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On the distribution of stresses in circular infinite cylinder with cylindrical cavities

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Nonaxisymmetric boundary value problem of elasticity theory for multiconnected body with cylindrical boundaries is investigated. The solution is constructed as a superposition of basic solutions of the Lame equation in coordinate systems related to the centers of body boundary surfaces. Boundary conditions are satisfied exactly by the generalized Fourier method. The numerical analysis of stresses in the areas of their greatest concentration is carried out.
Keywords: multiconnected body, cylindrical boundaries, generalized Fourier method, stress concentration.

Николаев А. Г., Танчик Е. А., **О распределении напряжений в круговом бесконечном цилиндре с цилиндрическими полостями.** Исследована неосесимметричная краевая задача теории упругости для многосвязного тела с цилиндрическими границами. Решение строится в виде суперпозиции базисных решений уравнения Ламе в системах координат, отнесенных к центрам граничных поверхностей тела. Граничные условия удовлетворяются точно обобщенным методом Фурье. Проведен численный анализ напряжений в зонах их наибольшей концентрации.
Ключевые слова: многосвязное тело, цилиндрические границы, обобщенный метод Фурье, концентрация напряжений

Ніколаєв О. Г., Танчік Є. А., **Про розподіл напружень в круговому нескінченному циліндрі з циліндричними порожнинами.** Досліджено неосесиметричну крайову задачу теорії пружності для багатозв'язного тіла з циліндричними границями. Розв'язок будується у вигляді суперпозиції базисних розв'язків рівняння Ламе в системах координат, віднесених до центрів граничних поверхонь тіла. Граничні умови задовольняються точно узагальненим методом Фур'є. Проведено чисельний аналіз напружень в зонах їх найбільшої концентрації.
Ключові слова: багатозв'язне тіло, циліндричні границі, узагальнений метод Фур'є, концентрація напружень

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Introduction

Boundary value problems of the theory of elasticity for an infinite cylinder had been considered in classical works [1 – 4]. Their solutions had been obtained by the authors using the Fourier method. Problem for semi-infinite and finite cylinders had been investigated in articles [5 – 8] using various modifications of the Fourier method.

A method for determining the stress state of a finite cylinder, based on the principle of superposition and the expansion of the stress tensor in the Fourier and Bessel – Dini series was proposed in papers [9, 10]. The problem was reduced to an infinite system of linear algebraic equations.

The behavior of solutions for boundary value problems of elasticity theory for a space with a thin cylindrical inclusion is analyzed by asymptotic methods in article [11].

Several studies of the stress-strain state of elastic space in the neighborhood of a cylindrical cavity or inclusion has been associated with the construction of models of fibrous porous or composite materials. Typically, these models have the simple structure in the form of a cylindrical cavity or inclusions or two coaxial cylinders. An example of such research is the work [12]. In the case of plane stress analysis of stresses in multiply fiber composite was carried out using the theory of functions of complex variable at work [13].

Solutions of thermoelasticity problems for an infinite cylinder were considered in the works [15, 14]. Solutions are constructed as power series expansions, Fourier series, Fourier – Bessel series in these papers.

Transversely isotropic rod with a cylindrical inclusion with axisymmetric own deformations was studied in paper [16]. An analytical solution for the elastic displacements, stresses and elastic energy of the rod were obtained.

The distribution of stresses in a cylinder with two cylindrical cavities or inclusions was investigated in works [17, 18]. In these papers stresses are determined using the generalized Fourier method. Apparatus of generalized Fourier method had been developed in [19]. Its application to the doubly connected problems was given in the book [20].

It should be noted that in the scientific literature there are practically no studies on the distribution of stresses in non-axisymmetric elastic multiply connected bodies with non-compact boundary.

Analytical-numerical solution of the non-axisymmetric boundary value problem of elasticity theory for multi-body in the form of a cylinder with a cylindrical cavities is presented in this paper. The solution is constructed as a superposition of the exact basis solutions of the Lame equation for the cylinder in the coordinates systems related to the centers of the boundary surfaces of the body. The boundary conditions of the problem are satisfied exactly with the help of the apparatus of the generalized Fourier method. As a result, the original problem is reduced to an infinite system of linear algebraic equations. The theorem about Fredholm property of the system operator in Hilbert space l_2

is proved for this class of problems for the first time. Resolving system is solved numerically by the reduction method. The practical speed of convergence of the reduction method is investigated. The numerical analysis of stresses in the areas of their greatest concentration is carried out.

1 Problem statement

Let's consider an infinite elastic cylinder Ω_0 containing N cylindrical cavities Ω_j $(j = 1 \div N)$, whose axes are parallel to the cylinder axis. Denote by O_j $(j = 0 \div N)$ points belonging to the axes of the original cylinder and cavities located in the plane perpendicular to the generatrix of cylinder. Without loss of generality, assume that points O_j $(j = 1 \div N)$ form a certain structure on the plane, in particular, centered hexagonal, and the point O_1 coincides with the point O_0 (Fig. 1).

We will use cylindrical coordinates systems (ρ_j, φ_j, z_j) with the same orientation, for which origins are related with the points O_j $(j = 1 \div N)$. Radii of cylinders Ω_j are equal to R_j, boundaries of cylinders Γ_j are described by the equations $\rho_j = R_j$. It is assumed that the cavities are located within original cylinder and the boundaries do not intersect each other.

