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The limit set of the Henstock-Kurzweil integral sums of a vector-valued function

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We introduce the notion of the limit set $I_{H-K}(f)$ of the Henstock-Kurzweil integral sums of a function $f : [0,1] \to X$, where X is a Banach space, and study its properties. In particular, we construct an example of function f, which is not integrable, but its limit set consists exactly of one point. We find sufficient conditions that guarantee the convexity of the limit set.
Keywords: Henstock-Kurzweil integral, Banach space, limit set of integral sums.

Костянко А. Г. **Множина граничних точок інтегральних сум Хенстока-Курцвейля векторнозначної функції.** Ми вводимо поняття множини граничних точок $I_{H-K}(f)$ інтегральних сум Хенстока-Курцвейля функції $f : [0,1] \to X$, де X - банахів простір, і вивчаємо його властивості. Зокрема, ми будуємо приклад неінтегрованої функції f, множина граничних точок котрої складається рівно з однієї точки. Також ми знаходимо достатні умови, що гарантують опуклість множини граничних точок.
Ключові слова: інтеграл Курцвейля-Хенстока, банахів простір, множина граничних точок інтегральних сум.

Костянко А. Г. **Множество предельных точек интегральных сумм Хенстока-Курцвейля векторнозначной функции.** Мы вводим понятие множества предельных точек $I_{H-K}(f)$ интегральных сумм Хенстока-Курцвейля функции $f : [0,1] \to X$, где X - банахово пространство, и изучаем его свойства. В частности, мы строим пример неинтегрируемой функции f, множество предельных точек которой состоит ровно из одной точки. Также мы находим достаточные условия, которые гарантируют выпуклость множетсва предельных точек.
Ключевые слова: интеграл Курцвейля-Хенстока, банахово пространство, множество предельных точек интегральных сумм.
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1. Introduction

The Henstock-Kurzweil integral was discovered in 1957. It generalizes Riemann integral and is used for integration of highly oscillatory functions which occur in quantum theory and nonlinear analysis (see [8, Chapter 4]). Moreover, all Lebesgue integrable functions are Henstock-Kurzweil integrable, and one of the advantages of the latter is that it does not rely on measure theory. Also one may consider integral and differential equations using Henstock-Kurzweil integral (see [2]). For functions which are not integrable we introduce the notion of the limit set of the Henstock-Kurzweil integral sums $I_{H-K}(f)$ and study its properties. Similar notion of a limit set $I(f)$ for Riemann integral and its properties is studied in [4, Appendix].

Our main result is construction of a function for which limit set $I_{H-K}(f)$ contains only 1 point but the function is not Henstock-Kurzweil integrable (see Theorem 3). Similar result for Riemann integral is established in [4, Appendix]. However in our case construction of such an example is more sophisticated. It appears that properties of the limit set of Riemann integral sums (as well as Henstock-Kurzweil integral sums) depend significantly on the properties of the space of values of a function under consideration. For example, if function takes values in a separable space then its limit set $I(f)$ associated with Riemann integral is not empty (see [1]). However the full description of such spaces is not known. Also for bounded functions with values in separable spaces it is known that $I(f)$ is a star-shaped set (see [5] and [4]). Conditions for convexity for $I(f)$ in the case of Riemann integral are given in [6] (see also [4]). In particular there are conditions which can be easily described when a considered function takes values in so called B-convex space. We establish analogues of these results for the limit set of Henstock-Kurzweil integral $I_{H-K}(f)$ (see Theorem 4, Theorem 5). In general situation we can not expect convexity of the limit set (see [3]).

The work is organised as follows. In Section 2 we recall the notion of Henstock-Kurzweil integral and introduce a notion of a limit set for Henstock-Kurzweil integral. In the beginning of Section 3 we reformulate basic definitions in terms of net convergence and give basic properties of the limit set $I_{H-K}(f)$ (see Theorem 1, Theorem 2). The main result is stated in Theorem 3. Results concerning convexity of the limit set $I_{H-K}(f)$ that generalize results obtained in [6] are given in Theorems 4 and 5.

