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Korobov's controllability function method applied to finite-time stabilization of the Rössler system via bounded controls

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The problem of stabilizing the Rössler system in finite time by bounded control is considered. We employ V. I. Korobov's controllability function method, which involves a Lyapunov-type function. The controllability function is the solution of an implicit equation. A family of bounded controls which solve the problem is explicitly computed. Besides, the time that it takes the trajectory to reach the desired equilibrium is estimated.
Keywords: Rössler system; Korobov's controllability function; bounded control; finite time stabilization.

Чоке-Ріверо А. Е., Гонзалес Грасіела А., Круз Муллісака Е. **Метод функції керованості Коробова, застосований до стабілізації системи Росслера за обмежений час за допомогою обмежених керувань.** Розглянуто задачу стабілізації системи Росслера за скінченний час за допомогою обмежених керувань. Ми застосовуємо метод функції керованості В. І. Коробова, який використовує функцію типу Ляпунова. Функція керованості є розв'язком неявного рівняння. Запропоновано сім'ю явно обчислюваних обмежених керувань, які розв'язують задачу синтезу. Окрім того, оцінюється час руху, потрібний для досягнення точки рівноваги.
Ключові слова: Система Росслера; функція керованості Коробова; обмежене керування; стабілізація за скінченний час.

Чоке-Риверо А. Э., Гонзалес Грасиела А., Круз Муллисака Э. **Подход функции упраляемости Коробова приложенный к стабилизации системы Росслера за конечное время с помощью ограниченных управлений.** Рассматривается задача стабилизации системы Росслера за конечное время при ограниченном управлении. Используем метод функций управляемости В. И. Коробова, являющийся функцией типа Ляпунова. Функция управляемости является решением неявного уравнения. Предлагается семейство явно вычисляемых ограниченных управлений, которые решают задачу синтеза. Кроме того, оценивается время движения, необходимое для достижения точки покоя.

Ключевые слова: Система Росслера; функция упраляемости Коробова; ограниченное управление; стабилизации за конечное время.

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1. Introduction

Rössler system has become one of the reference chaotic systems. Its novelty when introduced in [25], being that exhibits a chaotic attractor generated by a simpler set of nonlinear differential equations than Lorenz system. It is given by:

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_3, \\ \dot{x}_2 &= x_1 + \alpha x_2, \\ \dot{x}_3 &= \beta + x_3(x_1 - \gamma), \end{aligned} \tag{1}$$

and it develops chaotic behaviour for certain values of the parameter triplet (α, β, γ). The issue of controlling Rössler system by stabilizing one of its unstable equilibrium points has been previously dealt with in the literature. A feedback controller is designed in [12] stabilizing a chosen equilibrium point with exponential convergence and estimating the negative Lyapunov exponent. In [2], a sliding mode control is proposed by which global stabilization of an arbitrary given equilibrium point is achieved, In [23], an optimal control strategy that directs the chaotic motion to any desired equilibrium point is proposed. Both stability and optimality are obtained in [24] by applying linear feedback controllers to the chaotic Rössler system. A suboptimal feedback controller has been tested on the Rössler system in [27]. The synchronization approach and bifurcation diagram have been used in [18] to control the Rössler system. In this work, control of the Rössler system is stated by putting:

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_3, \\ \dot{x}_2 &= x_1 + \alpha x_2, \\ \dot{x}_3 &= \beta + x_3(x_1 - \gamma) + u, \end{aligned} \tag{2}$$

and considering the synthesis problem. Let $x = (x_1, x_2, x_3)^{\intercal}$. The synthesis problem consists in constructing a positional control $u = u(x)$ with $|u(x)| \leq u_1$ such that for any x^0 belonging to a certain neighborhood of the equilibrium point of the system (1), the trajectory $x(t)$ initiated in x^0 arrives at this equilibrium point in finite time. Namely, by using V. I. Korobov's method, also called the controllability function method, a family of bounded positional controls that solve the synthesis problem for the Rössler system is proposed. We mainly use two ingredients. The first one concerns the general theory of the controllability function [14]. The second ingredient is the family of bounded positional controls that was obtained in [7]. Note that the finite-time stabilization of control systems was studied in [8], [5], [6], [19], [20] and references therein. Different from previous works on finite-time stabilization [20], [21], we propose an explicit family of bounded controls constructed by taking into account the only nonlinearity of the Rössler system, which is a quadratic function.

