{e}^{\mathcal{A}t}$, $\mathcal{B} \in \mathscr{L}(U, X)$, Slemrod [29] obtained the first result in this way:

The condition

$$\exists \delta > 0,\ \forall x \in X: \quad \int_0^T \|\mathcal{B}^* \mathrm{e}^{-\mathcal{A}^* \tau} x\|^2 \, \mathrm{d}\tau \geq \delta \|x\|^2 \tag{6}$$

implies complete stabilizability

$$\forall \omega > 0, \exists \mathcal{F} \in \mathscr{L}(X, U), \quad \|\mathrm{e}^{(\mathcal{A}+\mathcal{B}\mathcal{F})t}\| \leq M_\omega \mathrm{e}^{-\omega t}, \quad M_\omega \geq 1,$$

with a linear bounded feedback $\mathcal{F}$ given by $\mathcal{F} = -\mathcal{B}^* K_\omega^{-1}(T)$, where

$$K_\omega(T) = \int_0^T \mathrm{e}^{-2\omega t} \mathrm{e}^{-\mathcal{A}\tau} \mathcal{B}\mathcal{B}^* \mathrm{e}^{-\mathcal{A}^* \tau} \, \mathrm{d}\tau.$$

This operator $K_\omega(T)$ is, in fact, the Gramian of the system

$$\dot{x} = \mathcal{A}x + \omega x + \mathcal{B}u,$$

and the group $S_\omega(t) = \mathrm{e}^{\omega t} \mathrm{e}^{\mathcal{A}t}$ has the infinitesimal generator $(\mathcal{A} + \omega I)$.

The inequality in (6) is the condition of exact controllability at time T in the case of a group, and in Hilbert spaces. It corresponds to the Kalman integral criterion:

$$K_0(T) = \int_0^T \mathrm{e}^{-\mathcal{A}\tau} \mathcal{B}\mathcal{B}^* \mathrm{e}^{-\mathcal{A}^* \tau} \, \mathrm{d}\tau$$

is bounded invertible. The fact that this is a necessary and sufficient condition of exact controllability in Hilbert spaces was published first (to the best of our knowledge) in [14] for the case of a bounded operator $\mathcal{A}$, but the proof, based on Theorem 3 (see [4]), can easily be rewritten in the case of a group $\mathrm{e}^{\mathcal{A}t}$ with unbounded operator $\mathcal{A}$, or directly from (4).

We can formulate this result as follows.

Theorem 5. *For System (1), where $\mathcal{A}$ is the infinitesimal generator of a group, U and X being Hilbert spaces, exact controllability implies complete stabilizability.*

Доведення. This a short proof, see [29] for more details. Let $\omega > 0$ be given. Let us take the feedback control

$$u(t) = \mathcal{F}x(t) = -\mathcal{B}^* K_\omega^{-1}(T)x(t).$$

We obtain then the closed loop system

$$\dot{x}(t) = (\mathcal{A} + \omega\mathcal{I} - \mathcal{B}\mathcal{B}^* K_\omega^{-1}(T))x(t),$$

where $\mathcal{I}$ is the identity operator, and the solution is given by the group $S^*_{\omega\mathcal{F}}(t)$ generated by the operator

$$\mathcal{A} + \omega\mathcal{I} - \mathcal{B}\mathcal{B}^* K_\omega^{-1}(T).$$

Let $x_0 \in \mathcal{D}(\mathcal{A}^*)$, then a simple calculation gives (see Appendix in [29]):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle S^*_{\omega\mathcal{F}}(t)x_0, K_\omega(T)^{-1}S^*_{\omega\mathcal{F}}(t)x_0\right\rangle = \\ -\|\mathrm{e}^{-\omega T}\mathcal{B}^*\mathrm{e}^{-\mathcal{A}^*T}S^*_{\omega\mathcal{F}}(t)x_0\| - \|\mathcal{B}^*S^*_{\omega\mathcal{F}}(t)x_0\|.$$