2. Basic definitions

We consider functions $f : [0,1] \to X$, where X is a Banach space.
Let P be a tagged partition of $[0,1]$, i.e.

$$P = \{(\xi_i, (t_{i-1}, t_i)), \text{where } 0 = t_0 < t_1 < \cdots < t_n = 1, \xi_i \in [t_{i-1}, t_i]\};$$

and $\delta : [0,1] \in (0,\infty)$ be a positive function, which is called gauge. We say, that P is δ-fine if $\xi_i \in [t_{i-1}, t_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for all $i = 1, \ldots, n$. We denote this by writing P is a δ-fine tagged partition of $[0,1]$. We define the Henstock-Kurzweil integral sums of the function f as

$$S(f,P) = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}).$$

These integral sums are the same as for the Riemann integral, but they are considered in context of a very different convergence definition:

Definition 1 *A function $f : [0,1] \to X$ is said to be Henstock-Kurzweil integrable on $[0,1]$ if there is $x \in X$ such that for every $\varepsilon > 0$ there is a gauge δ on $[0,1]$ such that for every δ-fine tagged partition P of $[0,1]$*

$$||S(f,P) - x|| < \varepsilon.$$

This x is called the Henstock-Kurzweil integral of f.

For functions that are not Henstock-Kurzweil integrable the role of an integral may be played by the limit set of the Henstock-Kurzweil integral sums.

Definition 2 *We say, that for $f : [0,1] \to X$ a vector $x \in X$ is a Henstock-Kurzweil point (H-K point) if for every $\varepsilon > 0$ and for every gauge δ on $[0,1]$ there is a δ-fine tagged partition P of $[0,1]$ such that*

$$||S(f,P) - x|| < \varepsilon.$$

The set of all H-K points of a function $f : [0,1] \to X$, where X is a Banach space, we denote by $I_{H-K}(f)$. For a nonintegrable function its limit set $I_{H-K}(f)$ may be empty or contain many points.

3. Properties of the limit set $I_{H-K}(f)$

We start with reformulation of Definition 1 and Definition 2 in terms of net convergence.

Let $(\Gamma, \succ)$ be the directed set, where $\Gamma = \{(\gamma = \delta, P) : \delta$ is a gauge on $[0,1]$ and P is a δ-fine tagged partition of $[0,1]\}$.

Definition 3 *We say, that pair (δ_1, P_1) follows pair (δ_2, P_2) (we denote it by $(\delta_1, P_1) \succ (\delta_2, P_2)$), if $\delta_1 \leq \delta_2$ on $[0,1]$.*

Define the net $F = F_f : \Gamma \to X$ by the rule $F((\delta, P)) = S(P, f)$. Then the following propositions are obvious

Proposition 1 *Let X be a Banach space and a function $f : [0,1] \to X$. Then the following conditions are equivalent:*
i) $x \in X$ is the integral of f on $[0,1]$,
ii) $x = \lim_\Gamma F$.

Proposition 2 *The limit set of the Henstock-Kurzweil integral sums coincides with the limit set of the net $F = F_f$.*

Remark 1 *Let X be a Banach space, then for a function $f : [0,1] \to X$, its limit set $I_{H-K}(f)$ is closed. This is a general result for the limit set of nets (see [?, Chapter 2]).*

Now we proceed to prove other properties of $I_{H-K}(f)$

Theorem 1 *Let X be a Banach space, $f : [0,1] \to X$ and $g : [0,1] \to X$ be a Henstock-Kurzweil integrable function. Then*

$$I_{H-K}(f+g) = I_{H-K}(f) + \int_0^1 g(t)dt.$$

Proof. *i*) Let us prove first the inclusion $I_{H-K}(f+g) \subset I_{H-K}(f) + \int_0^1 g(t)dt$. Take an arbitrary $x \in I_{H-K}(f+g)$ and denote $x_2 = \int_0^1 g(t)dt$. We are going to show that there exists $x_1 \in I_{H-K}(f)$ such that $x = x_1 + x_2$, i. e. we have to show that $x - x_2 \in I_{H-K}(f)$.