2. Dynamical features of the Rössler system

The equilibrium point of the system (1) for the triplet $(0, \beta, \gamma)$ with $\gamma \neq 0$ is $\bar{x} := \left(0, -\frac{\beta}{\gamma}, \frac{\beta}{\gamma}\right)^{\mathsf{T}}$. For the triplet (α, β, γ) with $\alpha \neq 0$ and $\gamma^2 - 4\alpha\beta = 0$, there is only one equilibrium point: $\bar{x} := \left(\frac{\gamma}{2}, \; -\frac{\gamma}{2\alpha}, \; \frac{\gamma}{2\alpha} \right)^{\mathsf{T}}$. While if $\gamma^2 - 4\alpha\beta > 0$, there are two equilibrium points:

$$\bar{x}_{\pm} := \left(\frac{\gamma \pm \Delta}{2}, \; -\frac{\gamma \pm \Delta}{2\alpha}, \; \frac{\gamma \pm \Delta}{2\alpha} \right)^{\mathsf{T}}, \tag{3}$$

where $\Delta := \sqrt{\gamma^2 - 4\alpha\beta}$. For any other case, system (1) has no equilibrium point. The typical chaotic Rössler system is determined by $\alpha = \beta = \frac{1}{5}$ and $\gamma = \frac{57}{10}$, resulting: $\bar{x}_{-} = \left(\frac{1}{20}\left(57 - \sqrt{3233}\,\right), \; \frac{1}{4}\left(\sqrt{3233} - 57\right), \; \frac{1}{4}\left(57 - \sqrt{3233}\,\right) \right)$ and $\bar{x}_{+} = \left(\frac{1}{20}\left(57 + \sqrt{3233}\right), \frac{1}{4}\left(-57 - \sqrt{3233}\right), \frac{1}{4}\left(57 + \sqrt{3233}\right) \right)$.

The stability exponents of $\bar{x}_{-}$ are $\{-5.686, 0.0970 \pm i0.9951\}$ so it is a saddle-focus with a two-dimensional unstable manifold. Besides, this point is placed in the nearness of the attractor. Instead, the stability exponents of $\bar{x}_{+}$ are $\{0.1929, -4.596 \times 10^{-6} \pm i5.428\}$. Hence, this equilibrium point has a two-dimensional stable manifold but it is outside the region of the chaotic attractor. For details, see [1], [22] or [10], where information about dynamical behavior of this system for other parameter values is provided.

3. Canonical controllable form

Let us consider the case $\alpha \neq 0$. Introducing $y = x - \bar{x}_{\pm}$, system (2) takes the following form:

$$\dot{y} = A_{\pm} y + \begin{pmatrix} 0 \\ 0 \\ y_1 y_3 \end{pmatrix} + bu, \tag{4}$$

being $y := (y_1, y_2, y_3)^{\mathsf{T}}$, $b := (0, 0, 1)^{\mathsf{T}}$ and

$$A_{\pm} := \begin{pmatrix} 0 & -1 & -1 \\ 1 & \alpha & 0 \\ \frac{\gamma \pm \Delta}{2a} & 0 & -\frac{\gamma \mp \Delta}{2} \end{pmatrix} \tag{5}$$

Let us note that the linear part of (4) results a completely controllable system. Then, there exists a coordinate change to transform it into its canonical controllable form [11], [13]. This coordinates change is given by $z = Fy$ with

$$F := \begin{pmatrix} 0 & -1 & 0 \\ -1 & -\alpha & 0 \\ -\alpha & 1 - \alpha^2 & 1 \end{pmatrix}, \tag{6}$$

and system (4) can be rewritten as follows:

$$\dot{z} = A_0 z + b p_{\pm}^{\intercal} z + bu + \begin{pmatrix} 0 \\ 0 \\ (\alpha z_1 - z_2)(z_1 - \alpha z_2 + z_3) \end{pmatrix}, \tag{7}$$

being $z := (z_1, z_2, z_3)^{\intercal}$. Here

$$A_0 := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{8}$$

and

$$p_{\pm} := \begin{pmatrix} \pm\Delta \\ \dfrac{(\gamma \mp \Delta)\alpha^2 - 2\alpha - \gamma \mp \Delta}{2\alpha} \\ \alpha - \dfrac{\gamma}{2} \pm \dfrac{\Delta}{2} \end{pmatrix}. \tag{9}$$

Remark 1.1. *For the case $\alpha = 0$, the matrix (5) is given by*

$$\begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ \frac{\beta}{\gamma} & 0 & -\gamma \end{pmatrix}.$$

The matrix F of the transformation $z = Fy$ is equal to $\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. The nonlinear part of (7) is given by $\begin{pmatrix} 0 \\ 0 \\ -z_2(z_1 + z_3) \end{pmatrix}$ and the vector (9) can be written as $\begin{pmatrix} -\gamma, & -\frac{\beta}{\gamma} - 1, & -\gamma \end{pmatrix}^{\intercal}$.

4. The controllability function method

Consider the canonical 3-dimensional control system

$$\dot{z} = f(z, u),\ z \in \mathbb{R}^n,\ u \in \Omega \subset \mathbb{R}, \tag{10}$$

where Ω is a closed interval of $\mathbb{R}$.