By density of $D(\mathcal{A}^*)$, this implies the uniform boundedness of $S_{\omega\mathcal{F}}(t)$

$$\exists M \geq 1, \quad \|S_{\omega\mathcal{F}}(t)\| \leq M, \quad t \geq 0.$$

This means that for all $\omega > 0$, there exists a feedback law $\mathcal{F}$, such that

$$\|S_\mathcal{F}(t)\| = \|\mathrm{e}^{(\mathcal{A}+\mathcal{B}\mathcal{F})t}\| \leq M\mathrm{e}^{-\omega t}, \quad t \geq 0.$$

This means that the system is completely stabilizable. □

Following Slemrod and using some results by Lions, Komornik [9] proved an analogous result for the case when $\mathcal{B}$ is unbounded. This was applied to systems governed by PDE with boundary control.

Shortly, with some conditions on the operator $\mathcal{B}$, complete stabilizability is obtained from exact controllability and the feedback is

$$\mathcal{F} = -J\mathcal{B}^*K_\omega^{-1}, \qquad K_\omega = \int_0^T \mathrm{e}^{-2\omega t}\mathrm{e}^{-\mathcal{A}\tau}\mathcal{B}J\mathcal{B}^*\mathrm{e}^{-\mathcal{A}^*\tau}\,\mathrm{d}\tau,$$

where J is the canonical Riesz isomorphism.

Example 2. This allows to obtain the complete stabilization of systems like

$$\begin{cases} \ddot{w}(t,\xi) - \nabla w(t,\xi) = 0, & (t,\xi) \in \mathbb{R}^+ \times \Omega \\ w(0,\xi) = f(\xi), & \xi \in \Omega \\ \dot{w}(0,\xi) = g(\xi), & \xi \in \Omega \\ w(t,\xi) = u(t,\xi) & \xi \in \partial\Omega. \end{cases}$$

The boundary control is given by the formula

$$u(t,\xi) = \mathcal{F}w(t,\xi) = -J\mathcal{B}^* K_\omega^{-1}(w(t,\xi), \dot{w}(t,\xi))$$

defined in the space $H_{-1}(\Omega) \times L_2(\Omega)$:

$$u(t,\xi) = \partial_\nu(Pw(t,\xi) + Q\dot{w}(t,\xi)),$$

where P and Q are linear and bounded. □

In the general case of a semigroup, we have the following implication:

Theorem 6. *For System (1) in Hilbert spaces, with bounded operator $\mathcal{B}$, exact controllability to zero implies complete stabilizability by bounded feedback $\mathcal{F}$.*

This is a consequence of the following result implicitly contained in the paper by Datko [3] (see also [34, Ch. III. 4]).

Theorem 7. *Exactly null controllable systems are exponentially stabilizable.*

The latter and the following remark give Theorem 6.

Remark 1. *Controllability of System (1) is equivalent to controllability of the shifted system:*

$$\dot{x} = \mathcal{A}x + \omega x + \mathcal{B}u.$$

This result may be extended to the case of an unbounded control operator.

We consider the same system (1) in Hilbert spaces, with possible unbounded $\mathcal{B}$ and $\mathcal{F}$, they are supposed to be admissible in a certain sense:

$\mathcal{B}$ bounded from U to the space X_{-1} which is the completion of X with the norm $\|(\lambda I - A)^{-1}x\|$.

Theorem 8 ([24])**.** *If system (1), in Hilbert spaces, with admissible operator $\mathcal{B}$ is exactly null controllable in X_{-1}, then it is completely stabilizable by an admissible feedback and then, finally, by a bounded feedback $\mathcal{F}$.*

The converse is not true, and there are different situations [24].

Example 3. In the space $L_2(0,+\infty)$, consider the semigroup

$$S(t)f(\xi) = \mathrm{e}^{-\frac{t^2}{2}-\xi t} f(\xi+t), \quad t \geq 0, \quad \xi \geq 0.$$

It is not difficult to see that for this semigroup:

$$\forall \omega > 0, \quad \exists M_\omega \geq 1, \quad \|S(t)\| \leq M_\omega \mathrm{e}^{-\omega t}.$$

We have $\sigma(S(t)) = \{0\}$ and then $\sigma(\mathcal{A}) = \emptyset$.

