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On constructing single-input non-autonomous systems of full rank

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For a nonlinear system of differential equations $\dot{x} = f(x)$, a method of constructing a system of full rank $\dot{x} = f(x)+g(x)u$ is studied for vector fields of the class C^k, $1 \le k < \infty$, in the case when $f(x) \neq 0$. A method for constructing a non-autonomous system of full rank is proposed in the case when the vector field $f(x)$ can vanish.
Keywords: nonlinear control system; accessible system; system of full rank; non-autonomous system; the straightening theorem for vector fields.

Андреєва Д. М., Ігнатович С. Ю. **Про побудову неавтономних систем повного рангу з одним керуванням.** Для нелінійної системи диференціальних рівнянь вигляду $\dot{x} = f(x)$ досліджено метод конструювання системи повного рангу $\dot{x} = f(x) + g(x)u$ для векторних полів класу C^k, $1 \le k < \infty$, у випадку, коли $f(x) \neq 0$. Запропоновано метод конструювання неавтономної системи повного рангу у випадку, коли векторне поле $f(x)$ може обертатися на нуль.
Ключові слова: нелінійна керована система; досяжна система; система повного рангу; неавтономна система; теорема про випрямлення векторного поля.

Андреева Д. Н., Игнатович С. Ю. **О построении неавтономных систем полного ранга с одним управлением.** Для нелинейной системы дифференциальных уравнений вида $\dot{x} = f(x)$ исследован метод конструирования системы полного ранга $\dot{x} = f(x) + g(x)u$ для векторных полей класса C^k, $1 \le k < \infty$, в случае, когда $f(x) \neq 0$. Предложен метод конструирования неавтономной системы полного ранга в случае, когда векторное поле $f(x)$ может обращаться в нуль.
Ключевые слова: нелинейная управляемая система; достижимая система; система полного ранга; неавтономная система; теорема о выпрямлении векторного поля.

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1. Introduction

In the paper [1] the following problem was considered. Let a system

$$\dot{x} = f(x) \tag{1}$$

be given, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a (real analytic) vector field, $n > 1$; the problem is to find conditions under which there exists a vector field $g(x) : \mathbb{R}^n \to \mathbb{R}^n$ such that the system

$$\dot{x} = f(x) + g(x)u \tag{2}$$

has full rank. Systems of full rank are also called accessible. This property for the system (2) means that the dimension of the span of all Lie brackets of vector fields $\mathrm{ad}_f^k g(x)$, $k \geq 0$, equals n.

The important result was obtained in [1], which implies that *an accessible system (2) can be constructed if and only if* $f(x) \neq 0$. Moreover, a method for constructing a system of full rank was proposed, which was based on the straightening theorem for vector fields [2].

Let us recall the main points of the construction [1]. Suppose $f(x) \neq 0$. According to the straightening theorem, there exist local coordinates $z_1, \ldots, z_n$ in which the vector field f takes the form $f = (0, \ldots, 0, 1)^T$, i.e., in the new coordinates the system (1) has the form

$$\dot{z}_1 = 0,\ \ldots,\ \dot{z}_{n-1} = 0,\ \dot{z}_n = 1. \tag{3}$$

Without loss of generality assume $f_n(x) \neq 0$; then a straightening diffeomorphism $z = \eta(x) = (\eta_1(x), \ldots, \eta_n(x))^T$ can be found from the system of partial differential equations

$$\begin{cases} \frac{\partial \eta_i}{\partial x_n} = -\frac{1}{f_n}\left(\frac{\partial \eta_i}{\partial x_1} f_1 + \ldots + \frac{\partial \eta_i}{\partial x_{n-1}} f_{n-1}\right),\ i = \overline{1, n-1}, \\ \frac{\partial \eta_n}{\partial x_n} = \frac{1}{f_n}\left(1 - \frac{\partial \eta_n}{\partial x_1} f_1 - \ldots - \frac{\partial \eta_n}{\partial x_{n-1}} f_{n-1}\right). \end{cases} \tag{4}$$