Considering the synthesis problem for the system (10), in 1979, V. I. Korobov [14] created the controllability function (CF) $\theta(x)$. The CF is a Lyapunov-type function, i.e., $\theta(x) > 0$ for $x \neq 0$ and $\theta(0) = 0$. The CF satisfies the following inequality:

$$\sum_{i=1}^{n} \frac{\partial \theta(z)}{\partial z_i} f_i(z, u(z)) \leq -\varphi(\theta(z)), \tag{11}$$

where $\varphi(\theta) > 0$ for $\theta \neq 0$, $\varphi(0) = 0$ and

$$\int_0^{\bar{\theta}} \frac{d\theta}{\varphi(\theta)} < \infty, \quad \bar{\theta} > 0. \tag{12}$$

Let us consider the canonical control system

$$\dot{z} = A_0 z + bw, \quad |w| \leq w_1, \tag{13}$$

where A_0 is given in Equation (8). Following [7], a family of bounded positional controls $w(z)$ that stabilize the system (13) at finite time can be constructed. In particular, the value of the CF at the given initial position z^0 is exactly $T(z^0)$ the time that the trajectory from z^0 takes to arrive at the origin. As in previous works of V. I. Korobov and coauthors [15], [16], [17], the CF $\theta(z)$ is proposed in [7] as the solution of the following implicit equation

$$2a_0\theta = (K(\theta)z, z). \tag{14}$$

Here $(\cdot, \cdot)$ is the canonical inner product while a_0 is a positive number to be determined and $K(\theta)$ is a 3×3 positive definite matrix for $\theta > 0$ defined as $K(\theta) := D(\theta)K_1D(\theta)$ where

$$D(\theta) := \begin{pmatrix} \theta^{-\frac{5}{2}} & 0 & 0 \\ 0 & \theta^{-\frac{3}{2}} & 0 \\ 0 & 0 & \theta^{-\frac{1}{2}} \end{pmatrix},$$

and

$$K_1 := \begin{pmatrix} \frac{40a_1}{a_1+30} & -\frac{240-12a_1}{a_1+30} & -\frac{120}{a_1+30} \\ -\frac{240-12a_1}{a_1+30} & -\frac{180-4a_1}{a_1+30} & -\frac{60}{a_1+30} \\ -\frac{120}{a_1+30} & -\frac{60}{a_1+30} & -\frac{12}{a_1+30} \end{pmatrix}, \tag{15}$$

for

$$a_1 < -40. \tag{16}$$

Furthermore,

$$\frac{1}{\theta}K - \frac{d}{d\theta}K = \frac{1}{\theta}D(\theta)K_2D(\theta),$$

$$K_2 := \begin{pmatrix} \frac{240a_1}{a_1+30} & -\frac{5(240-12a_1)}{a_1+30} & -\frac{480}{a_1+30} \\ -\frac{5(240-12a_1)}{a_1+30} & -\frac{4(180-4a_1)}{a_1+30} & -\frac{180}{a_1+30} \\ -\frac{480}{a_1+30} & -\frac{180}{a_1+30} & -\frac{24}{a_1+30} \end{pmatrix} \tag{17}$$

and

$$2a_0 \leq \frac{36}{a_1^2 + 12a_1 + 360} w_1^2. \tag{18}$$

The bounded positional control $w(z)$ that solves the synthesis problem for the system (13) is given by

$$w(z) = a^{\intercal}(\theta(z))z, \tag{19}$$

where

$$a(\theta) := \begin{pmatrix} \frac{a_1}{\theta^3} \\ \left(\frac{a_1}{3} - 10\right)\frac{1}{\theta^2} \\ -\frac{6}{\theta} \end{pmatrix} \tag{20}$$

and $\theta(z)$ is the solution of the implicit equation (14).

The fact that the value $\theta(z^0)$ coincides with $T(z^0)$ is guaranteed by the equality

$$\dot{\theta} = -1, \tag{21}$$

which in turn is a special case of the inequality (11).

In terms of the matrices A_0, $K = K(\theta)$, the vector b and $a = a(\theta)$, Equality (21) is equivalent to the following matrix equation

$$KA_0 + A_0^{\intercal}K + ab^{\intercal}K + Kba^{\intercal} + \frac{1}{\theta}K - \frac{d}{d\theta}K = 0_3.$$

Here 0_3 is the 3×3 null matrix.

5. The CF for nonlinear control systems

The controllability function method for nonlinear control systems with non controllable linear part was considered in [3], [4]. In the case when the linear part of the nonlinear control system is completely controllable, the general solution of the synthesis problem was proposed by V. I. Korobov in [14]. This is the case for the controlled Rössler system (2), so, we develop for it a rather specific family of bounded controls based on the control (19). We also focus on the specific form of the nonlinear part of the control system (2).