It is easy to see that, for some $f \neq 0$ we have $S(t)f \neq 0$ for any $t \geq 0$. The system is completely stabilizable ($\mathcal{B} = 0$) but not controllable. □

Example 4. In the space $L_2(0,1)$, consider the semigroup

$$S(t)f(\xi) = \begin{cases} f(\xi+t) & 0 \leq t+\xi \leq 1, \\ 0 & t+\xi > 1. \end{cases}$$

It is not difficult to see that for this semigroup

$$\forall \omega > 0, \quad \exists M_\omega \geq 1, \quad \|S(t)\| \leq M_\omega \mathrm{e}^{-\omega t}.$$

We have $\sigma(S(t)) = \{0\}$ and then $\sigma(\mathcal{A}) = \emptyset$.

For any initial function $f \in L_2(0,1)$, we have $S(t)f(x) = 0$ for $t > 2$. This means that $S(t) = 0$, for all $t > 2$. Then, for any control operator $\mathcal{B}$, the system is exactly null controllable at time $T > 2$ with the trivial control $u = 0$. □

4. Complete stabilizability implies exact controllability

In this Section we consider how complete stabilizability may imply exact (null) controllability, and is there an equivalence in some situation as in finite dimensional spaces.

4.1. Banach space, bounded case

Consider System (1) where operators $\mathcal{A}$ and $\mathcal{B}$ are linear, bounded: $\mathcal{A} \in \mathscr{L}(X)$ and $\mathcal{B} \in \mathscr{L}(U,X)$. This means that it is not the case of partial differential or functional-differential equations (with delay). Anyway, it is of historical and theoretical interest.

An important result on complete stabilizability was obtained in [28]. First we formulate result about exact controllability.

Theorem 9 (Korobov-Rabah (1979), [10])**.** *System (1) is exactly controllable if and only if there exists $k \in \mathbb{N}$ such that*

$$\operatorname{Im}[\mathcal{B}\ \mathcal{A}\mathcal{B}\ \cdots\ \mathcal{A}^k\mathcal{B}] = X, \tag{7}$$

where $[\mathcal{B}\ \mathcal{A}\mathcal{B}\ \cdots\ \mathcal{A}^k\mathcal{B}]$ is acting from $U \times \cdots \times U$ to X and the condition is independent on the class of controls.

In fact the exact controllability holds in the class of piece-wise controls, when the condition of the theorem is verified.

Theorem 10 (Sklyar (1982), [28])**.** *System* (1) *with bounded operators is completely stabilizable if and only if the block operator* $[\mathcal{B}\ \mathcal{A}\mathcal{B}\ \cdots\ \mathcal{A}^k\mathcal{B}]$ *is right invertible for some* k*:*

$$\exists \mathcal{P}_i \in \mathscr{L}(X,U),\ i=0,1,\dots,k: \qquad \mathcal{B}\mathcal{P}_0 + \mathcal{A}\mathcal{B}\mathcal{P}_1 + \cdots + \mathcal{A}^k\mathcal{B}\mathcal{P}_k = I. \tag{8}$$

There is a spectral formulation, which is a consequence of the following result.

Theorem 11 (Kaashoek and al. (1983), [7])**.** *The operator* $[\mathcal{B}\ \mathcal{A}\mathcal{B}\ \cdots\ \mathcal{A}^k\mathcal{B}]$ *is right invertible for some* k *if and only if operators* $[\lambda\mathcal{I}-\mathcal{A}\quad \mathcal{B}]$*, from* $X\times U$ *to* X *are right invertible for all* $\lambda\in\mathbb{C}$*, i.e.*

$$\forall\lambda\in\mathbb{C},\ \exists \mathcal{Q}_\lambda\in\mathscr{L}(X),\ \exists P_\lambda\in\mathscr{L}(X,U): \quad (\lambda\mathcal{I}-\mathcal{A})\mathcal{Q}_\lambda + \mathcal{B}\mathcal{P}_\lambda = \mathcal{I}. \tag{9}$$

This gives a spectral formulation of the characterization of complete stabilization.