Then, let us apply the polynomial transformation $\xi = \psi(z)$ of the form

$$\begin{cases} \xi_1 = \frac{z_1}{0!} + \frac{z_2 z_n}{1!} + \frac{z_3 z_n^2}{2!} + \cdots + \frac{z_{n-1} z_n^{n-2}}{(n-2)!} + \frac{z_n^n}{n!}, \\ \xi_2 = \frac{z_2}{0!} + \frac{z_3 z_n}{1!} + \frac{z_4 z_n^2}{2!} + \cdots + \frac{z_{n-1} z_n^{n-3}}{(n-3)!} + \frac{z_n^{n-1}}{(n-1)!}, \\ \ldots \\ \xi_{n-1} = \frac{z_{n-1}}{0!} + \frac{z_n^2}{2!}, \\ \xi_n = \frac{z_n}{1!}, \end{cases} \tag{5}$$

and obtain the system (1) in ξ-coordinates:

$$\dot{\xi}_1 = \xi_2,\ \dot{\xi}_2 = \xi_3,\ \ldots,\ \dot{\xi}_{n-1} = \xi_n,\ \dot{\xi}_n = 1. \tag{6}$$

Now, we choose a vector field $g(\xi) : \mathbb{R}^n \to \mathbb{R}^n$ in the form $g(\xi) = (0, \ldots, 0, 1)^T$; then we obtain a linear system of full rank

$$\dot{\xi}_1 = \xi_2,\ \dot{\xi}_2 = \xi_3,\ \ldots,\ \dot{\xi}_{n-1} = \xi_n,\ \dot{\xi}_n = 1 + u. \tag{7}$$

Finally, by the inverse coordinate transformation

$$x = \eta^{-1}(\psi^{-1}(\xi)), \tag{8}$$

the system (7) is transformed to the form (2). Since the full rank property is invariant under the transformation (8), the obtained system (2) has full rank.

Due to using the straightening theorem, this construction is local, that is, a vector field $g(x)$ is built in a neighborhood of any point where $f(x) \neq 0$.

In the present paper we consider two generalizations. First, we consider the case when the vector field $f(x)$ belongs to the class C^k and find out what class of smoothness is guaranteed for a vector field $g(x)$. In particular, this question is related to the linearizability of control systems in the class C^1 [3]. Second, we consider the case when the vector field $f(x)$ can vanish and propose a method of constructing a non-autonomous vector field $g(t, x)$.

2. The case of a vector field $f(x)$ from the class C^k

Let us consider a vector field $f(x)$ which is finitely many times differentiable in a certain domain,

$$f(x) \in C^k(Q), \tag{9}$$

where $1 \le k < \infty$, and analyze the class of smoothness of transformations in the method of [1] recalled above.

Theorem 1 *Let a vector field $f(x) \in C^k(Q)$ be given, $Q \subset \mathbb{R}^n$, $n > 1$, and $f(x_0) \neq 0$ where $x_0 \in Q$. There exists a neighborhood $U(x_0)$ and a vector field $g(x) \in C^{k-1}(U(x_0))$ such that the system (2) has full rank (in $U(x_0)$).*

Proof. This theorem is a consequence of the construction of [1] and the straightening theorem. We give a formal proof.

Let us apply the method of [1] described above. By the straightening theorem, a straightening diffeomorphism $z = \eta(x)$ defined by the system (4) exists in some neighborhood $U(x_0)$ and belongs to the class $C^k(U(x_0))$ [2].

Now, let us observe that the system (7) is linear and the change of variables $\xi = \psi(z)$ is polynomial, hence, the system (7) in z-coordinates takes the form

$$\dot{z} = \widetilde{f}(z) + \widetilde{g}(z)u,$$

where $\widetilde{f}(z)$, $\widetilde{g}(z)$ are real analytic. Then for the system in the initial coordinates we get

$$\begin{gathered} \dot{x} = (\eta^{-1}(z))'\dot{z}_{\,|z=\eta(x)} = \\ = (\eta^{-1}(z))'\widetilde{f}(z)_{\,|z=\eta(x)} + (\eta^{-1}(z))'\widetilde{g}(z)_{\,|z=\eta(x)}u = \\ = f(x) + (\eta^{-1}(z))'\widetilde{g}(z)_{\,|z=\eta(x)}u. \end{gathered}$$

As we mentioned above, $\eta(x) \in C^k(U(x_0))$, hence, $(\eta^{-1}(z))'$ belongs to the class C^{k-1}. Thus, the vector field $g(x) = (\eta^{-1}(z))'\widetilde{g}(z)_{\,|z=\eta(x)}$ is from the class $C^{k-1}(U(x_0))$. The theorem is proved.