Let's consider the first boundary value problem of elasticity theory for a specified domain. It is assumed that outer boundary is under the load $\mathbf{f}(\varphi_0, z_0)$, which has an absolutely and uniformly convergent series expansion and integral representation

$$\mathbf{f}(\varphi_0, z_0) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \Big[f_{x,m}(\lambda)\mathbf{e}_x + f_{y,m}(\lambda)\mathbf{e}_y + f_{z,m}(\lambda)\mathbf{e}_z \Big] e^{i\lambda z + im\varphi} d\lambda. \tag{1}$$

where $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ — are unit vectors of the Cartesian coordinate system, which are co-directional with inserting cylindrical coordinates systems.

It is considered that the vector function $\mathbf{f}$ satisfies static conditions on the surface $\rho_1 = R_0$.

Elastic displacement vector satisfies the following boundary value problem for the Lame equation:

$$\nabla^2 \mathbf{U} + \frac{1}{1 - 2\sigma} \nabla \mathrm{div} \mathbf{U} = 0, \tag{2}$$

with boundary conditions: on the outer boundary

$$\mathbf{FU}|_{\Gamma_0} = \mathbf{f}(\varphi_0, z_0), \tag{3}$$

and on the cavities boundaries

$$\mathbf{FU}|_{\Gamma_j} = 0, \tag{4}$$

where $\mathbf{U}$ — displacement vector, $\mathbf{FU}$ — corresponding stress vector on the boundary surface, σ — Poisson's ratio.

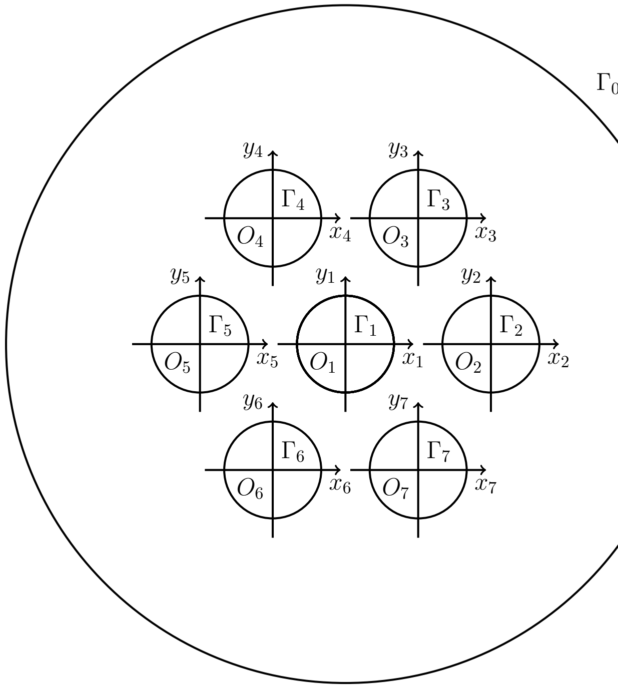

Fig. 1: Schematic representation of the problem

2 Solution of the problem

General solution of the boundary value problem (2) — (4) in the domain $\Omega_0 \setminus \bigcup\limits_{j=1}^{N} \Omega_j$ constructed as a superposition of basic solutions of the Lame equation for cylinder in the coordinate system related to the centers of cavities

$$\mathbf{U} = \sum_{j=1}^{N} \sum_{s=1}^{3} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} A_{s,m}^{(j)}(\lambda) \mathbf{U}_{s,\lambda,m}^{+(3)}(\rho_j, \varphi_j, z_j) d\lambda + \\ + \sum_{s=1}^{3} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} A_{s,m}^{(0)}(\lambda) \mathbf{U}_{s,\lambda,m}^{-(3)}(\rho_0, \varphi_0, z_0) d\lambda, \quad (5)$$

where $A_{s,m}^{(j)}(\lambda)$ — unknown functions to be determined; $\mathbf{U}_{s,\lambda,m}^{\pm(3)}(\rho, \varphi, z)$ — basic solutions of the Lame equation for the cylinder (sign + (−) matches the external (internal) solution) were introduced in [19]. In the article [21] was introduced the concept of a basis system of solutions of the Lame equation and proved basis property of systems $\left\{\mathbf{U}_{s,\lambda,m}^{+(3)}\right\}_{s=1,m=-\infty,\lambda=-\infty}^{3,\infty,\infty}$, $\left\{\mathbf{U}_{s,\lambda,m}^{-(3)}\right\}_{s=1,m=-\infty,\lambda=-\infty}^{3,\infty,\infty}$.

Let's consider the explicit form of these solutions

$$\mathbf{U}_{s,\lambda,m}^{\pm(3)}(\rho,\varphi,z) = \lambda^{-1}\mathbf{D}_s u_{\lambda,m}^{\pm(3)}(\rho,\varphi,z); \quad s=1,3; \tag{6}$$

$$\mathbf{U}_{2,\lambda,m}^{\pm(3)}(\rho,\varphi,z) = \lambda^{-1}\mathbf{B}_2 u_{\lambda,m}^{\pm(3)}(\rho,\varphi,z), \tag{7}$$

where

$$\mathbf{D}_1 = \nabla, \quad \mathbf{D}_2 = z\nabla - \chi\mathbf{e}_z, \quad \mathbf{D}_3 = i[\nabla\times\mathbf{e}_z],$$

$$\mathbf{B}_2 = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\nabla - \chi\Big[\mathbf{e}_z\times[\nabla\times\mathbf{e}_z]\Big];$$

$$u_{\lambda,m}^{+(3)}(\rho,\varphi,z) = e^{i\lambda z+im\varphi}\tilde{K}_m(\lambda\rho), \quad u_{\lambda,m}^{-(3)}(\rho,\varphi,z) = e^{i\lambda z+im\varphi}I_m(\lambda\rho),$$

$I_m(x)$ — modified Bessel function, $\tilde{K}_m(x) = (\mathrm{sign}x)^m K_m(|x|)$, $K_m(x)$ — Macdonald function; $\chi = 3-4\sigma$, $u_{\lambda,m}^{\pm(3)}$ — complete set of particular solutions of the Laplace equation in cylindrical coordinates, i — imaginary unit.