Theorem 12. *System* (1) *is completely stabilizable if and only if operators* $[\lambda\mathcal{I}-\mathcal{A}\quad \mathcal{B}]$ *are right invertible for all* $\lambda\in\mathbb{C}$*.*

If U, X are Banach spaces, not isomorphic to Hilbert spaces, the condition (8) (or the spectral formulation (9)) is more than exact controllability, it is equivalent to the following two conditions:

1. $\operatorname{Im}[\mathcal{B}\ \mathcal{A}\mathcal{B}\ \cdots\ \mathcal{A}^k\mathcal{B}] = X$, i.e. exact controllability by Theorem 9.

2. $\operatorname{Ker}[\mathcal{B}\ \mathcal{A}\mathcal{B}\ \cdots\ \mathcal{A}^k\mathcal{B}]$ is complemented.

This means that in Banach spaces (not isomorphic to Hilbert spaces) there are exactly controllable systems, generated by a group, which are not completely stabilizable.

Example 5 (Sklyar (1982))**.** Consider the Banach space U which has at least a subspace L which cannot be complemented in U. Let X be the factor space U/L and $\mathcal{B}$ be the canonical projection of U onto $X = U/L$. The operator $\mathcal{B}$ is then onto (surjective). This means that the system

$$\dot{x} = x + \mathcal{B}u,$$

is exactly controllable because $\operatorname{Im}\mathcal{B} = X$ (here $\mathcal{A} = \mathcal{I}$). If we suppose that it is completely stabilizable, then, according to Theorem 8, there exists $\mathcal{P}\in\mathscr{L}(X,U)$ such that $\mathcal{F}\mathcal{P} = \mathcal{I}$, and $\operatorname{Ker}\mathcal{P} = L$ can be complemented. This is in contradiction with the construction of L. □

The result of Theorem 10 was proved in [13] without citing the original result [28].

The situation is different in Hilbert spaces: every closed subspace can be complemented and the equivalence between range condition (7) and right invertibility (8) can be given by Douglas theorem.

Remark 2. *When this paper was under review, appeared a conference paper [12] on exact controllability and complete stabilizability (in fact arbitrary assignability of the Bohl exponent) of a class of linear non-stationary discrete systems in Banach spaces. The main result concerns a case when the state operator is constant, the control operator is periodic, the space X is a reflexive Banach space and U is a separable Hilbert space: it is claimed that exact controllability implies arbitrary assignability of Bohl exponents. No proofs are given in this paper.*

4.2. Hilbert spaces, bounded case, spectrum assignment

In Hilbert spaces, exact controllability is equivalent to complete stabilizability, but more precisely we have a pole assignment result which may be interesting, at least in operator theory.

Theorem 13 (Eckstein (1981) [5])**.** *The system (1) is exactly controllable if and only if for every nonempty compact set $\Lambda \subset \mathbb{C}$, there is a bounded linear feedback $\mathcal{F}$, such that*

$$\sigma(\mathcal{A} + \mathcal{B}\mathcal{F}) = \Lambda,$$

i.e. the problem of spectrum assignment is solvable.

The proof uses explicitly the Hilbert space structure, in particular the fact that the subspace $\operatorname{Ker}[\mathcal{B}\ \mathcal{A}\mathcal{B}\ \cdots\ \mathcal{A}^k\mathcal{B}]$ can always be complemented. There is also a dual formulation of this result in [7] and a constructive approach of the dual operator corresponding to $\mathcal{F}$: let $\mathcal{C}$ be an operator from X to the Hilbert space Y, find $\mathcal{K}$ such that $\sigma(\mathcal{A} + \mathcal{K}\mathcal{C}) = \Lambda$. This appears in the problem of detectability and construction of asymptotic observers.