Example 1. We give an example of a control system in the case when the right-hand side $f(x)$ is from the class C^1. Consider the system

$$\begin{cases} \dot{x_1} = 0, \\ \dot{x}_2 = 1 + x_1|x_1|, \end{cases}$$

where $x \in Q = \{x \in \mathbb{R}^2 : x_1 > -1\}$. We find a straightening diffeomorphism from the following system of partial differential equations

$$\begin{cases} \frac{\partial \eta_1}{\partial x_2} = 0, \\ \frac{\partial \eta_2}{\partial x_2} = \frac{1}{1+x_1|x_1|}. \end{cases} \tag{10}$$

Solving this system we get a transformation $z = \eta(x)$ of the form

$$\begin{cases} z_1 = x_1, \\ z_2 = \frac{x_2}{1+x_1|x_1|}. \end{cases} \tag{11}$$

The initial system in z-coordinates has the form

$$\dot{z}_1 = 0, \ \dot{z}_2 = 1,$$

and in ξ-coordinates, where $\xi_1 = z_1 + \frac{z_2^2}{2}$, $\xi_2 = z_2$, it is

$$\dot{\xi}_1 = \xi_2, \ \dot{\xi}_2 = 1.$$

Let us choose $g(\xi) = (0,1)^T$, then the system

$$\dot{\xi}_1 = \xi_2, \ \dot{\xi}_2 = 1 + u \tag{12}$$

has full rank. The system of full rank (12) in z-coordinates takes the form

$$\dot{z}_1 = -z_2 u, \ \dot{z}_2 = 1 + u.$$

By the inverse coordinate transformation

$$\begin{cases} x_1 = z_1, \\ x_2 = z_2(1 + z_1|z_1|) \end{cases}$$

we obtain the system of full rank in the initial coordinates:

$$\begin{cases} \dot{x}_1 = -\frac{x_2}{1+x_1|x_1|} u, \\ \dot{x}_2 = 1 + x_1|x_1| + (1 + x_1|x_1| - \frac{2|x_1|x_2^2}{(1+x_1|x_1|)^2}) u. \end{cases} \tag{13}$$

Thus, the vector field $g(x)$ is chosen as

$$g(x) = \begin{pmatrix} -\frac{x_2}{1+x_1|x_1|} \\ 1 + x_1|x_1| - \frac{2|x_1|x_2^2}{(1+x_1|x_1|)^2} \end{pmatrix}. \tag{14}$$

This example illustrates Theorem 1. We can see that the vector field

$$f(x) = \begin{pmatrix} 0 \\ 1 + x_1|x_1| \end{pmatrix}$$

is indeed from the class C^1 while the smoothness class of the constructed vector field $g(x)$ of the form (14) is one less, i.e., $g(x)$ is only continuous (it is not differentiable at the points where $x_1 = 0$, $x_2 \neq 0$).

3. Constructing a system of full rank in the case $f(x_0) = 0$

Now, let us consider a method of constructing a system of full rank in a neighborhood of a point x_0 such that $f(x_0) = 0$. In this case we admit non-autonomous vector fields, that is, we choose a vector field g in the form $g = g(t, x)$. For a system

$$\dot{x} = f(x) + g(t, x)u,$$

the full rank property means that the dimension of the span of all Lie brackets of vector fields $\mathrm{ad}^k_{\widehat{f}} g(t, x)$, $k \geq 0$, equals n, where the vector field $\widehat{f}$ corresponds to the differential operator $\frac{\partial}{\partial t} + f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}$.

Theorem 2 *Let a vector field $f(x) \in C^k(Q)$ be given, $Q \subset \mathbb{R}^n$, $n > 1$, and $x_0 \in Q$. There exists a neighborhood $U(\hat{x}_0)$, where $\hat{x}_0 = (0, x_0)^T \in \mathbb{R}^{n+1}$, and a vector field $g(t, x) \in C^{k-1}(U(\hat{x}_0))$ such that the system*

$$\dot{x} = f(x) + g(t, x)u \tag{15}$$

has full rank (in $U(\hat{x}_0)$).