To this end fix $\varepsilon > 0$. From integrability of $g(t)$ and Proposition 1 we get that for every $\gamma \in \Gamma$ there is a $\tilde{\gamma} \succ \gamma$ such that for every $\gamma_1 \succ \tilde{\gamma}$

$$||x_2 - F_g(\gamma_1)|| < \frac{\varepsilon}{2}.$$

Using condition $x \in I_{H-K}(f+g)$ and Proposition 2, we obtain that for $\tilde{\gamma}$ as above there is $\gamma_1 \succ \tilde{\gamma}$ such that

$$||x - F_{f+g}(\gamma_1)|| < \frac{\varepsilon}{2}.$$

So, for every $\gamma \in \Gamma$ there is a $\gamma_1 \succ \gamma$ such that

$$||x - x_2 - F_f(\gamma_1)|| \le ||x - F_{f+g}(\gamma_1)|| + ||x_2 - F_g(\gamma_1)|| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which means that $x - x_2 \in I_{H-K}(f)$.
ii) Applying *i*) with f instead of $f+g$ and $-g$ instead of g we obtain

$$I_{H-K}(f) \subset I_{H-K}(f+g) + \int_0^1 -g(t)dt.$$

After adding $\int_0^1 -g(t)dt$ to both sides of this expression we get

$$I_{H-K}(f) + \int_0^1 g(t)dt \subset I_{H-K}(f+g),$$

which was to be proved. □

Theorem 2 *Let $f : [0,1] \to X$ and $g : [0,1] \to X$, where X is a Banach space, and the image of f or of g is relatively compact in X, then*

$$I_{H-K}(f+g) \subset I_{H-K}(f) + I_{H-K}(g).$$

Proof. Let us prove that $I_{H-K}(f+g) \subset I_{H-K}(f) + I_{H-K}(g)$. If $I_{H-K}(f+g)$ is empty, the inclusion is satisfied. Let us assume $I_{H-K}(f+g)$ is not empty fix an $x \in I_{H-K}(f+g)$ and define a new directed set

$$\tilde{\Gamma} = \{(\varepsilon, \delta, P) : \varepsilon > 0, ||x - F_{f+g}(\delta, P)|| < \varepsilon\}.$$

We say that $(\varepsilon_1, \delta_1, P_1)$ follows $(\varepsilon_2, \delta_2, P_2)$ if $\varepsilon_1 \leq \varepsilon_2$ and $\delta_1 \leq \delta_2$.

Let us introduce net $\tilde{F}_{f+g}((\varepsilon, \delta, P)) = F_{f+g}((\delta, P))$ then $x = \lim_{\tilde{\Gamma}} \tilde{F}_{f+g}(\tilde{\gamma})$, i.e. for every $\varepsilon > 0$ there is $\tilde{\gamma} \in \tilde{\Gamma}$ such that for every $\tilde{\gamma}_1 \succ \tilde{\gamma}$

$$||x - \tilde{F}_{f+g}(\tilde{\gamma}_1)|| < \varepsilon.$$

Let image $f([0,1])$ be relatively compact, then $\tilde{F}_f(\tilde{\Gamma})$ is also relatively compact, i.e. for $\tilde{F}_f$ there exists a limit point x_1. Let $\tilde{\gamma}$ be as in the above condition. Then for every $\tilde{\gamma}_2 \in \Gamma$ there is a $\tilde{\gamma}_3$ that follows both $\tilde{\gamma}$ and $\tilde{\gamma}_2$. Since x_1 is a limit point for $\tilde{F}_f$, there is a $\tilde{\gamma}_1 \succ \tilde{\gamma}_3$ such that

$$||x_1 - \tilde{F}_f(\tilde{\gamma}_1)|| < \varepsilon.$$

Using previous estimates we obtain: for every $\varepsilon > 0$ and for every $\tilde{\gamma}_2$ there is a $\tilde{\gamma}_1 \succ \tilde{\gamma}_2$ such that

$$||x - x_1 - \tilde{F}_g(\tilde{\gamma}_1)|| \leq ||x_1 - \tilde{F}_f(\tilde{\gamma}_1)|| + ||x - \tilde{F}_{f+g}(\tilde{\gamma}_1)|| < 2\varepsilon.$$

We have demonstrated that $x_2 = x - x_1 \in I_{H-K}(g)$. $\square$

However the inverse inclusion may not be true and our next example shows that. By $\lambda^*(C)$ we denote the outer measure of $C \subset [0,1]$.