To summarize the characterization in Hilbert spaces, when the operators are bounded, we can formulate as follows.

Theorem 14. *Let U and X be Hilbert spaces, $\mathcal{A}$ and $\mathcal{B}$ be linear bounded operators. The following statements are equivalent*

1. *The system* (1) *is exactly controllable in the class L_2.*

2. *For every nonempty compact set $\Lambda \subset \mathbb{C}$, there is a bounded linear feedback $\mathcal{F}$, such that*

   $$\sigma(\mathcal{A} + \mathcal{B}\mathcal{F}) = \Lambda,$$

   i.e. the problem of spectrum assignment is solvable.

3. $\operatorname{Im}[\mathcal{B}\ \mathcal{A}\mathcal{B}\ \cdots\ \mathcal{A}^k\mathcal{B}] = X$ *for some $k \in \mathbb{N}$,*

4. $\operatorname{Im}[\lambda\mathcal{I}-\mathcal{A} \quad \mathcal{B}] = X$ *for all* $\lambda \in \mathbb{C}$.

This is a complete analog of Theorem 1.

4.3. Unbounded case in Hilbert spaces

We return to the system (1) with unbounded $\mathcal{A}$ and bounded $\mathcal{B}$.

Theorem 15 (Megan (1975), Zabczyk (1976), see [34])**.** *If* $\mathrm{e}^{\mathcal{A}t}$ *is a group and the system (1) in Hilbert spaces is completely stabilizable, then it is exactly controllable.*

This result was extended to the case of surjective operators $\mathrm{e}^{\mathcal{A}t}$, $t \geq 0$ by Rabah & Karrakchou (1997) and Zeng & Xie & Guo (2013), see [23, 35].

We can summarize as:

Theorem 16. *The system (1) in Hilbert spaces, with a bounded operator* $\mathcal{B}$, *is exactly controllable in the class* L_2 *if and only if*

1. *Operators* $\mathrm{e}^{\mathcal{A}t}$ *are surjective for* $t \geq 0$.
2. *The system is completely stabilizable.*

Exact controllability is equivalent to exact null controllability when $\mathrm{e}^{\mathcal{A}t}$ *are surjective for* $t \geq 0$.

But what about exact controllability in the class L_1 in this context? Surjectivity of operators $\mathrm{e}^{\mathcal{A}t}$ is not needed. Let us return to Example 1.

Example 6 (Example 1 continued)**.** As we saw, the system is exactly controllable in the class L_1 but not in the class L_p for $p > 1$. Let us show that it is completely stabilizable by a bounded feedback. Let us take arbitrary $\omega > 0$ and $n \in \mathbb{N}$ such that $n > \omega$. Define operator $\mathcal{F}$ from $X = \ell_2$ to $U = \ell_2$ by

$$\mathcal{F}x = \sum_{i=1}^{n-1}(i-n)\langle x, e_i\rangle e_i.$$

Then

$$(\mathcal{A}+\mathcal{B}\mathcal{F})x = \sum_{i=1}^{n-1} -n\langle x, e_i\rangle e_i + \sum_{i=n}^{\infty} -i\langle x, e_i\rangle e_i,$$

which gives for the semigroup $S_{\mathcal{F}}(t) = \mathrm{e}^{(\mathcal{A}+\mathcal{B}\mathcal{F})t}$

$$\|\mathrm{e}^{(\mathcal{A}+\mathcal{B}\mathcal{F})}x\|^2 = \sum_{i=1}^{n-1}\mathrm{e}^{-2nt}|\langle x, e_i\rangle|^2 + \sum_{i=n}^{\infty}\mathrm{e}^{-2it}|\langle x, e_i\rangle|^2 \leq \mathrm{e}^{-2nt}\|x\|^2.$$