In some cases (for example, if $f(x_0) \neq 0$) a vector field g can be chosen as $g = g(x)$. However, the case $f(x) \equiv 0$ demonstrates that this is not true in the general case (if $n > 1$).

Proof. Supplementing the system (1) by the equation $\dot{t} = 1$ we obtain the following system

$$\dot{x}_1 = f_1(x),\ \ldots,\ \dot{x}_n = f_n(x),\ \dot{t} = 1, \tag{16}$$

where the right-hand side is nonzero for any x and t. Now we find a straightening diffeomorphism $z = \eta(t, x)$ from the system

$$\begin{cases} \frac{\partial \eta_i}{\partial t} = -\frac{\partial \eta_i}{\partial x_1} f_1 - \ldots - \frac{\partial \eta_i}{\partial x_n} f_n,\ i = \overline{1, n}, \\ \eta_{n+1} = t. \end{cases} \tag{17}$$

As a result of straightening the vector field, the system (16) takes the form

$$\dot{z}_1 = 0,\ \ldots,\ \dot{z}_n = 0,\ \dot{z}_{n+1} = 1. \tag{18}$$

Let us choose the vector field g in the form $g = g(t) = (1, t, t^2, \ldots, t^{n-1}, 0)^T$; then we obtain the system of full rank of the form

$$\dot{z}_1 = u, \ \dot{z}_2 = tu, \ \ldots, \ \dot{z}_n = t^{n-1}u, \ \dot{z}_{n+1} = 1. \tag{19}$$

Applying the inverse coordinate transformation $(x, t) = \eta^{-1}(z)$ (and dropping the equation $\dot{t} = 1$) we obtain the system of full rank in the initial coordinates:

$$\dot{x} = f(x) + g(t, x)u.$$

Arguing analogously to the proof of Theorem 1 we get that $g(t, x) \in C^{k-1}(U(\hat{x}_0))$. The theorem is proved.

Example 2. Consider the linear system

$$\begin{cases} \dot{x_1} = 0, \\ \dot{x}_2 = x_1. \end{cases}$$

Since we cannot use the result of [1] at the point with $x_1 = 0$, we apply the method described above. Namely, we add the equation $\dot{t} = 1$ to the given system and obtain the equivalent system

$$\begin{cases} \dot{x_1} = 0, \\ \dot{x}_2 = x_1, \\ \dot{t} = 1. \end{cases} \tag{20}$$

Now we can straighten the vector field and find a straightening diffeomorphism from the following system of partial differential equations

$$\begin{cases} \frac{\partial \eta_i}{\partial t} = -\frac{\partial \eta_i}{\partial x_2} x_1, \ i = 1, 2, \\ \eta_3 = t. \end{cases} \tag{21}$$

Solving this system we obtain a transformation of the form

$$\begin{cases} z_1 = x_1, \\ z_2 = tx_1 - x_2, \\ z_3 = t. \end{cases} \tag{22}$$

We perform the straightening diffeomorphism, then the system (20) in z-coordinates takes the form

$$\dot{z}_1 = 0, \ \dot{z}_2 = 0, \ \dot{t} = 1.$$

Let us choose the vector field $g(t) = (1, t, 0)^T$, then the resulting system

$$\dot{z}_1 = u, \ \dot{z}_2 = tu, \ \dot{t} = 1 \tag{23}$$

has full rank. Performing the inverse change of variables

$$\begin{cases} x_1 = z_1, \\ x_2 = tz_1 - z_2, \\ t = z_3, \end{cases}$$

and dropping the trivial equation $\dot{t} = 1$ we get the system of full rank in the initial coordinates

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1. \end{cases} \tag{24}$$

Thus, the vector field g is chosen as $g = (1, 0)^T$, therefore, it is constant and in particular does not depend on t.