Note that the nonlinear part of the translated system (4)

$$g(y) := (0, 0, y_1 y_3)^{\intercal}$$

is a Lipschitz function in a neighborhood of the origin; consequently, a positive number C_1 exists such that

$$||g(y)|| \leq C_1||y||. \tag{22}$$

Let us introduce the positional control given by

$$u = w - p^{\intercal}z, \tag{23}$$

where w is defined as in (19). To deal with the linear control part of the system (7) as if it were the canonical control system (13), we look for the restriction on control w. Here we use the same idea as in [14]. We set

$$w_1 := u_1 - u_2 \sum_{j=1}^{3} |p_j|. \tag{24}$$

We assume that $u_2 < \frac{u_1}{\sum\limits_{j=1}^{3} |p_j|}$. As in [14], we require that the system (7) is considered in the neighborhood

$$Q := \{z : |z_j| \leq u_2, \quad j = 1, 2, 3\}. \tag{25}$$

Note that the linear part of the system (2) at equilibrium points described in Section 2 is completely controllable.

In the following result, we calculate the time derivative of the CF θ with respect to the system (7). Our goal is to verify the inequality (11) for some function φ.

Notation. Let S be an $n \times n$ matrix. The norm of S is defined by

$$||S|| := \max_{1 \leq j \leq n} \sum_{i=1}^{m} |s_{ij}|.$$

The number $\lambda_{\min,S}$ is the smallest eigenvalue of matrix S. Here we suppose that S is a symmetric matrix.

Theorem 1. *Let K_1, K_2 and C_1 be as in (15), (17) and (22). The following inequality is valid:*

$$\dot{\theta} \leq -1 + 2\theta \frac{C_1 ||K_1||}{\lambda_{\min,K_2}}. \tag{26}$$

Proof. By taking the derivative of the equality (14) with respect to time t and the system (7), we have

$$\begin{aligned} \dot{\theta} =& \frac{((KA_0 + A_0^\intercal K + ab^\intercal K + Kba^\intercal)z, z)}{((\frac{1}{\theta}K - \frac{d}{d\theta}K)z, z)} + 2\frac{(Kz, Fg(F^{-1}z))}{((\frac{1}{\theta}K - \frac{d}{d\theta}K)z, z)} \\ =& -1 + 2\frac{(Kz, Fg(F^{-1}z))}{((\frac{1}{\theta}K - \frac{d}{d\theta}K)z, z)} \\ \leq& -1 + 2\theta\frac{C_1||K_1||}{\lambda_{\min,K_2}}. \end{aligned} \tag{27}$$

In the last inequality, we used the obvious inequality

$$\frac{(Kz, Fg(F^{-1}z))}{((\frac{1}{\theta}K - \frac{d}{d\theta}K)z, z)} \leq \theta \frac{C_1||K_1||}{\lambda_{\min,K_2}}.$$

□

Note that inequality (12) is satisfied if $\varphi(\theta) = 1 - M\theta$, for some positive M:

$$\int\limits_0^{\bar{\theta}} \frac{1}{1 - M\theta} d\theta = -\frac{1}{M} \ln(1 - M\bar{\theta}), \quad 1 \geq \bar{\theta}M.$$

By employing inequality (26), the following remark yields. A similar remark appeared in [9].

Remark 1.2. *Let* $\widehat{\theta} > 0$, $C_2 > 0$ *such that for* $\theta \leq \widehat{\theta}$

$$-1 + \theta \frac{C_1 ||K_1||}{\lambda_{\min, K_2}} \leq -C_2. \tag{28}$$

Then, the following inequality is valid:

$$\dot{\theta} \leq -C_2. \tag{29}$$

And the next bound on the arriving time is obtained

$$T(z) \leq \frac{\theta_0}{C_2}. \tag{30}$$

Proof. In view of (26), the inequality (29) readily follows. To prove (30), we integrate (29) on the trajectory $z = z(t)$. We attain $\theta(z(t)) - \theta_0 \leq -C_2 t$. By employing [14, page 552], we have that $z(T) = 0$. This implies $\theta(z(T)) = 0$; thus we obtain inequality (30).

Remark 1.3. *The following optimization problem will be useful for improving the size of the neighborhood of the origin where initial conditions must be chosen to achieve the control objective:*

$$\max_{\theta > 0} \chi(z, \theta)$$

for $||z|| \leq C$ *and such that* $\chi < 0$, *where*

$$\chi(z, \theta) := -1 + 2 \frac{(Kz, Fg(F^{-1}z))}{((\frac{1}{\theta} K - \frac{d}{d\theta} K) z, z)}.$$

Let $\widehat{\theta}$ *be the value at which the maximum of* χ *is achieved. This value of* $\widehat{\theta}$ *can be employed instead of* $\widehat{\theta}$ *of Remark 1.2. On the other hand, for applications, available software should be more adequate.*

The existence of such $\widehat{\theta}$ is verified by Remark 1.2. The proof of this remark can be carried out by using the Lagrange multipliers.