And this means that

$$\|\mathrm{e}^{(\mathcal{A}+\mathcal{B}\mathcal{F})}\| \leq \mathrm{e}^{-nt} \leq \mathrm{e}^{-\omega t},$$

i.e. the system is completely stabilizable. □

5. The case of neutral type systems

We consider some large class of neutral type systems

$$\dot{z}(t) = A_{-1}\dot{z}(t-1) + \mathcal{L}z_t(\cdot) + Bu(t), \tag{10}$$

where

$$\mathcal{L}z_t(\cdot) = \int_{-1}^{0} [A_2(\theta)\dot{z}(t+\theta) + A_3(\theta)z(t+\theta)] \ \mathrm{d}\theta,$$

with $z(t)$ taking values in $\mathbb{R}^n$ and $u(t)$ in $\mathbb{R}^p$. The system can be written as

$$\dot{x} = \mathcal{A}x + \mathcal{B}u, \tag{11}$$

where $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $\mathrm{e}^{\mathcal{A}t}$ given in the product space $M_2(-1,0;\mathbb{R}^n) \stackrel{\mathrm{def}}{=} \mathbb{R}^n \times L_2(-1,0;\mathbb{R}^n)$, noted shortly M_2, and $\mathcal{A}, \mathcal{B}$ are defined by

$$\mathcal{A}x(t) = \begin{pmatrix} \mathcal{L}z_t(\cdot) \\ \dfrac{\mathrm{d}z_t(\theta)}{\mathrm{d}\theta} \end{pmatrix}, \quad x(t) = \begin{pmatrix} z(t) \\ z_t(\cdot) \end{pmatrix}, \quad \mathcal{B}u(t) = \begin{pmatrix} u(t) \\ 0 \end{pmatrix},$$

with the domain $D(\mathcal{A})$ given by

$$D(\mathcal{A}) = \left\{(v,\varphi) \in M_2 : \varphi(\cdot) \in H^1, v = \varphi(0) - A_{-1}\varphi(-1)\right\}.$$

The characteristic matrix is noted by

$$\Delta_{\mathcal{A}}(\lambda) = \lambda I - A_{-1}\mathrm{e}^{-\lambda} - \lambda \int_{-1}^{0} A_2(\theta)\mathrm{e}^{s\theta}\mathrm{d}\theta - \int_{-1}^{0} A_3(\theta)\mathrm{e}^{s\theta}\ \mathrm{d}\theta.$$

The behavior of the dynamical system is described by the spectrum:

$$\sigma(\mathcal{A}) = \{\lambda : \det \Delta_{\mathcal{A}}(\lambda) = 0\}.$$

The set $\sigma(\mathcal{A})$ is formed by eigenvalues of finite multiplicity, see for example [21] and references therein.

5.1. Complete stabilizability

It is characterized by two conditions: one on the spectrum $\sigma(\mathcal{A})$, the second on the neutral term (here A_{-1}).

Theorem 17 ([20]). *The system* (10)*, in abstract form* (11)*, is completely stabilizable if and only if*

1. *For all* $\lambda \in \mathbb{C}$, $\quad \operatorname{rank} \begin{bmatrix} \Delta_{\mathcal{A}}(\lambda) & B \end{bmatrix} = n$.
2. *For all* $\mu \in \mathbb{C}$, $\mu \neq 0$, $\quad \operatorname{rank} \begin{bmatrix} \mu I - A_{-1} & B \end{bmatrix} = n$.

The condition 2 may be formulated as $\mathrm{Im}\, A_{-1} \subset \mathrm{Im}\, [B \;\, A_{-1}B \;\, \cdots \;\, \mathcal{A}_{-1}^{n-1}B]$.

Proof. It is important to check the meaning of the second condition. A large part of the spectrum of $\mathcal{A}$ may be obtained from the non zero spectrum of the matrix A_{-1}:

$$\lambda \;=\; \log|\mu| + \mathrm{i}(\arg\mu + 2k\pi) + o(1/|k|), \quad k \in \mathbb{Z}.$$

The eigenvalues are in some circles centered at $\log\mu$ with decreasing radii, when $|k|$ is growing [21]. The pole assignment is possible in the interior of the circles under Condition 1, which means the controllability of poles.