Example 1 *There exist functions $f(t), g(t) : [0,1] \to \mathbb{R}$ such that their images are relatively compact in X, but*

$$I_{H-K}(f+g) \neq I_{H-K}(f) + I_{H-K}(g).$$

Define f and $g : [0,1] \to \mathbb{R}$ by the rules

$$f(t) = \begin{cases} 1 & \text{if } t \in A, \\ -1 & \text{if } t \in B, \end{cases} \qquad g(t) = \begin{cases} -1 & \text{if } t \in A, \\ 1 & \text{if } t \in B, \end{cases}$$

where A and B are non-measurable sets, $\lambda^*(A) = \lambda^*(B) = 1$, $A \cup B = [0,1]$ and $A \cap B = \emptyset$. It is not difficult to see that $f + g = 0$, and $I_{H-K}(f+g) = \{0\}$, but $I_{H-K}(f) + I_{H-K}(g) = \{-2, 0, 2\}$.

The next property of $I_{H-K}(f)$ is obvious, so we state it without proof.

Proposition 3 *Let $f : [0,1] \to X$, where X is a Banach space, T be a continuous linear map and $x \in I_{H-K}(f)$, then $Tx \in I_{H-K}(Tf)$.*

Our next theorem is the general result for limits of nets (see [**?**, Chapter 2]).

Proposition 4 *Let X be a Banach space and for a function $f : [0,1] \to X$ its image $f([0,1])$ is relatively compact in X. Then f is integrable if and only if its limit set $I_{H-K}(f)$ consists exactly of one point and under this assumption its integral is exactly this point.*

It is easy to see that the assumption image $f([0,1])$ is relatively compact in X implies $F(\delta, P)$ is relatively compact in X. Thus under this assumption the limit set of net contains at least one point (see [**?**, Chapter 2]). Hence the limit set of the Henstock-Kurzweil integral of f is not empty. Let us show that compactness condition can not be replaced by boundedness condition.

Recall that $\ell_1[0,1]$ is the space of real-valued functions defined on the segment $[0,1]$, having at most countable support and such that $\sum_{\alpha\in[0,1]} |g(\alpha)| < \infty$. The norm in $\ell_1[0,1]$ is $||g|| = \sum_{\alpha\in[0,1]} |g(\alpha)|$. The standard basic vectors of the space $\ell_1[0,1]$ have the following form

$$e_t(\alpha) = \begin{cases} 1 & \text{if } \alpha = t, \\ 0 & \text{if } \alpha \neq t. \end{cases}$$

Then $||e_t|| = 1$ for all $t \in [0,1]$. Any element $g \in \ell_1[0,1]$ can be represented in the form $g = \sum_{i=1}^{\infty} a_i e_{t_i}$, and $|| \sum_{i=1}^{\infty} a_i e_{t_i}|| = \sum_{i=1}^{\infty} |a_i|$.

Function $f : [0,1] \to \ell_1[0,1]$, which acts by the rule $f(t) = e_t$, is an example of a function that has an empty limit set $I_{H-K}(f)$.

Our next goal is to construct an example which shows that a one-point limit set does not guarantee the existence of the integral. Further we need the following technical result.

Proposition 5 *Let δ be a gauge on $[0,1]$, $C \subset [0,1]$ and $\lambda^*(C) = 1$. Then, for every $\varepsilon > 0$ there is a δ-fine tagged partition P of $[0,1]$ such that the sum of lengths of segments whose tag points lie in C (we denote them by $(\tilde{\tau}_k, \tau_k)$ for $k = 1, \ldots, n$) obeys the following inequality:*

$$\sum_{k=1}^{n} (\tau_k - \tilde{\tau}_k) > 1 - \varepsilon.$$

Proof. Step 1. For all $t \in C$ denote $\Delta_t = (t - \delta(t), t + \delta(t))$. Let us consider properties of the set $\Delta = \cup_{t\in C} \Delta_t$.

(1) Δ is an open set, and consequently it may be represented in the following form:

$$\Delta = \bigsqcup_{k=1}^{\infty} (a_k, b_k);$$

(2) C is a subset of Δ.

Using 1 and 2, we can conclude that $\sum_{k=1}^{\infty}(b_k - a_k) \geq 1$.
Step 2. Pick $\varepsilon_k > 0$ such that $\sum_{k=1}^{\infty}\varepsilon_k < \varepsilon$. Notice that we can represent (a_k, b_k) in the form $\cup_{t\in(a_k,b_k)\cap C}\Delta_t$ for all k. After a small decrease of intervals, we obtain

$$[a_k + \frac{\varepsilon_k}{2}, b_k - \frac{\varepsilon_k}{2}] \subset \bigcup_{t\in(a_k,b_k)\cap C} \Delta_t.$$