Remark 1.4. *Considering the control system (10), in the case when the origin is an equilibrium point of (10), according to [14, Theorem 1], the state variables* $z_k(t)$, *for* $k = 1, 2, 3$ *do not leave a certain neighborhood of the origin and approach the equilibrium point as* $t \to T$. *For* $t > T$ *the trajectory* $z(t)$ *stays at the equilibrium point. Both these phenomena are explained by the fact that the control is a positional control that stabilizes the system at finite time.*

Lemma 1. *Let* $\bar{\bar{x}}$ *be one of the equilibrium points described in Section 2. Let* $a := \left(a_1, (\frac{a_1}{3} - 10), -6\right)^{\intercal}$, *with* $a_1 < -40$. *Furthermore, let* $(k_{j,\ell})_{j,\ell=1}^{3} := K_1$, *and let the parameter* a_0 *satisfy (18). Thus,* $\theta(x - \bar{\bar{x}})$ *is the unique positive solution of*

$$\mathcal{E}(x, \theta, \bar{\bar{x}}) = 0 \tag{31}$$

with

$$\mathcal{E}(x,\theta,\bar{\bar{x}}) := 2a_0\theta^6 - \sum_{j,\ell=1}^{3} k_{j,\ell}\theta^{j+\ell-2}(c, A^{j-1}(x-\bar{\bar{x}}))(c, A^{\ell-1}(x-\bar{\bar{x}})), \tag{32}$$

where vector c is such that $(b,c)=0$, $(Ab,c)=0$ and $(A^2b,c)=1$.

The proof of this lemma repeats the proof of the first part of [14, page 540].

Remark 1.5. *Fixed θ, the set $E := \{x \in \mathbb{R}^3 : \mathcal{E}(x,\theta,\bar{\bar{x}})=0\}$ is an ellipsoid. The trajectories of the system (2) starting from the volume embraced by E or on E do not escape from this set. For $t \to T$ the trajectory of the system approaches the equilibrium point $\bar{\bar{x}}$.*

Let Q_1 be the domain in $\mathbb{R}^3$ that corresponds to (25), i.e., after the transformation $y = F^{-1}z$ and the translation $y = x - \bar{\bar{x}}$, that is, $Q_1 := \{x \in \mathbb{R}^3 : x = F^{-1}z + \bar{\bar{x}},\ \ z \in Q\}$. Define

$$Q_2 := \{x \in \mathbb{R}^3 | \theta(x-\bar{\bar{x}}) < \widetilde{\theta}\} \tag{33}$$

where $\widetilde{\theta} \leq \widehat{\theta}$ and such that $Q_2 \subset Q_1$. The main result of our work is seen below.

Theorem 2. *Let p be defined as in (9). Under the conditions of Lemma 1, let*

$$u(x,\bar{\bar{x}}) = \sum_{j=1}^{3} a_j(\theta^{j-3}(x-\bar{\bar{x}}))(c, A^{j-1}(x-\bar{\bar{x}})) - \sum_{j=1}^{3} p_j(x-\bar{\bar{x}})(c, A^{j-1}(x-\bar{\bar{x}})). \tag{34}$$

Suppose that $x^0 = (x_1^0, x_2^0, x_3^0)$ belongs to Q_2.
Thus, a) the control (34) satisfies the condition $|u(x)| \leq u_1$ and solves the synthesis problem. b) The time taken by the trajectory from x^0 to the equilibrium point $\bar{\bar{x}}$ satisfies the following inequality:

$$T(x^0,\bar{\bar{x}}) \leq \frac{\theta_0}{C_2}. \tag{35}$$

Proof. Part a) is proven by employing (13), (23) and (25). Part b) readily follows from (30), the transformation $z = Fy$ and the translation $y = x - \bar{\bar{x}}$. Recall that u_1 is a number that indicates the boundaries of the control set.

Remark 1.6. *The parameter a_1 determines the vector (20). For each $a_1 < -40$, a positional control which solves the synthesis problem is computed by (34). Besides, both the neighborhood Q_2 where the initial conditions must be taken to achieve control objective and an upper bound of the time to reach the equilibrium, are also obtained (formulae (33) and (35), respectively).*

Remark 1.7. *The value $\widehat{\theta}$ as introduced in Remark 1.4, involves to solve on optimization problem but it provides a better optimization of Q_2 than if the value $\widehat{\theta}$ of Remark 1.6 is applied.*