The spectrum of the operator $\mathcal{A}$ is close to some vertical axes defined by the spectrum of the neutral term, the matrix A_{-1}. We need to move these axes by the feedback in order to obtain complete stabilizability, and then the corresponding values μ must be controllable. The necessity of both conditions is then clear enough.

Let us shortly consider the sufficient part. The condition 2 allows to move arbitrary all the non zeros eigenvalues of A_{-1}, and to make it simple. The condition 1 allows to control an eigenvalue in the interior of circles (for k large). It remains then to move a finite number of eigenvalues, it is possible as shown in [22, 26].

It is of interest to show the form of the feedback needed for exponential and then for complete stabilizability:

$$u(t) = \mathcal{F}x(t) = F_{-1}\dot{z}(t-1) + \int_0^T [F_2(\theta)\dot{z}(t+\theta) + F_3(\theta)z(t+\theta)]\; \mathrm{d}\theta.$$

This means that $\mathcal{F}$ is not bounded and domains of $\mathcal{A} + \mathcal{BF}$ and $\mathcal{A}$ are not the same. However this form is necessary in order to move the spectrum [16, 18].

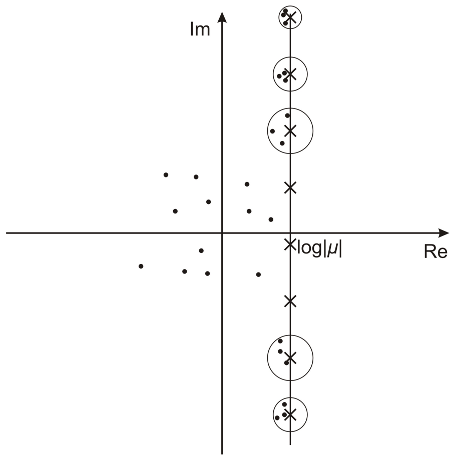

Fig. 1. Spectrum of $\mathcal{A}$, for each $\mu \in \sigma(A_{-1}), \mu \neq 0$.

5.2. Complete stabilizability and exact controllability

The notion of exact controllability for System (10) must be modified. As here the control u is finite dimensional, exact controllability in the sense of Definition 1 is not possible. In fact it may be shown that, in this case, the reachability set $\mathcal{R}_T$ is in $D(\mathcal{A})$ [25, 26]. Then by exact controllability at time T, we mean $\mathcal{R}_T = D(\mathcal{A})$, and exact null controllability means to reach zero from all initial condition in $D(\mathcal{A})$. Results given in [25, 26] may be summarized in the following theorem.

Theorem 18. *If $\mathcal{A}$ is the infinitesimal generator of a group or, equivalently, if the matrix A_{-1} is not singular, $0 \notin \sigma(A_{-1})$, then exact controllability is equivalent to complete stabilizability.*

Proof. If $\mu = 0$ is not in $\sigma(A_{-1})$, conditions 1 and 2 of Theorem 17 give a criteria of exact controllability. We refer to [25, 26] for detailed proofs. In our case of System (10), for exact null controllability we can obtain only a partial result.

Theorem 19. *Exact controllability to zero of System* (10) *implies complete stabilizability.*

Proof. If Condition 1 in Theorem 17 is not verified, it means that some eigenvalues are not controllable. Taking a corresponding eigenvectors as an initial condition, we can see that exact controllability does not hold. To prove Condition 2 in Theorem 17, we can proceed by derivation of the solution $z(t) = 0$ for $t \geq T \geq n$. The calculation gives $\operatorname{Im} A_{-1} \subset \operatorname{Im} [B \ A_{-1}B \ \cdots \ \mathcal{A}_{-1}^{n-1}B]$ (see [24]), which means that Condition 2 in Theorem 17 is verified.

The converse (in fact the criteria of exact null controllability without relation with stabilizability) has been proved in some particular cases of neutral type systems with discrete delays [15, 8] and in the case of retarded systems [17]. In the case of our general system (10), the question is still open, see Conjecture in [24] and [2].