By the Heine-Borel theorem there exist such points $t_{k_1} < t_{k_2} < \cdots < t_{k_{N_k}}$ that

$$[a_k + \frac{\varepsilon_k}{2}, b_k - \frac{\varepsilon_k}{2}] \subset \bigcup_{j=1}^{N_k} \Delta_{t_{k_j}}.$$

Step 3. We are going to introduce smaller intervals $\tilde{\Delta}(t_{k_j}) \subset \Delta_{t_{k_j}}$ in such a way that, if $\tilde{\Delta}(t_{k_j}) \neq \emptyset$, then $t_j \in \tilde{\Delta}_{t_{k_j}}$; intersection of interiors of $\tilde{\Delta}_{t_{k_j}}$ and $\tilde{\Delta}_{t_{k_i}}$ is empty for $j \neq i$ and

$$[a_k + \frac{\varepsilon_k}{2}, b_k - \frac{\varepsilon_k}{2}] \subset \bigcup_{j=1}^{N_k} \tilde{\Delta}_{t_{k_j}}.$$

To this end let us consider four cases:
(1) $t_{k_2} - \delta(t_{k_2}) < a_k + \frac{\varepsilon_k}{2}$, then we skip point t_{k_1} and $\tilde{\Delta}(t_{k_1}) = \emptyset$;
(2) $t_{k_1} > t_{k_2} - \delta(t_{k_2})$, then we may choose $\tilde{\Delta}(t_{k_1})$ as follows $\tilde{\Delta}(t_{k_1}) = [a_k + \frac{\varepsilon_k}{2}, t_{k_1}]$;
(3) $t_{k_1} \leq t_{k_2} - \delta(t_{k_2})$ and $t_{k_1} + \delta(t_{k_1}) \leq t_{k_2}$, then $\tilde{\Delta}(t_{k_1})$ may have the form $\tilde{\Delta}(t_{k_1}) = [a_k + \frac{\varepsilon_k}{2}, t_{k_1} + \frac{t_{k_1}+\delta(t_{k_1})-t_{k_2}+\delta(t_{k_2})}{2}]$;
(4) $t_{k_1} \leq t_{k_2} - \delta(t_{k_2})$ and $t_{k_1} + \delta(t_{k_1}) > t_{k_2}$, then $\tilde{\Delta}(t_{k_1}) = [a_k + \frac{\varepsilon_k}{2}, t_{k_2}]$
Now we consider the segment $[a_k + \frac{\varepsilon_k}{2}, b_k - \frac{\varepsilon_k}{2}] \setminus \tilde{\Delta}(t_{k_1})$ and go over to the point t_{k_2}, for it we check the similar four cases. Then we do the same for all points t_{k_j} for all k and j.

As result, we obtain

$$\sum_{k=1}^{\infty}\sum_{j=1}^{N_k} \tilde{\Delta}(t_{k_j}) > 1 - \varepsilon,$$

which proves the statement. □

Theorem 3 *There exists a function $f : [0,1] \to \ell_1[0,1]$ such that its limit set $I_{H-K}(f)$ consists exactly of one point, but this function is not Henstock-Kurzweil integrable.*

Proof. Define $f : [0,1] \to \ell_1[0,1]$ by the rule

$$f(t) = \begin{cases} e_t & \text{if } t \in A, \\ e_0 & \text{if } t \in B, \end{cases}$$

where A and B are non-measurable sets, $\lambda^*(A) = \lambda^*(B) = 1$, $A \cup B = [0,1]$ and $A \cap B = \emptyset$.

Set B obeys conditions of Proposition 5, therefore for every $\varepsilon > 0$ and for every gauge δ on $[0,1]$ there is a δ-fine tagged partition P of $[0,1]$ such that

$$||S(f,P) - e_0|| < \varepsilon.$$

On the other hand, set A also fulfils conditions of Proposition 5, and so for the same $\varepsilon > 0$ and gauge δ on $[0,1]$ there is a δ-fine tagged partition P of $[0,1]$ such that for corresponding ξ_i (almost all of which are in A) and t_i

$$||S(f,P) - e_0|| = ||\sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) - e_0|| =$$

$$= ||\sum_{i=1}^{m} e_{\xi_i}(t_{k_i} - t_{k_{i-1}}) + \sum_{i=m+1}^{n} e_0(t_{k_i} - t_{k_{i-1}}) - e_0|| > 2|1-\varepsilon|,$$

as result, e_0 is not the integral of f.