In the expanded coordinate form basic solutions (6), (7) are of the form:

$$\mathbf{U}_{1,\lambda,m}^{\pm(3)}(\rho,\varphi,z) = \mp u_{\lambda,m-1}^{\pm(3)}\mathbf{e}_{-1} \mp u_{\lambda,m+1}^{\pm(3)}\mathbf{e}_1 + iu_{\lambda,m}^{\pm(3)}\mathbf{e}_0, \tag{8}$$

$$\mathbf{U}_{2,\lambda,m}^{\pm(3)}(\rho,\varphi,z) = \mp(D-\chi)\left[u_{\lambda,m-1}^{\pm(3)}\mathbf{e}_{-1} + u_{\lambda,m+1}^{\pm(3)}\mathbf{e}_1\right] + iDu_{\lambda,m}^{\pm(3)}\mathbf{e}_0, \tag{9}$$

$$\mathbf{U}_{3,\lambda,m}^{\pm(3)}(\rho,\varphi,z) = \pm u_{\lambda,m-1}^{\pm(3)}\mathbf{e}_{-1} \mp u_{\lambda,m+1}^{\pm(3)}\mathbf{e}_1, \tag{10}$$

где $D = \rho\dfrac{\partial}{\partial\rho}$, $\mathbf{e}_{-1} = \dfrac{1}{2}(\mathbf{e}_x + i\mathbf{e}_y)$, $\mathbf{e}_1 = \dfrac{1}{2}(\mathbf{e}_x - i\mathbf{e}_y)$, $\mathbf{e}_0 = \mathbf{e}_z$.

Stress vector at the site with the normal $\mathbf{n}$ has the form:

$$\mathbf{F}\mathbf{U} = 2G\left[\frac{\sigma}{1-2\sigma}\mathbf{n}\,\mathrm{div}\mathbf{U} + \frac{\partial\mathbf{U}}{\partial\mathbf{n}} + \frac{1}{2}(\mathbf{n}\times\mathrm{rot}\mathbf{U})\right], \tag{11}$$

where G — shear modulus.

Applying the formulas (8) — (10) operator (11) at the site with the normal $\mathbf{n} = \mathbf{e}_\rho$ we obtain:

$$\mathbf{F}\mathbf{U}_{1,\lambda,m}^{\pm(3)} = \frac{2G}{\rho}\left\{\mp Du_{\lambda,m-1}^{\pm(3)}\mathbf{e}_{-1} \mp Du_{\lambda,m+1}^{\pm(3)}\mathbf{e}_1 + iDu_{\lambda,m}^{\pm(3)}\mathbf{e}_0\right\}; \tag{12}$$

$$\begin{aligned}\mathbf{F}\mathbf{U}_{2,\lambda,m}^{\pm(3)} = \frac{2G}{\rho}\Big\{ &\mp[(m-1)(m-1+2\sigma) + \lambda^2\rho^2 + (2\sigma-3)D]u_{\lambda,m-1}^{\pm(3)}\mathbf{e}_{-1}\mp \\ &\mp[(m+1)(m+1-2\sigma) + \lambda^2\rho^2 + (2\sigma-3)D]u_{\lambda,m+1}^{\pm(3)}\mathbf{e}_1 + \\ &i[m^2 + \lambda^2\rho^2(2\sigma-2)D]u_{\lambda,m}^{\pm(3)}\mathbf{e}_0\Big\},\end{aligned} \tag{13}$$

$$\mathbf{F}\mathbf{U}_{3,\lambda,m}^{\pm(3)} = \frac{G}{\rho}\left\{\pm(D+m-1)u_{\lambda,m-1}^{\pm(3)}\mathbf{e}_{-1} \mp (D-m-1)u_{\lambda,m+1}^{\pm(3)}\mathbf{e}_1 - imu_{\lambda,m}^{\pm(3)}\mathbf{e}_0\right\}, \tag{14}$$

3 Addition theorems

Basic solutions of the Lame equation in coordinate systems, combined with the centers pair of cylinders, are associated by the addition theorems. The following addition theorems are the case [19]:

$$\mathbf{U}^{+(3)}_{s,\lambda,m}(\rho_j,\varphi_j,z_j)=\sum_{t=1}^{3}\sum_{l=-\infty}^{\infty}\tilde{D}^{(j\alpha)}_{s,t}f^{(33)l,j,\alpha}_{1,\lambda,m}\mathbf{U}^{-(3)}_{t,\lambda,l}(\rho_\alpha,\varphi_\alpha,z_\alpha); \qquad (15)$$

$$\mathbf{U}^{+(3)}_{s,\lambda,m}(\rho_j,\varphi_j,z_j)=\sum_{t=1}^{3}\sum_{l=-\infty}^{\infty}\tilde{D}^{(j\alpha)}_{s,t}f^{+(33)l,j,\alpha}_{\lambda,m}\mathbf{U}^{+(3)}_{t,\lambda,l}(\rho_\alpha,\varphi_\alpha,z_\alpha); \qquad (16)$$