6. Graph of the trajectory and control

For a given initial point (x_1^0, x_2^0, x_3^0), to plot the graph of the trajectory $x(t)$, the differential equation (27) is extended as follows:

$$\begin{aligned}\dot{x}_1 &= -x_2 - x_3,\\ \dot{x}_2 &= x_1 + \alpha x_2,\\ \dot{x}_3 &= \beta + x_3(x_1-\gamma) + u(x,\theta,\bar{\bar{x}}),\\ \dot{\theta} &= -1 + 2\psi(x,\theta,\bar{\bar{x}})\end{aligned}$$

with initial conditions $x_1(0) = x_1^0$, $x_2(0) = x_2^0$, $x_3(0) = x_3^0$ and $\theta(0) = \theta_0$. Here θ_0 is the root equation (31), and

$$\psi(x,\theta,\bar{\bar{x}}) := \frac{(D(\theta)K_1D(\theta)(x-\bar{\bar{x}}), Fg(F^{-1}(x-\bar{\bar{x}})))}{\frac{1}{\theta}(D(\theta)K_2D(\theta)(x-\bar{\bar{x}}),(x-\bar{\bar{x}}))} \tag{36}$$

with $\theta = \theta(x-\bar{\bar{x}})$.

Example 1. For $\alpha = \beta = 0.2$ and $\gamma = 5.7$, we have the Rössler chaotic system and $\bar{x}_+ = (5.69297, -28.4649, 28.4649)$ is one of the corresponding equilibrium points. Let $u_1 = 3.2$ and let $a_1 = -45$. The positional control has the form $u(x) = -\frac{6(0.2(5.69297-x_1)+0.96(x_2+28.4649)+x_3-28.4649)}{\theta} + \frac{45(x_2+28.4649)}{\theta^3} - \frac{-25x_1-5x_2}{\theta^2} - 29.4249x_1 - 0.392x_2 - 0.192974x_3 + 161.85$. The graphs of $x_1(t)-\bar{x}_+^1$, $x_2(t)-\bar{x}_+^2$ and $x_3(t)-\bar{x}_+^3$ are shown in Fig. 1.

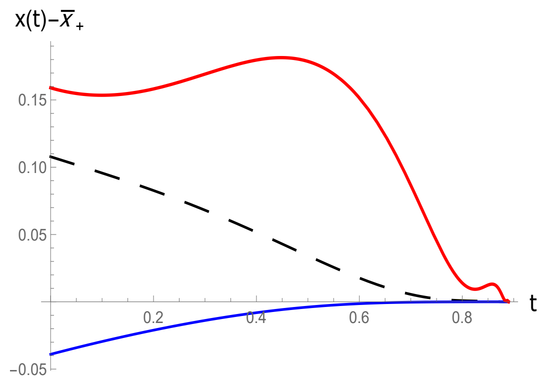

Fig. 1. Trajectories of $x_1(t)-\bar{x}_+^1$, $x_2(t)-\bar{x}_+^2$ and $x_3(t)-\bar{x}_+^3$.

The graph of the position control $u(x(t))$ is displayed in Fig. 2.

The controllability function on the trajectory $\theta(x(t))$ is shown in Fig. 3.

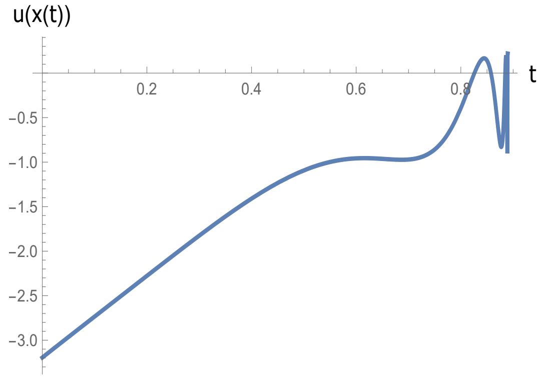

Fig. 2. The positional control $u(x(t))$.

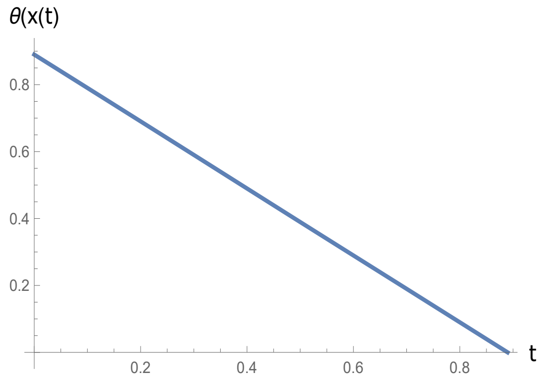

Fig. 3. The controllability function $\theta(x(t))$.

By using Wolfram Mathematica, we have calculated that the time of arriving from $x^0 = (5.80077379, -28.5038689, 28.6238689)$ to the equilibrium point $\bar{x}_+$ is $T(x^0, \bar{x}_+) = 0.8898539650858471$ and that $|x_1(T) - 5.69297| \leq 1.05197 * 10^{-9}$, $x_2(T) = -28.4649$ and $|x_3(T) - 28.6238689| \leq 1.177185658 * 10^{-5}$.