It is easy to show, that there is no other limit points of f, and we complete the proof. □

Let us recall the following definition. X has infratype p if there exists a constant $C > 0$ such that an inequality

$$\min_{\alpha_i=\pm 1} ||\sum_{i=1}^{n} \alpha_i x_i|| \le C(\sum_{i=1}^{n} ||x_i||^p)^{1/p}$$

holds for any finite collection $\{x_i\}_{i=1}^n$ of elements of X. The basic properties of spaces with infratype $p > 1$ can be found in [4, Chapter 5] .

Theorem 4 *Let $f : [0,1] \to X$, where X is a Banach space, and $f([0,1])$ is relatively compact in X, then $I_{H-K}(f)$ is convex.*

Proof. Notice that f is bounded and denote $M = \sup\{||f(t)||, t \in [0,1]\}$. By relative compactness of $K = f([0,1])$ for every $\varepsilon > 0$ there is a finite ε-net A_ε for K. Denote by Y the linear span of A_ε. Since Y is finite dimensional, it has infratype $p = 2$.

Let x_1 and x_2 be two points in $I_{H-K}(f)$. Since $I_{H-K}(f)$ is closed (see Remark (1)), it is sufficient to show that $\frac{1}{2}(x_1 + x_2) \in I_{H-K}(f)$. To this end fix N. Since $x_1, x_2 \in I_{H-K}(f)$ then for every $\varepsilon > 0$ and for every $\gamma \in \Gamma$ there are $\gamma_1 \succ \gamma$ and $\gamma_2 \succ \gamma$ such that $||x_1 - F_f(\gamma_1)|| < \varepsilon$ and $||x_2 - F_f(\gamma_2)|| < \varepsilon$, also γ_1 and γ_1 may be chosen in such a way that points k/N, where $k = 0, 1, \dots, N$, belong to the set of endpoints of the correspondent partition. Denote by F_i^k, $i = 1, 2$, $k = 1, \dots, N$, the part of the integral sum $F_f(\gamma_i)$ corresponding to the segments of the partition that lie in $[k/N, (k+1)/N]$. Now for each of the segments $[k/N, (k+1)/N]$ we choose in arbitrarily manner either the sum F_1^k or F_2^k. After this we can formally write 2^N different integral sums of the function f in the following form:

$$F\left(\sum_{k=1}^{N} \alpha_k\right) = \sum_{k=1}^{N} \left(\frac{1+\alpha_k}{2} F_1^k + \frac{1-\alpha_k}{2} F_2^k\right),$$

where $\alpha_k = \pm 1$ are arbitrarily. Let us show that one of these sums lies close enough to $\frac{1}{2}(x_1 + x_2)$. Indeed,

$$\begin{aligned}||F\left(\sum_{k=1}^{N}\alpha_k\right) &- \frac{1}{2}(x_1+x_2)|| \le \varepsilon + ||\frac{1}{2}(F_1+F_2) - F\left(\sum_{k=1}^{N}\alpha_k\right)|| = \\ &= \varepsilon + ||\frac{1}{2}(F_1+F_2) - \sum_{k=1}^{N}\left(\frac{1+\alpha_k}{2}F_1^k + \frac{1-\alpha_k}{2}F_2^k\right)|| = \\ &= \varepsilon + \frac{1}{2}||\sum_{k=1}^{N}\alpha_k(F_1^k - F_2^k)||.\end{aligned}$$

For every element $f(\xi^i_{k_j})$ from the sums $F_i^k = \sum_{j=1}^{n_k} f(\xi^i_{k_j})(t_{k_j} - t_{k_{j-1}})$, $i = 1, 2$, there is the nearest element from ε-net, let us denote it by $g(\xi^i_{k_j})$. Then

$$\frac{1}{2}||\sum_{k=1}^{N}\alpha_k(F_1^k - F_2^k)|| \le$$

$$\le \varepsilon + \frac{1}{2}||\sum_{k=1}^{N}\alpha_k\left(\sum_{j=1}^{n_k} g(\xi^1_{k_j})(t_{k_j} - t_{k_{j-1}}) + \sum_{j=1}^{n_k} g(\xi^2_{k_j})(t_{k_j} - t_{k_{j-1}})\right)||.$$