$$\mathbf{U}^{-(3)}_{s,\lambda,m}(\rho_j,\varphi_j,z_j)=\sum_{t=1}^{3}\sum_{l=-\infty}^{\infty}\tilde{D}^{(j\alpha)}_{s,t}f^{-(33)l,j,\alpha}_{\lambda,m}\mathbf{U}^{-(3)}_{t,\lambda,l}(\rho_\alpha,\varphi_\alpha,z_\alpha), \qquad (17)$$

$$f^{(33)l,j,\alpha}_{1,\lambda,m}=(-1)^l u^{+(3)}_{\lambda,m-l}(\rho_{j\alpha},\varphi_{j\alpha},z_{j\alpha}),\quad f^{\pm(33)l,j,\alpha}_{\lambda,m}=u^{-(3)}_{\lambda,m-l}(\rho_{j\alpha},\varphi_{j\alpha},z_{j\alpha}),$$

$$\tilde{D}^{(j\alpha)}_{s,t}=\left[\delta_{st}+\delta_{t1}\delta_{s2}\rho_{j\alpha}\frac{\partial}{\partial\rho_{j\alpha}}\right],\quad j\neq\alpha,$$

where $(\rho_{j\alpha},\varphi_{j\alpha},z_{j\alpha})$ — cylindrical coordinates of the point O_α in the coordinate system (ρ_j,φ_j,z_j), δ_{st} — Kronecker delta.

4 Resolving system of equations

Using the addition theorems (15) — (17), we represent the displacement vector $\mathbf{U}$ in the coordinate system with the origin at O_1 near borders Γ_0 и Γ_1:

$$\mathbf{U}=\sum_{s=1}^{3}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}A^{(1)}_{s,m}(\lambda)\mathbf{U}^{+(3)}_{s,\lambda,m}(\rho_1,\varphi_1,z_1)d\lambda+$$
$$+\sum_{s=1}^{3}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}A^{(0)}_{s,m}(\lambda)\mathbf{U}^{-(3)}_{s,\lambda,m}(\rho_1,\varphi_1,z_1)d\lambda+\sum_{j=2}^{N}\sum_{s=1}^{3}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}A^{(j)}_{s,m}(\lambda)\times$$
$$\times\sum_{t=1}^{3}\sum_{l=-\infty}^{\infty}\tilde{D}^{(j1)}_{s,t}u^{-(3)}_{\lambda,m-l}(\rho_{j1},\varphi_{j1},z_{j1})\mathbf{U}^{+(3)}_{t,\lambda,l}(\rho_1,\varphi_1,z_1)d\lambda, \qquad (18)$$

$$\mathbf{U}=\sum_{s=1}^{3}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}A_{s,m}^{(1)}(\lambda)\mathbf{U}_{s,\lambda,m}^{+(3)}(\rho_1,\varphi_1,z_1)d\lambda+$$
$$+\sum_{s=1}^{3}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}A_{s,m}^{(0)}(\lambda)\mathbf{U}_{s,\lambda,m}^{-(3)}(\rho_1,\varphi_1,z_1)d\lambda+\sum_{j=2}^{N}\sum_{s=1}^{3}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}A_{s,m}^{(j)}(\lambda)\times$$
$$\times\sum_{t=1}^{3}\sum_{l=-\infty}^{\infty}(-1)^l\tilde{D}_{s,t}^{(j1)}u_{\lambda,m-l}^{+(3)}(\rho_{j1},\varphi_{j1},z_{j1})\mathbf{U}_{t,\lambda,l}^{+(3)}(\rho_1,\varphi_1,z_1)d\lambda, \quad (19)$$

and with origin in the point O_j $(j=2\div N)$:

$$\mathbf{U}=\sum_{s=1}^{3}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}A_{s,m}^{(j)}(\lambda)\mathbf{U}_{s,\lambda,m}^{+(3)}(\rho_j,\varphi_j,z_j)d\lambda+\sum_{\alpha=1,\alpha\neq j}^{N}\sum_{s=1}^{3}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}A_{s,m}^{(\alpha)}(\lambda)\times$$
$$\times\sum_{t=1}^{3}\sum_{l=-\infty}^{\infty}(-1)^l\tilde{D}_{s,t}^{(\alpha j)}u_{\lambda,m-l}^{+(3)}(\rho_{\alpha j},\varphi_{\alpha j},z_{\alpha j})\mathbf{U}_{t,\lambda,l}^{-(3)}(\rho_j,\varphi_j,z_j)d\lambda+$$
$$+\sum_{s=1}^{3}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}A_{s,m}^{(0)}(\lambda)\sum_{t=1}^{3}\sum_{l=-\infty}^{\infty}\tilde{D}_{s,t}^{(1j)}u_{\lambda,m-l}^{-(3)}(\rho_{1j},\varphi_{1j},z_{1j})\mathbf{U}_{t,\lambda,l}^{-(3)}(\rho_j,\varphi_j,z_j)d\lambda. \quad (20)$$