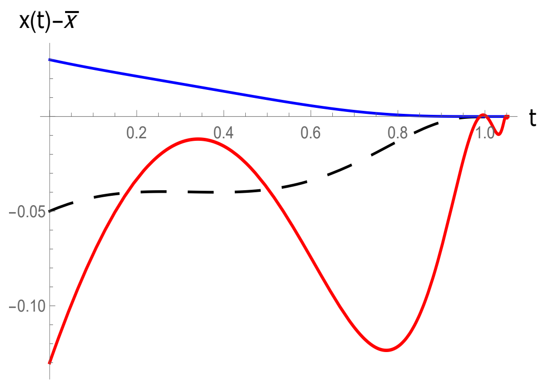

Fig. 4. Trajectories of $x_1(t) - \bar{x}^1$, $x_2(t) - \bar{x}^3$ and $x_3(t) - \bar{x}^3$.

Example 2. Let $\alpha = 0, \beta = 1/5$ and $\gamma = 1$. The corresponding equilibrium point is equal to $\bar{x} = (0, -\frac{1}{5}, \frac{1}{5})$. Let $u_1 = 1.1$ and let $a_1 = -45$. The positional control has the form $u(x) = \frac{45\left(x_2+\frac{1}{5}\right)}{\theta^3} + \frac{25x_1}{\theta^2} - \frac{6(x_2+x_3)}{\theta} - \frac{6x_1}{5} + x_3 - \frac{1}{5}$. The graphs of $x_1(t) - \bar{x}^1$, $x_2(t) - \bar{x}^3$ and $x_3(t) - \bar{x}^3$ are shown in Fig. 4.

The graph of the position control $u(x(t))$ is seen in Fig. 5.

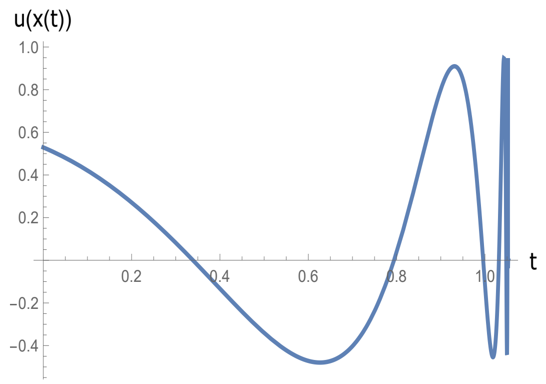

Fig. 5. The positional control $u(x(t))$.

The controllability function on the trajectory $\theta(x(t))$ is shown in Fig. 6.

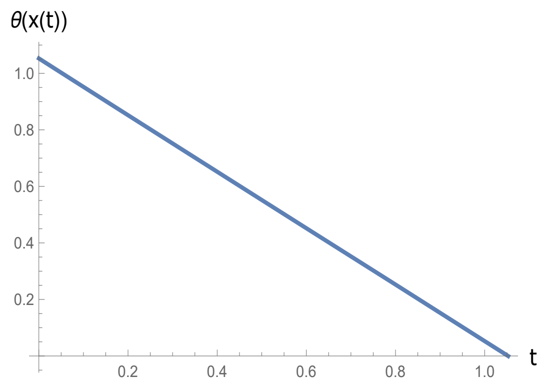

Fig. 6. The controllability function $\theta(x(t))$.

By using Wolfram Mathematica, we have calculated that the time of arriving from $x^0 = (-0.05, -0.17, 0.07)$ to the equilibrium point $\bar{x}$ is $T(x^0, \bar{x}) = 1.0528937566$ and that $|x_1(T)| \leq 3.19427 * 10^{-11}$, $|x_2(T) + \frac{1}{5}| \leq 6.57807 * 10^{-15}$ and $|x_3(T) - \frac{1}{5}| \leq 3.48898 * 10^{-8}$.

7. Conclusion

In this paper, a family of bounded finite-time stabilizing positional controls for the Rössler system is built. By using the controllability function method, which is a Lyapunov-type function, the finite time to reach the desired equilibrium point is estimated. This is obtained for an arbitrary given control bound and an adequate set Q_2 of initial conditions to achieve the control objective is computed. Let us note that this proposal may also be developed for any controlled system of the form:

$$\dot{x} = f(x) + bu$$

being $f(x) = A(x - \bar{x}) + g(x)$, $\bar{x}$ an equilibrium point of f, A the jacobian matrix of f evaluated in $\bar{x}$ and g the corresponding nonlinear part of f such that $\{A, b\}$ is completely controllable and g is a lipschitzian function in a neighborhood of $\bar{x}$. We claim that the smaller the constant bound is, the more reduced is the set of initial conditions for which stabilization is guaranteed; see (14), (18) and (23). Moreover, the smaller the bound is, the longer is the time to arrive at the equilibrium point from the the same initial point; see (14), (18).