Example 3. Consider the nonlinear system

$$\begin{cases} \dot{x}_1 = x_1^2, \\ \dot{x}_2 = x_2. \end{cases} \tag{25}$$

Let us construct a system of full rank in a neighborhood of the point where $f(x) = 0$, i.e., $x_0 = 0$. To this end, we supplement the system (25) with the equation $\dot{t} = 1$ and get

$$\begin{cases} \dot{x_1} = x_1^2, \\ \dot{x}_2 = x_2, \\ \dot{t} = 1. \end{cases} \tag{26}$$

A straightening diffeomorphism for the system (26) can be found from the following system of partial differential equations

$$\begin{cases} \frac{\partial \eta_i}{\partial t} = -(\frac{\partial \eta_i}{\partial x_1} x_1^2 + \frac{\partial \eta_i}{\partial x_2} x_2), \ i = 1, 2, \\ \eta_3 = t, \end{cases} \tag{27}$$

then as a transformation $z = \eta(x, t)$ we can take

$$\begin{cases} z_1 = \frac{x_1}{x_1 t + 1}, \\ z_2 = x_2 e^{-t}, \\ z_3 = t, \end{cases} \tag{28}$$

which is defined in a neighborhood of the origin. The system (26) in z-coordinates has the form

$$\dot{z}_1 = 0, \ \dot{z}_2 = 0, \ \dot{t} = 1.$$

We choose the vector field $g(t) = (1, t, 0)^T$ and obtain the system of full rank in z-coordinates

$$\dot{z}_1 = u, \ \dot{z}_2 = tu, \ \dot{t} = 1.$$

Performing the inverse change of variables

$$\begin{cases} x_1 = \frac{z_1}{1-z_1 t}, \\ x_2 = z_2 e^t, \\ t = z_3, \end{cases} \tag{29}$$

and removing the trivial equation $\dot{t} = 1$ we obtain the system in x-coordinates

$$\begin{cases} \dot{x_1} = x_1^2 + (x_1 t + 1)^2 u, \\ \dot{x}_2 = x_2 + t e^t u, \end{cases}$$

which has full rank in a neighborhood of the point $x_0 = 0$ where the vector field f equals zero.

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Андреєва Д. М., Ігнатович С. Ю. **Про побудову неавтономних систем повного рангу з одним керуванням.** У статті розвинено метод конструювання системи повного рангу, який було запропоновано в роботі [Y. Kawano, Ü. Kotta, C.H. Moog. Any dynamical system is fully accessible through one single actuator, and related problems, Intern. J. of Robust and Nonlinear Control, – 2016. – 8. V.**26**. – P. 1748-1754.]. Задача полягає в наступному: для заданого векторного поля $f(x)$ знайти таке векторне поле $g(x)$, що отримана афінна керована система $\dot{x} = f(x) + g(x)u$ буде повного рангу. У вищевказаній роботі було показано, що таке $g(x)$ існує в околі точки x, якщо $f(x) \neq 0$, та було запропоновано метод конструювання $g(x)$. Як основний інструмент було застосовано теорему про випрямлення векторного поля; фактично, випрямляючи векторне поле $f(x)$, ми конструюємо лінійну керовану систему. Проте, було розглянуто тільки випадок дійсно аналітичних векторних полів. В даній роботі ми розглядаємо два узагальнення. По-перше, ми вивчаємо дане питання для векторних полів $f(x) \in C^k$, $1 \leq k < \infty$. Ми показуємо, що запропонований метод можна застосувати, проте векторне поле $g(x)$, взагалі кажучи, буде належати тільки класу C^{k-1}. Ми наводимо приклад векторного поля $f(x) \in C^1$, а саме, $f(x) = (0, 1/(1 + x_1|x_1|))^T$, для якого метод дає недиференційовне (хоча й неперервне) векторне поле $g(x)$. По-друге, ми розглядаємо випадок, коли $f(x)$ обертається на нуль, та описуємо метод конструювання векторного поля $g(t, x)$, яке, взагалі кажучи,

є неавтономним, такого, що система $\dot{x} = f(x) + g(t,x)u$ є повного рангу. Ми застосовуємо теорему про випрямлення векторного поля, але для розширеної системи, в якій час є додатковою координатою. Ми наводимо приклад лінійного векторного поля $f(x) = (0, x_1)^T$ в околі початку координат, в якому отримане векторне поле є автономним, а саме $g(x) = (1, 0)^T$. Також ми наводимо приклад нелінійного векторного поля $f(x) = (x_1^2, x_2)^T$ в околі початку координат; відповідне неавтономне векторне поле має вигляд $g(t,x) = ((x_1 t + 1)^2, te^t)^T$.
Ключові слова: нелінійна керована система; досяжна система; система повного рангу; неавтономна система; теорема про випрямлення векторного поля.

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