After satisfaction of the boundary conditions, the problem reduces to an infinite system of linear algebraic equations for the unknown coefficients $A_{s,m}^{(j)}(\lambda)$:

$$\sum_{s=1}^{3}\Bigg\{A_{s,m}^{(0)}(\lambda)\mathbf{G}_{s,\lambda,m}^{-(3)}(R_0)+A_{s,m}^{(1)}(\lambda)\mathbf{G}_{s,\lambda,m}^{+(3)}(R_0)+$$
$$+\mathbf{G}_{s,\lambda,m}^{+(3)}(R_0)\sum_{j=2}^{N}\sum_{t=1}^{3}\sum_{l=-\infty}^{\infty}A_{t,l}^{(j)}(\lambda)\tilde{D}_{t,s}^{(j1)}u_{\lambda,l-m}^{-(3)}(\rho_{j1},\varphi_{j1},z_{j1})\Bigg\}=$$
$$=\Big(f_{x,m-1}(\lambda)-if_{y,m-1}(\lambda),f_{x,m+1}(\lambda)+if_{y,m+1}(\lambda),f_{z,m}(\lambda)\Big), \quad (21)$$

$$\sum_{s=1}^{3}\Bigg\{A_{s,m}^{(0)}(\lambda)\mathbf{G}_{s,\lambda,m}^{-(3)}(R_1)+A_{s,m}^{(1)}(\lambda)\mathbf{G}_{s,\lambda,m}^{+(3)}(R_1)+$$
$$+\mathbf{G}_{s,\lambda,m}^{-(3)}(R_1)\sum_{j=2}^{N}\sum_{t=1}^{3}\sum_{l=-\infty}^{\infty}A_{t,l}^{(j)}(\lambda)(-1)^m\tilde{D}_{t,s}^{(j1)}u_{\lambda,l-m}^{+(3)}(\rho_{j1},\varphi_{j1},z_{j1})\Bigg\}=0, \quad (22)$$

$$\sum_{s=1}^{3}\Bigg\{A_{s,m}^{(j)}(\lambda)\mathbf{G}_{s,\lambda,m}^{+(3)}(R_j)+$$
$$+\mathbf{G}_{s,\lambda,m}^{-(3)}(R_j)\sum_{\alpha=1,\alpha\neq j}^{N}\sum_{t=1}^{3}\sum_{-\infty}^{\infty}A_{t,l}^{(\alpha)}(\lambda)(-1)^m\tilde{D}_{t,s}^{(\alpha j)}u_{\lambda,l-m}^{+(3)}(\rho_{\alpha j},\varphi_{\alpha j},z_{\alpha j})+$$
$$+\mathbf{G}_{s,\lambda,m}^{-(3)}(R_j)\sum_{t=1}^{3}\sum_{l=-\infty}^{\infty}\tilde{D}_{t,s}^{(1j)}u_{\lambda,l-m}^{-(3)}(\rho_{1j},\varphi_{1j},z_{1j})\Bigg\}=0, \quad (23)$$

$$j=2\div N;\quad m\in\mathbb{Z};\quad \lambda\in\mathbb{R},\quad \lambda\neq 0,$$

где $\mathbf{G}_{s,\lambda,m}^{\pm(3)}(R)=\left(G_{s,\lambda,m}^{\pm(-1)},G_{s,\lambda,m}^{\pm(1)},G_{s,\lambda,m}^{\pm(0)}\right)$;

$$G_{1,\lambda,m}^{\pm(-1)}(R)=\mp\frac{2G}{R}D\tilde{u}_{\lambda,m-1}^{\pm(3)}(R),\quad G_{1,\lambda,m}^{\pm(1)}(R)=\pm\frac{2G}{R}D\tilde{u}_{\lambda,m+1}^{\pm(3)}(R),$$

$$G_{1,\lambda,m}^{\pm(0)}(R)=\frac{2G}{R}iD\tilde{u}_{\lambda,m}^{\pm(3)}(R),\quad G_{3,\lambda,m}^{\pm(0)}(R)=-\frac{G}{R}im\tilde{u}_{\lambda,m}^{\pm(3)}(R),$$

$$G_{3,\lambda,m}^{\pm(1)}(R)=\mp\frac{G}{R}(D-m-1)\tilde{u}_{\lambda,m+1}^{\pm(3)}(R),$$

$$G_{3,\lambda,m}^{\pm(-1)}(R)=\pm\frac{G}{R}(D+m-1)\tilde{u}_{\lambda,m-1}^{\pm(3)}(R),$$

$$G_{2,\lambda,m}^{\pm(0)}(R)=\frac{2G}{R}i\left[m^2+\lambda^2R^2+(2\sigma-2)D\right]\tilde{u}_{\lambda,m}^{\pm(3)}(R),$$

$$G_{2,\lambda,m}^{\pm(-1)}(R)=\mp\frac{2G}{R}\left[(m-1)(m-1+2\sigma)+\lambda^2R^2+(2\sigma-3)D\right]\tilde{u}_{\lambda,m-1}^{\pm(3)}(R),$$

$$G_{2,\lambda,m}^{\pm(1)}(R)=\mp\frac{2G}{R}\left[(m+1)(m+1-2\sigma)+\lambda^2R^2+(2\sigma-3)D\right]\tilde{u}_{\lambda,m+1}^{\pm(3)}(R),$$

$$\tilde{u}_{\lambda,m}^{+(3)}(R)=\tilde{K}_m(\lambda R),\quad \tilde{u}_{\lambda,m}^{-(3)}(R)=I_m(\lambda R).$$

5 Analysis of the resolving system

Theorem 1 *For each $\lambda\neq 0$ system operator* (21) – (23) *is Fredholm in the Hilbert space l_2 under the conditions $R_j+R_\alpha<\rho_{j\alpha}$ ($j\neq\alpha$; $j,\alpha=1\div N$), $\rho_{1\alpha}+R_\alpha<R_0$ ($\alpha=2\div N$).*