For the cases in which the Rössler system is chaotic, this technique may be implemented as a tool for control chaos. Indeed, if the equilibrium point is embedded in the strange attractor, a trajectory initiated in the basin of attraction of the attractor can reach the region Q_2 and by this moment, this finite-time control strategy can be activated, so, the equilibrium point will be reached in finite time. The use of finite-time stabilizing control for control chaos or for chaos synchronization is not new (see for example [26]). Hence, the introduction of this control strategy in these scenarios promises interesting future research.

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Чоке-Ріверо А. Е., Гонзалес Грасіела А., Круз Муллісака Е. **Метод функції керованості Коробова, застосований до стабілізації системи Росслера за обмежений час за допомогою обмежених керувань.** Система Росслера стала однією з референтних хаотичних систем. Її новизна при введенні, була в тому, що вона демонструє хаотичний атрактор, породжений більш простим набором нелінійних диференціальних рівнянь, ніж система Лоренца. Ця система за певних значень її триплета параметрів демонструє хаотичну поведінку. Питання керування системою Росслера шляхом стабілізації однієї з її нестійких точок рівноваги раніше розглядалося в літературі. У цій роботі запропоновано керування системою Росслера на основі задачі синтезу. Для заданої системи та однієї з її точок рівноваги, задача синтезу полягає у побудові обмеженого позиційного керування таким чином, що для будь-якого x^0, що належить певному околу точки рівноваги, траєкторія $x(t)$, що починається в x^0, дістається до цієї точки рівноваги за скінченний час. А саме, з використанням методу В. І. Коробова, який також називають методом функції керованості, пропонується сім'я обмежених позиційних керувань, які розв'язують задачу синтезу для системи Росслера. В основному ми використовуємо два компоненти. Перший стосується загальної теорії функції керованості. Другий компонент - це сім'я обмежених позиційних керувань, яка будується в цій роботі. На відміну від попередніх робіт щодо стабілізації за скінчений час, ми пропонуємо явну сім'ю обмежених керувань, побудовану з урахуванням лише нелінійності системи Росслера, яка є квадратичною функцією. За допомогою методу функції керованості, яка є функцією типу Ляпунова, оцінюється скінченний час, потрібний для досягнення бажаної точки рівноваги. Цю оцінку отримано для довільно заданої межі керування,а також наведено відповідну множину початкових умов для досягнення мети керування. Цей підхід може бути також розвинутий для будь-якої керованої системи, лінійна частина якої є повністю керованою, а її відповідна нелінійна частина - ліпшицевою функцією в околі точки рівноваги. У свою чергу, ця техніка може бути реалізована як інструмент керування хаосом.
Ключові слова: Система Росслера; функція керованості Коробова; обмежене керування; стабілізація за скінченний час.

A. E. Choque-Rivero, Graciela A. González, E. Cruz Mullisaca. **Korobov's controllability function method applied to finite-time stabilization of the Rössler system via bounded controls.** Rössler system has become one of the reference chaotic systems. Its novelty when introduced, being that exhibits a chaotic attractor generated by a simpler set of nonlinear differential equations than Lorenz system. It develops chaotic behaviour for certain values of its parameter triplet. The issue of controlling Rössler system by stabilizing one of its unstable equilibrium points has been previously dealt with in the literature. In this work, control of the Rössler system is stated by considering the synthesis problem. Given a system and one of its equilibrium points, the synthesis problem consists in constructing a bounded positional control such that for any x^0 belonging to a certain neighborhood of the equilibrium point, the trajectory $x(t)$ initi-

ated in x^0 arrives at this equilibrium point in finite time. Namely, by using V. I. Korobov's method, also called the controllability function method, a family of bounded positional controls that solve the synthesis problem for the Rössler system is proposed. We mainly use two ingredients. The first one concerns the general theory of the controllability function The second ingredient is a family of bounded positional controls that was obtained in. Different from previous works on finite-time stabilization we propose an explicit family of bounded controls constructed by taking into account the only nonlinearity of the Rössler system, which is a quadratic function. By using the controllability function method, which is a Lyapunov-type function, the finite time to reach the desired equilibrium point is estimated. This is obtained for an arbitrary given control bound and an adequate set of initial conditions to achieve the control objective is computed. This proposal may also be developed for any controlled system for which its linear part is completely controllable and its corresponding nonlinear part is a lipschitzian function in a neighborhood of the equilibrium point. In turn, this technique may be implemented as a tool for control chaos.
Keywords: Rössler system; Korobov's controllability function; bounded control; finite time stabilization.

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