6. Conclusion

We gave a short review of the relations between exact controllability and pole assignment (complete stabilizability). There are a lot of papers concerning different classes of systems, in particular, governed by partial differential equations. This paper gives our point of view to some questions concerning abstract infinite dimensional systems, based on results close to ours, in particular for functional differential equations of neutral type. Several problems are still open concerning the general abstract case or some particular classes.

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Про точну керованість та повну стабілізацію для лінійних систем

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У цій роботі ми розглядаємо лінійні системи з керуванням, описані рівнянням $\dot{x} = \mathcal{A}x + \mathcal{B}u$, де функції u та x приймають значення U та X відповідно. Для такого об'єкта подано короткий огляд результатів, що стосуються зв'язків між точною керованістю та повною стабілізацією (стабілізація з довільною швидкістю спаду). Аналіз проводиться в різних ситуаціях: обмежений чи необмежений стан та оператори керування $\mathcal{A}$ та $\mathcal{B}$, простори Банаха або Гільберта U та X.

Добре відома еквівалентність між повною керованістю та розташуванням полюсів у ситуації скінченновимірних просторів в загальному випадку не має місця в нескінченномірних просторах. Точної керованості недостатньо для повної стабілізації, якщо U і X є банаховими просторами. У постановці Гільбертового простору цей результат є справедливим. Зворотньє твердження також не є простим: в деяких ситуаціях повна стабілізація передбачає точну керованість (у просторі Банаха з обмеженими операторами), в інших ситуаціях це не відповідає дійсності. Відповідні результати наведені з деякими ідеями доведення. Технічні деталі можна знайти в цитованій літературі. Наведено кілька прикладів. Особлива увага приділяється випадку нескінченновимірних систем, збудованих за системами із загаюваннями нейтрального типу із розподіленими загаюваннями. Більш детально досліджується питання про зв'язок між точною нуль-керованістю та повною стабілізацією. Загалом між цими поняттями немає еквівалентності. Однак для деяких класів рівнянь нейтрального типу існує еквівалентність. Питання про те, чи є еквівалентність для більш загальних систем, залишається відкритим. Це короткий і невичерпний огляд деяких досліджень теорії керування у нескінченновимірних просторах. Наші роботи в цій сфері були ініційовані В. І. Коробовим протягом 70-х років минулого сторіччя у Харківському державному університеті.

Ключові слова: Точна керованість; повна стабілізація; нескінченновимірні системи; нейтральний тип.

On exact controllability and complete stabilizability for linear systems

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In this paper we consider linear systems with control described by the equation $\dot{x} = \mathcal{A}x + \mathcal{B}u$ where functions u and x take values in U and X respectively. For such object, a short review of results concerning relations between exact controllability and complete stabilizability (stabilizability with arbitrary decay rate) is given. The analysis is done in various situations: bounded or unbounded state and control operators $\mathcal{A}$ and $\mathcal{B}$, Banach or Hilbert spaces U and X.

The well known equivalence between complete controllability and pole assignment in the situation of finite dimensional spaces is no longer true in general in infinite dimensional spaces. Exact controllability is not sufficient for complete stabilizability if U and X are Banach spaces. In Hilbert space setting this implication holds true. The converse also is not so simple: in some situations, complete stabilizability implies exact controllability (Banach space setting with bounded operators), in other situation, it is not true. The corresponding results are given with some ideas for the proofs. Complete technical development are indicated in the cited literature. Several examples are given. Special attention is paid to the case of infinite dimensional systems generated by delay systems of neutral type in some general form (distributed delays). The question of the relation between exact null controllability and complete stabilizability is more precisely investigated. In general there is no equivalence between the two notions. However for some classes of neutral type equations there is an equivalence. The question how the equivalence occurs for more general systems is still open. This is a short and non exhaustive review of some research on control theory for infinite dimensional spaces. Our works in this area were initiated by V. I. Korobov during the 70th of the past century in Kharkov State University.

Key words: Exact controllability; complete stabilizability; infinite dimensional systems; neutral type.

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