By renaming the unknown functions

$$A_{s,m}^{(j)}(\lambda)=\frac{\tilde{A}_{s,m}^{(j)}(\lambda)}{K_m(|\lambda|R_j)},\quad (j=1\div N),\quad A_{s,m}^{(0)}(\lambda)=\frac{\tilde{A}_{s,m}^{(0)}(\lambda)}{I_m(\lambda R_0)}, \qquad (24)$$

and solving system with respect to $\tilde{A}^{(j)}_{s,m}(\lambda)$, system (21), (23) can be represented in the form

$$\tilde{A}^{(\alpha)}_{s,m}(\lambda)+\sum_{j\neq\alpha}\sum_{p=1}^{3}\sum_{l=-\infty}^{\infty}T^{j,p,l}_{1,\alpha,s,m}\tilde{A}^{(j)}_{p,l}(\lambda)+\sum_{p=1}^{3}\sum_{l=-\infty}^{\infty}T^{p,l}_{2,\alpha,s,m}\tilde{A}^{(0)}_{p,l}(\lambda)=0, \qquad (25)$$

$$\tilde{A}^{(0)}_{s,m}(\lambda)+\sum_{j=1}^{N}\sum_{p=1}^{3}\sum_{l=-\infty}^{\infty}T^{j,p,l}_{3,s,m}\tilde{A}^{(j)}_{p,l}(\lambda)=F_{s,m}(\lambda). \qquad (26)$$

Omit the explicit entry of matrix coefficients. Note that matrix coefficients modules $\left|T^{j,p,l}_{1,\alpha,s,m}\right|$, $\left|T^{p,l}_{2,\alpha,s,m}\right|$, $\left|T^{j,p,l}_{3,s,m}\right|$ estimated from above by finite linear combinations of expressions like (27) — (29) respectively

$$\left|\frac{I_m(\lambda R_\alpha)}{K_l(|\lambda|R_j)}K_{m-l}(|\lambda|\rho_{j\alpha})\right|, \qquad (27)$$

$$\left|\frac{I_m(\lambda R_\alpha)}{I_l(\lambda R_0)}I_{m-l}(|\lambda|\rho_{1\alpha})\right|, \qquad (28)$$

$$\left|\frac{K_m(|\lambda|R_0)}{K_l(|\lambda|R_j)}I_{m-l}(\lambda\rho_{j1})\right|. \qquad (29)$$

Here were used the estimates of resolving systems determinants of the first boundary value problem of elasticity theory for interior and exterior of the cylinder, which were derived in the work [21].

To prove the theorem it is sufficient to show the fulfillment of following conditions for matrix coefficients of the system (25), (26):

$$\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\left|T^{j,p,l}_{1,\alpha,s,m}\right|^2<\infty, \qquad (30)$$

$$\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\left|T^{p,l}_{2,\alpha,s,m}\right|^2<\infty, \qquad (31)$$

$$\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\left|T^{j,p,l}_{3,s,m}\right|^2<\infty. \qquad (32)$$

Let's consider the addition theorem for harmonic functions [20]:

$$u^{+(3)}_{\lambda,m}(\rho_j,\varphi_j,z_j)=\sum_{l=-\infty}^{\infty}(-1)^l u^{+(3)}_{\lambda,m-l}(\rho_{j\alpha},\varphi_{j\alpha},z_{j\alpha})u^{-(3)}_{\lambda,l}(\rho_\alpha,\varphi_\alpha,z_\alpha). \qquad (33)$$

This expansion can be interpreted as Fourier series representation of the harmonic function $u^{+(3)}_{\lambda,m}(\rho_j,\varphi_j,z_j)$ the variable $\varphi_\alpha\in[0,2\pi]$. Then for this expansion Parseval equality has a place by

$$\sum_{l=-\infty}^{\infty}\left|K_{m-l}(|\lambda|\rho_{j\alpha})\right|^2\left|I_l(\lambda\rho_\alpha)\right|^2=\frac{1}{2\pi}\int_0^{2\pi}\left|K_m(|\lambda|\rho_j)\right|^2 d\varphi_\alpha. \tag{34}$$

By the estimates (27) — (29) to prove the theorem it is sufficient to show convergence of the series

$$\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\left|\frac{I_m(\lambda R_\alpha)}{K_l(|\lambda|R_j)}K_{m-l}(|\lambda|\rho_{j\alpha})\right|^2, \tag{35}$$

$$\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\left|\frac{I_m(\lambda R_\alpha)}{I_l(\lambda R_0)}I_{m-l}(|\lambda|\rho_{1\alpha})\right|^2, \tag{36}$$

$$\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\left|\frac{K_m(|\lambda|R_0)}{K_l(|\lambda|R_j)}I_{m-l}(\lambda\rho_{j1})\right|^2. \tag{37}$$

In the work [21] was proved the estimate

$$I_m(z)K_m(z)>\frac{c}{m^2+1}(1+2z)^{-1},\quad m\geq 0,\quad z>0, \tag{38}$$

where $c>0$ — some constant. Then the series (35) can be majorized by the series

$$\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\left|I_l(\lambda R_\alpha)I_m(\lambda R_j)K_{m-l}(|\lambda|\rho_{j\alpha})\right|^2.$$

Let the value $\rho_\alpha=R_\alpha$ be substituted in the identity (34), then multiply its both sides by $|I_m(\lambda R_j)|^2$ and sum up by m from $-\infty$ to ∞. As a result, we obtain

$$\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\left|I_l(\lambda R_\alpha)I_m(\lambda R_j)K_{m-l}(|\lambda|\rho_{j\alpha})\right|^2=$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\sum_{m=0}^{\infty}\left|I_m(\lambda R_j)\right|^2\left|K_m(|\lambda|\rho_j)\right|^2_{\big|_{\rho_\alpha=R_\alpha}}d\varphi_\alpha. \tag{39}$$