Using this inequality and definition of infratype, we obtain the required result

$$\min_{\alpha_i = \pm 1} ||F\left(\sum_{k=1}^{N}\alpha_k\right) - \frac{1}{2}(x_1+x_2)|| \le 2\varepsilon + CN^{-1/2}M.$$

Since $\varepsilon > 0$ can be made arbitrarily small and N arbitrarily large, we see that point $\frac{1}{2}(x_1 + x_2)$ lies in the limit set of the Henstock-Kurzweil integral, which completes the proof of the lemma. □

Theorem 5 *Let $f : [0, 1] \to X$, where X is a B-convex normed space, and f is dominated by some integrable function g, then $I_{H-K}(f)$ is convex.*

Proof. Recall that B-convexity of X is equivalent to existence of some infratype $p > 1$.

Let x_1 and x_2 be two points in $I_{H-K}(f)$. Let us prove that $\frac{1}{2}(x_1 + x_2) \in I_{H-K}(f)$. To this end fix N. Since g is integrable function ($\int_0^1 g(t)dt = M$), the interval $[0, 1]$ can be divided into N parts such that $\int_{t_{i-1}}^{t_i} g(t)dt = \frac{M}{N}$, where $0 = t_0 < t_1 < \cdots < t_N = 1$. From condition $x_1, x_2 \in I_{H-K}(f)$ we obtain: for every $\varepsilon > 0$ and for every $\gamma \in \Gamma$ there are $\gamma_1 \succ \gamma$ and $\gamma_2 \succ \gamma$ such that $||x_1 - F_f(\gamma_1)|| < \varepsilon$ and $||x_2 - F_f(\gamma_2)|| < \varepsilon$, also γ_1 and γ_2 may be chosen in such a way that points t_i, where $i = 0, \ldots, N$, belong to the set of endpoints of the correspondent partition.

Further applying similar arguments as in Theorem (4), we come to the inequality

$$||F\left(\sum_{k=1}^{N}\alpha_k\right)-\frac{1}{2}(x_1+x_2)||\le\varepsilon+\frac{1}{2}||\sum_{k=1}^{N}\alpha_k(F_1^k-F_2^k)||.$$

Using definition of infratype and taking into account that g dominates f, we obtain

$$\min_{\alpha_i=\pm1}||F\left(\sum_{k=1}^{N}\alpha_k\right)-\frac{1}{2}(x_1+x_2)||\le\varepsilon+CN^{1/p-1}M.$$

Since $\varepsilon>0$ can be made arbitrarily small and N arbitrarily large, we see that $\frac{1}{2}(x_1+x_2)\in I_{H-K}(f)$, which was to be proved. □

Remark, that a function with $I_{H-K}(f)\ne\emptyset$ (and even a Henstock-Kurzweil-integrable function) does not necessarily have an integrable majorant. Moreover there is no any restrictions on the behaviour of the function $||f(t)||$ for a Henstock-Kurzweil-integrable f, as the following proposition shows

Proposition 6 *Let $f:[0,1]\to\mathbb{R}^+$. Then there is Henstock-Kurzweil-integrable function $g:[0,1]\to\ell_\infty[0,1]$ such that $||g(t)||=f(t)$ for every $t\in[0,1]$.*

Proof. Define $g:[0,1]\to\ell_\infty[0,1]$ by the rule $g(t)=f(t)e_t$. It is obvious that $||g(t)||=f(t)$ for every $t\in[0,1]$.

Let us prove that $\int_0^1 g(t)dt=0$. Fix $\varepsilon>0$ and define gauge by the rule $\delta(t)=\frac{\varepsilon}{2(f(t)+1)}$. Choose intervals $[t_{k-1},t_k]$, $0=t_0<t_1<\dots<t_n=1$, in such a way that $[t_{k-1},t_k]\subset(\xi_k-\delta(\xi_k),\xi_k+\delta(\xi_k))$, where $\xi_k\in[t_{k-1},t_k]$. Then

$$||S(P,g)||=||\sum_{k=1}^{n}g(\xi_k)(t_k-t_{k-1})||=$$

$$=\max_k\{f(\xi_k)(t_k-t_{k-1})\}\le\max_k\frac{\varepsilon f(\xi_k)}{f(\xi_k)+1}<\varepsilon,$$

so $g(t)$ is integrable and $\int_0^1 g(t)dt=0$. □

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