From the asymptotic formulas as $m\to\infty$ [22]

$$I_m(z)=\left(\frac{z}{2}\right)^m\frac{1}{m!}\Big[1+O(m^{-1})\Big], \tag{40}$$

$$K_m(z)=\frac{2^{m-1}(m-1)!}{z^m}\Big[1+O(m^{-1})\Big] \tag{41}$$

follows that the series in the left side of (39) is convergent under condition $\rho_j>R_j$. Let's define minimal value ρ_j^{min} for arbitrary values of the angle φ_α. From the dependence between cylindrical coordinates with the origins O_j and O_α it follows that for $\rho_\alpha = R_\alpha$

$$\rho_j = \sqrt{\rho_{j\alpha}^2 + R_\alpha^2 + 2\rho_{j\alpha} R_\alpha \cos(\varphi_\alpha - \varphi_{j\alpha})}$$

and minimal value ρ_j is reached under condition $\varphi_\alpha - \varphi_{j\alpha} = \pi$ and is equal to $\rho_j^{min} = \rho_{j\alpha} - R_\alpha$ ($\rho_{j\alpha} > R_\alpha$ — natural geometrical condition in the problem statement).

Thus the condition of series convergence will be satisfied if $\rho_j^{min} > R_j$. It means that $R_j + R_\alpha < \rho_{j\alpha}$.

Similarly, we can write this equality

$$\sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \left| K_m(|\lambda| R_0) I_l(\lambda R_\alpha) I_{m-l}(\lambda \rho_{1\alpha}) \right|^2 =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \sum_{m=0}^{\infty} \left| K_m(|\lambda| R_0) \right|^2 \left| I_m(\lambda \rho_1) \right|^2_{|\rho_\alpha = R_\alpha} d\varphi_\alpha. \quad (42)$$

On the basis of asymptotics (40), (41) the series in formula (42) is convergent under condition $\rho_1 < R_0$. On the surface $\rho_\alpha = R_\alpha$ it is true $\rho_1^{max} = \rho_{1\alpha} + R_\alpha$. Thus condition of convergence of the series (42) is truthfulness of inequality $\rho_1^{max} < R_0$ or $\rho_{1\alpha} + R_\alpha < R_0$.

By the estimate (38) convergence of the series (37) under condition $\rho_{1j} + R_j < R_0$ follows from convergence of the series (42).

6 Analysis of numerical results

For numerical implementation we assume that the boundary of cylinder is under piecewise constant normal load.

$$\mathbf{FU}|_{\Gamma_0} = T\mathbf{e}_\rho = \begin{cases} T, & |z| \le h, \\ 0, & |z| > h. \end{cases} \quad (43)$$

Following values of parameters were selected: $R_j = R$, $R_0 = 10R$, $\sigma = 0.38$. Centered hexagonal packing of cylindrical cavities symmetrically located with respect to the axis of cylinder is considered (fig. 1).

The system (21) — (23) is numerically solved by reduction method relatively to the parameter m ($-m_{max} \le m \le m_{max}$) with fixed values of λ, which are nodes of Gauss-Laguerre quadrature formula.

On the figures 2 — 4 graphs of stresses σ_y/T, σ_x/T, σ_z/T on the line, which connects centers of cavities 6-th and 7-th (fig. 1), depending on relative distance between cavities a/R in planes $z = 0$ and $z = h$ are shown. Relative distance between boundaries of neighbor cavities is plotted along horizontal axis. Maximal

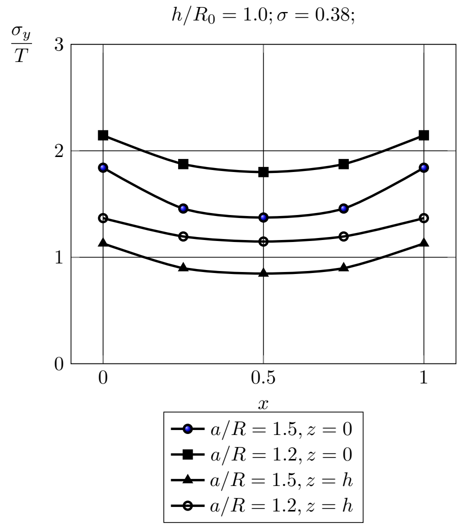

Fig. 2: Stresses σ_y/T on the line, which connects centers of cavities depending on relative distance between them

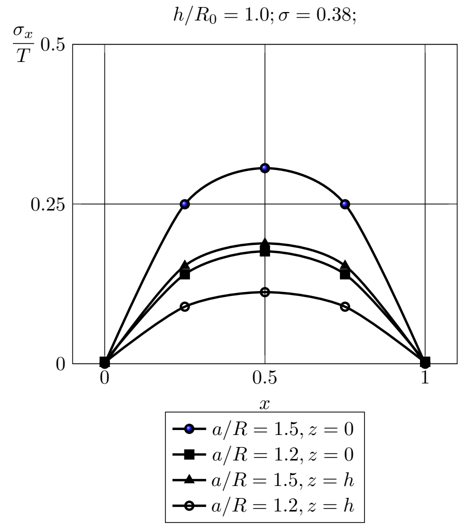

Fig. 3: Stresses σ_x/T on the line, which connects centers of cavities depending on relative distance between them

stress concentration σ_y/T is observed on the cavities boundaries, while for stresses σ_x/T is in the center of the line. It is characteristically that the signs of stresses σ_z/T are different on planes $z = 0$ and $z = h$.

On the figure 5 comparison of stresses σ_y/T on the line, which connects centers of cavities 6-th and 7-th (fig. 1), for centered hexagonal structure and on the corresponding line for centered tetragonal structure is given.

On the figures 6, 7 stresses σ_x/T and σ_y/T between 1-st and 5-th, 6-th and 7-th cavities are compared. Slight asymmetry of graphs of stresses relative to the point located in the middle between 1-st and 5-th cavities is observed.

The efficiency of proposed method can be seen by the rate of convergence of reduction method (table 1). The values of normal components of stress tensor in the middle point of the line, which connects centers of neighbor cavities, depending on size of reduced system if $a/R = 2.0$, $h/R_0 = 1.0$ are given in this table.

m_{max}	5	10	15
σ_x/T	0.506252	0.506847	0.506847
σ_y/T	1.16719	1.16695	1.16695
σ_z/T	-0.156974	-0.156848	-0.156848

Table 1: Convergence of reduction method

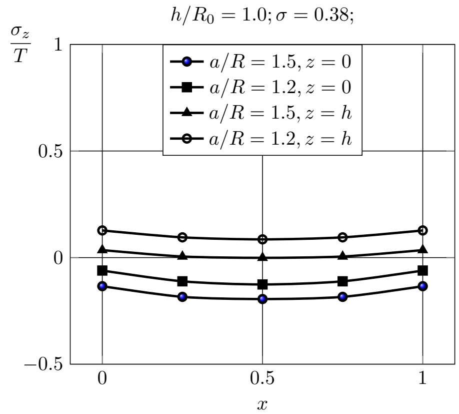

Fig. 4: Stresses σ_z/T on the line, which connects centers of cavities depending on relative distance between them

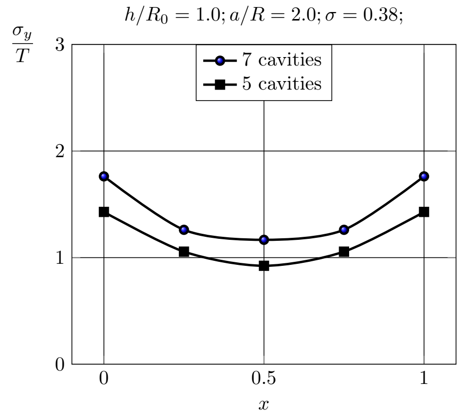

Fig. 5: Stresses σ_y/T on the line, which connects centers of cavities depending on the type of packing

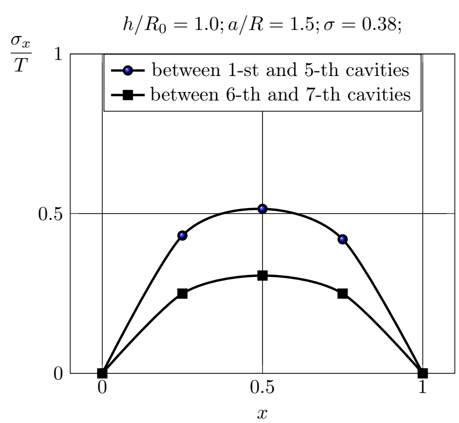

Fig. 6: Stresses σ_x/T on the line, which connects centers of 1-st and 5-th, 6-th and 7-th cavities

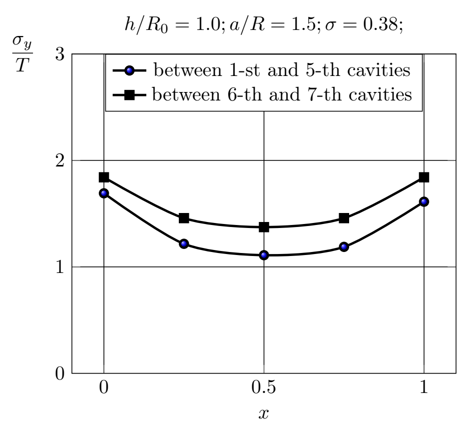

Fig. 7: Stresses σ_y/T on the line, which connects centers of 1-st and 5-th, 6-th and 7-th cavities

Conclusion

Development of analytical-numerical method of solution of boundary value problems of elasticity theory in the non-axisymmetrical multiconnected domains, which boundary is the system of unidirectional infinite circular cylinders is proposed. Solution is constructed in the form of superposition of exact basic solutions of Lame equation for cylinder in coordinate systems related to centers of boundary surfaces of the body. Boundary conditions are satisfied exactly by the generalized Fourier method apparatus. As a result initial problem is reduced to infinite linear algebraic system of equations with exponentially decreasing coefficients, which has Fredholm operator in the Hilbert space l_2. The last circumstance allows to apply reduction method for numerical solution of the system. It is well known [23] that solution of reduced system converges to exact solution of resolving system when $m_{max} \to \infty$. Practical rate of convergence of reduction method is investigated, which shows efficiency of method even for a large number of cavities. Numerical analysis of stresses in domains of their concentration is carried out. Reliability of results is proved by the comparison them for two cases: cylinder with seven and five cylindrical cavities.

Advantage of this approach is that this method allows to satisfy boundary conditions exactly reducing procedure of construction of numerical solution to inversion of linear algebraic system in contrast to well known methods, such as finite element analysis, boundary integral equations, finite differences and so on, which work poorly in domains with non-compact boundaries. Herewith approximate solution quickly converges to exact solution. This allows significantly increase accuracy of results using the same resources as in other methods.

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